Hamiltonian Dynamics Problem Sheet

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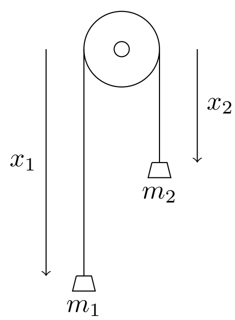
Problem 1. Write down Hamilton's equations for the following Hamiltonians

$$H(q_1, q_2, p_1, p_2; t) = \frac{1}{2}\left(p_1^2\left(p_2^2 + q_2^2\right) + q_1^2\right)$$

$$H = \frac{1}{2m}\left(p_r^2 + \frac{p_\theta^2}{r^2} + \frac{p_\phi^2}{r^2 \sin^2\theta}\right) - \frac{\mu m}{r}$$

Problem 2.



Two masses are hanging via a massless string from a frictionless pulley, The kinetic energy of the masses is

$$T = \frac{1}{2}m_1\dot{x}_1^2 + \frac{1}{2}m_2\dot{x}_2^2, \tag{1}$$

while the potential energy is

$$V = -m_1 g x_1 - m_2 g x_2. \tag{2}$$

We selected $V = 0$ at the centre of the pulley. The system is subjected to the constraint $x_1 + x_2 = l =$ constant. Write down the Lagrangian, convert to the Hamiltonian and write down Hamilton's equations.

Problem 3. Show that the following transformation from (q,p) to (Q,P) is canonical

$$P = 2(1 + \sqrt{q}\cos p)\sqrt{q}\sin p,\ Q = \ln(1 + \sqrt{q}\cos p)$$

by checking if the Poisson bracket $[Q, P]_{q,p} = 1$. Verify that the following type 3 generating function $F_3(p, Q)$ corresponds to this transformation.

$$F_3(p, Q) = -\left(e^Q - 1\right)^2 \tan p$$

Problem 4. An idealised kick rotator may be represented by the following discrete map

$$\theta_{n+1/2} = \theta_n + 0.5 * K * p_n \tag{3}$$

$$p_{n+1} = p_n - K\sin\theta_{n+1/2} \tag{4}$$

$$\theta_{n+1} = \theta_{n+1/2} + 0.5 * K * p_{n+1} \tag{5}$$

Write a code (e.g. in Python) to iterate this map a few hundred times starting with a set of starting coordinates (p_0, θ_0) that cover the range ($-\pi$, π) in both phase space coordinates. Plot all the coordinates after each iteration on a single phase space figure. Repeat for various values of K ($K << 1$, $K \sim 1$ and $K > 1$). You should observe bounded motion, chaos, islands of stability, fixed points etc.

Problem 5. The Lie transform for a hard edge quadrupole of length L and strength k can be written

$$f = -(L/2)\left(kx^2 + p_x^2\right) \tag{6}$$

Show that this transform is equivalent to the transfer matrix M for a quadrupole,

$$M = \begin{pmatrix} \cos\sqrt{k}L & \frac{1}{\sqrt{k}}\sin\sqrt{k}L \\ -\sqrt{k}\sin\sqrt{k}L & \cos\sqrt{k}L \end{pmatrix}$$