Lecture 8
Beams and Imperfections

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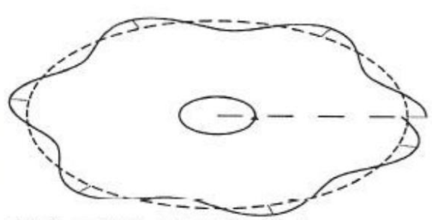
Resonance & Resonant Conditions

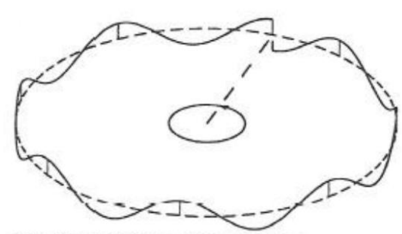
- After a certain number of turns around the machine the phase advance of the betatron oscillation is such that the oscillation repeats.

- For example:
  - If the phase advance per turn is 120º then the betatron oscillation will repeat itself after 3 turns.
  - This could correspond to tune Q = 3.333 or 3Q = 10.
  - But also Q = 2.333 or 3Q = 7.

- The order of a resonance is defined as 'n' in **n x Q = integer**

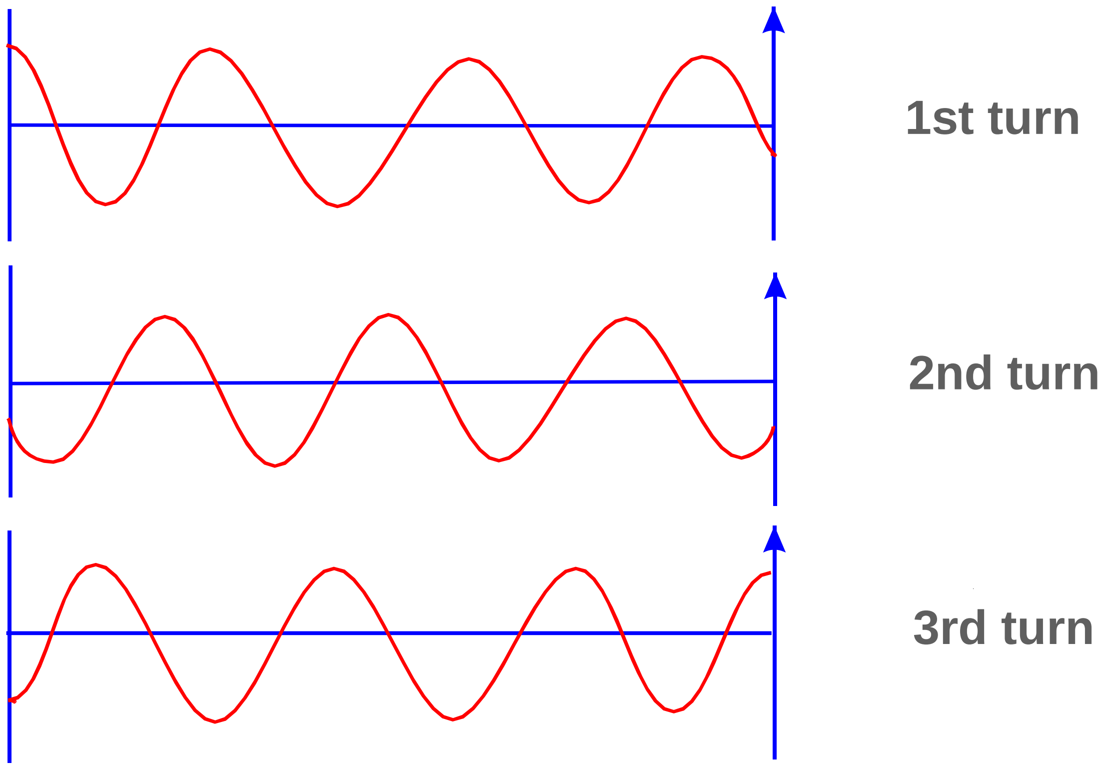


Horizontal Betatron Oscillation
with tune: $Q_h = 6.3$,
i.e., 6.3 oscillations per turn.



Vertical Betatron Oscillation
with tune: $Q_v = 7.5$,
i.e., 7.5 oscillations per turn.

Q = 3.333

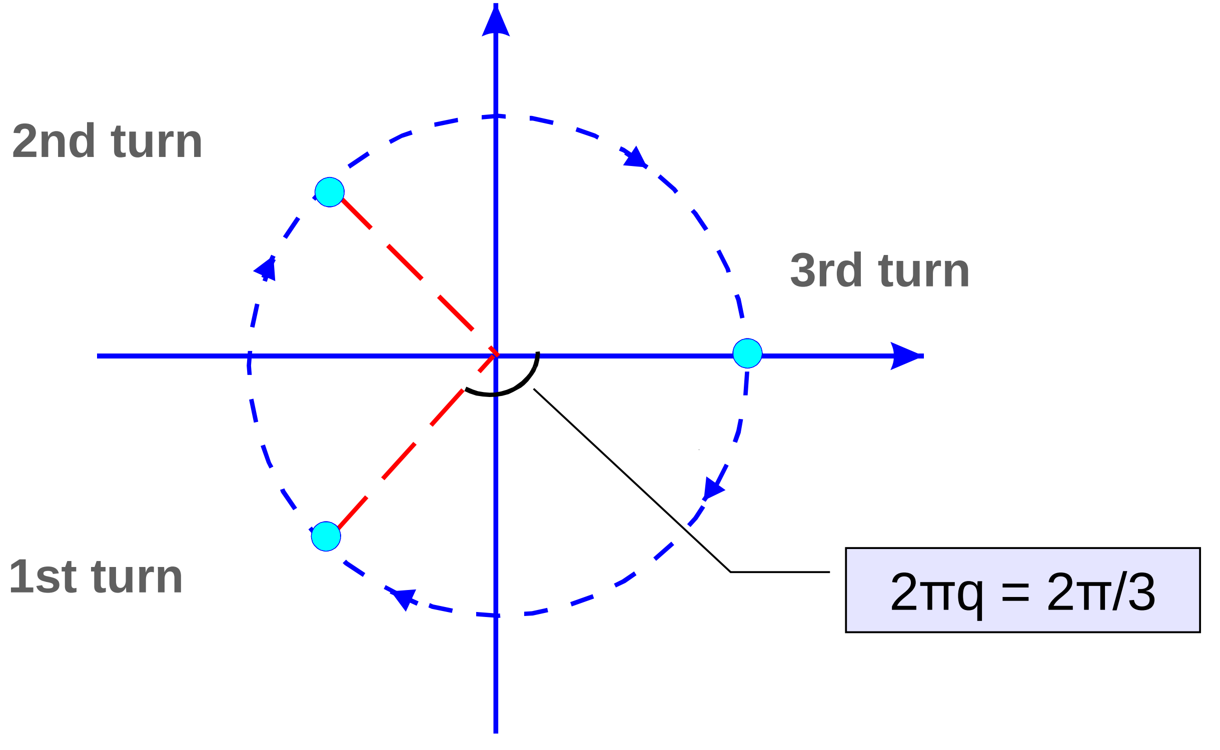


Third order resonant betatron oscillation

3Q = 10, Q = 3.333, q = 0.333

Q = 3.333 in Normalised Phase Space

- ✓ Third order resonance on a normalised phase space plot



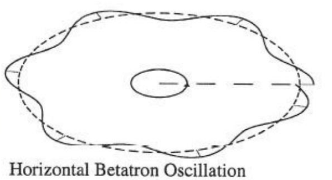
Resonant Conditions: A bit more detail

- Synchrotron is periodic focusing system, often made up of smaller periodic regions.
  - Can write down a periodic one-turn matrix as

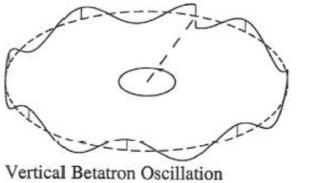
$$M = I\cos\Delta\phi_C + J\sin\Delta\phi_C \quad I = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \quad J = \begin{pmatrix} \alpha(s) & \beta(s) \\ -\gamma(s) & -\alpha(s) \end{pmatrix}$$

- Tune is defined as the total betatron phase advance in one revolution around the ring, divided by 2π

$$Q_{x,y} = \frac{\Delta\phi_{x,y}}{\Delta\theta} = \frac{1}{2\pi}\oint \frac{ds}{\beta_{x,y}(s)}$$



Horizontal Betatron Oscillation with tune: $Q_h = 6.3$, i.e., 6.3 oscillations per turn.



Vertical Betatron Oscillation with tune: $Q_v = 7.5$, i.e., 7.5 oscillations per turn.

Resonant Conditions: A bit more detail

- Tunes are both horizontal and vertical.
- Are direct indication of amount of focusing in an accelerator.
  - Higher tune means tighter focusing, lower $< \beta_{x,y}(s) >$

- Tunes are critical for accelerator performance
  - Linear stability depends upon phase advance.
  - Resonant instabilities can occur when $nQ_x + mQ_y = k$
  - Often adjusted using groups of quadrupoles

$$M_{one-turn} = I\cos(2\pi Q) + J\sin(2\pi Q)$$

Resonance & Resonant Conditions

- Resonance can be excited through various imperfections in the beamline.
  - The magnets themselves.
  - Unwanted higher-order field components in magnets.
  - Tilted magnets.
  - Experiment solenoids (LHC experiments).

- Aim is to reduce and compensate these effects as much as possible and then find some point in the tune diagramme where the beam is stable.

Machine Imperfections

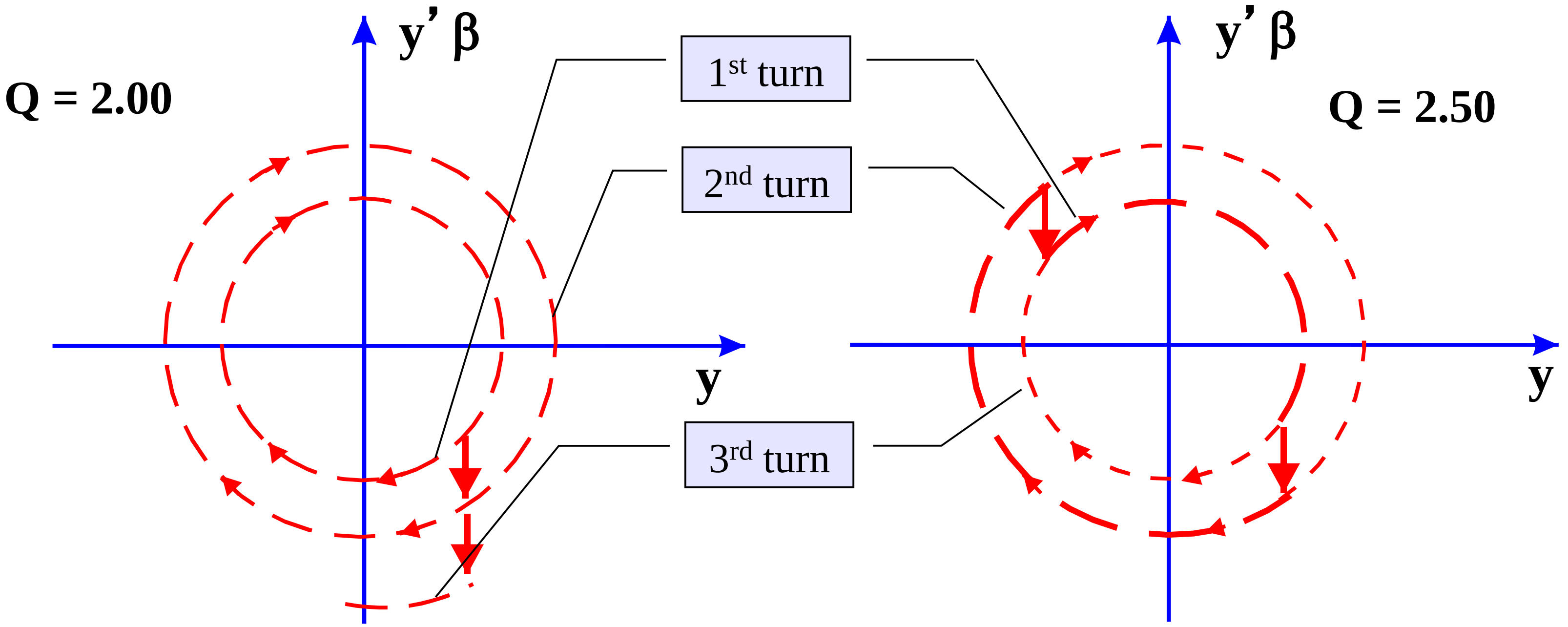
- It is not possible to construct a perfect machine.
  - Magnets can have imperfections.
  - The alignment in the machine has non-zero tolerance.
  - …
- So, have to ask:
  - What will happen to betatron oscillation due to various field errors.
  - Consider errors in dipoles, quadrupoles, sextupoles, etc…
- Study the beam behaviour as a function of ‘Q’.
- How is it influenced by these resonant conditions?

Machine Imperfections

- Various imperfections in the beamline will alter the tune in a periodic machine.

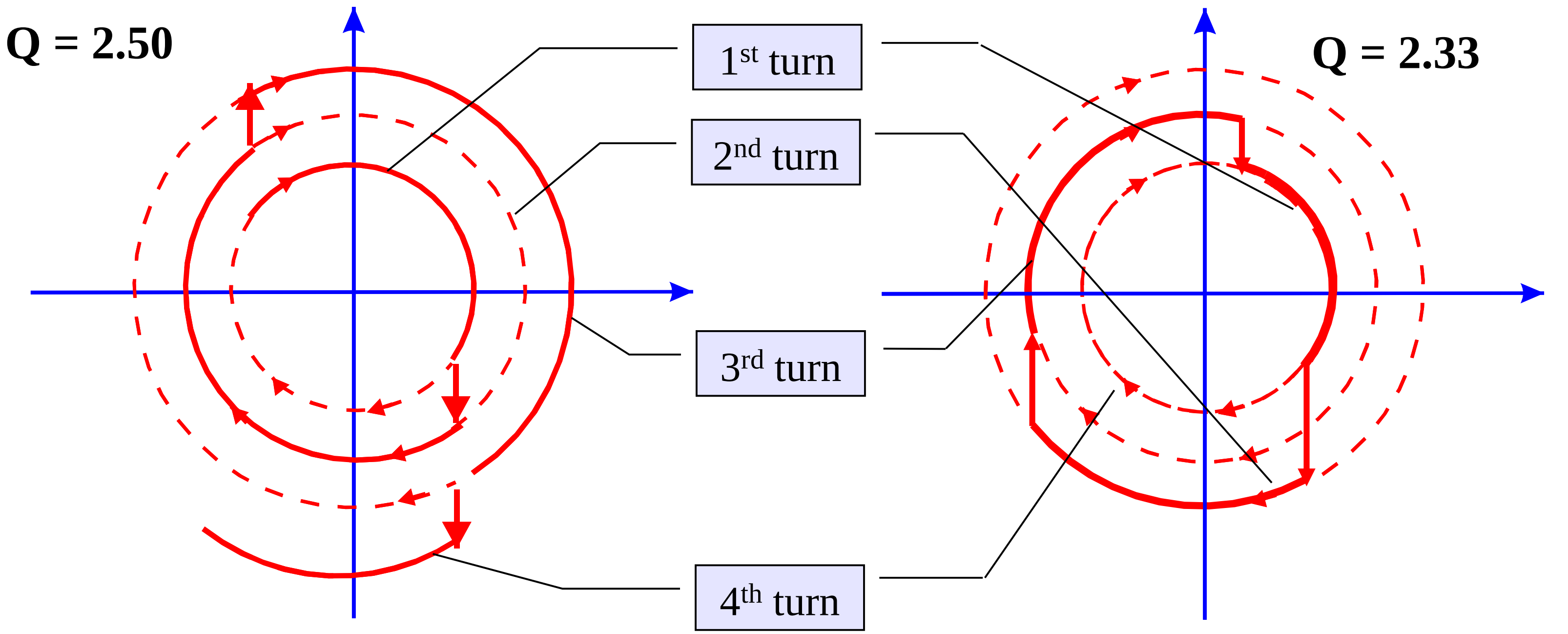
- One way to visualize the influence of these imperfections is by looking at what happens in the normalised phase space plot.

Dipole (deflection independent of position)



- ✓ For **Q = 2.00**: Oscillation induced by the **dipole kick** grows on each turn and the particle is lost (**1st order resonance Q = 2**).
- ✓ For **Q = 2.50**: Oscillation is cancelled out **every second turn**, and therefore the particle **motion is stable**.

Quadrupole (deflection ∝ position)

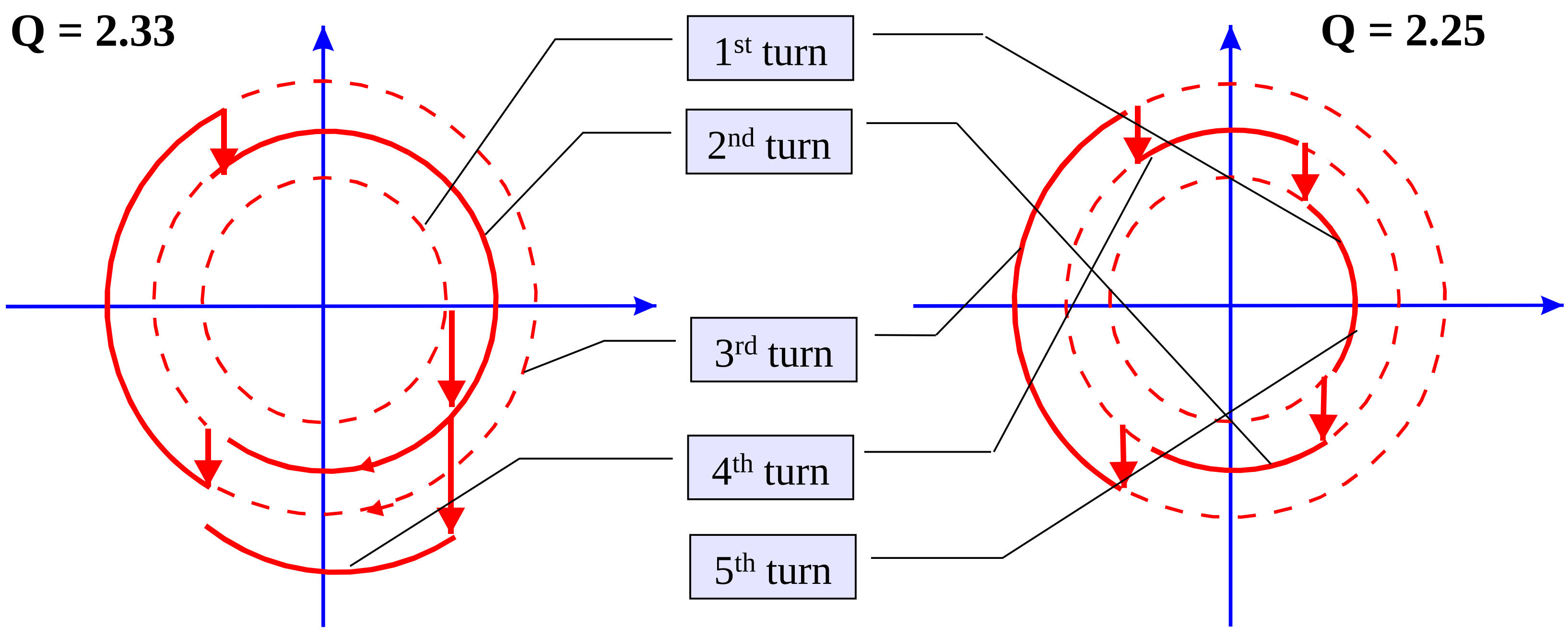


- ✓ For **Q = 2.50**: Oscillation induced by the **quadrupole kick** grows on each turn and the particle is lost

  (2nd order resonance 2Q = 5)

- ✓ For **Q = 2.33**: Oscillation is cancelled out **every third turn**, and therefore the particle **motion is stable**.

Sextupole (deflection ∝ position^2)



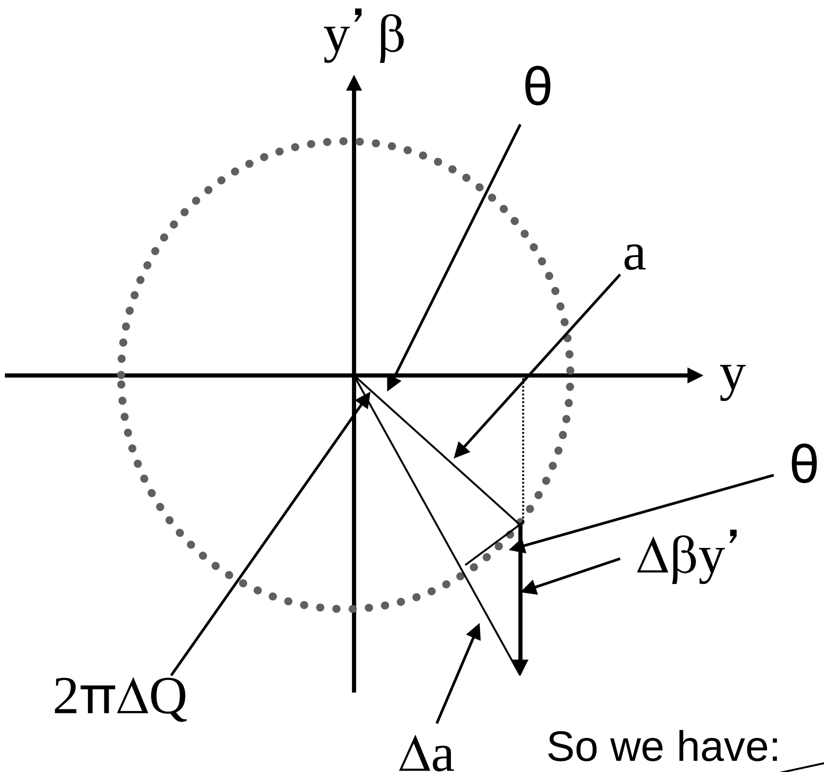
- ✓ For **Q = 2.33**: Oscillation induced by the **sextupole kick** grows on each turn and the particle is lost

(3rd order resonance 3Q = 7)

- ✓ For **Q = 2.25**: Oscillation is cancelled out **every fourth turn**, and therefore the particle **motion is stable**.

Resonant Condition - Quadrupole

- Let us try to a **mathematical expression** for **amplitude growth** in the case with a **quadrupole**:



$2\pi Q$ = phase angle over 1 turn = θ

$\Delta\beta y'$ = kick

a = old amplitude

Δa = change in amplitude

$2\pi\Delta Q$ = change in phase

y does not change at the kick

$y = a\cos(\theta)$

In a quadrupole $\Delta y' = lky$

Only if $2\pi\Delta Q$ is small

So we have:

$\Delta a = \beta\Delta y' \sin(\theta) = l\beta \sin(\theta)\, a\, k\cos(\theta)$

Resonant Condition - Quadrupole

- So have: $\Delta a = l \cdot \beta \cdot \sin(\theta)\ a \cdot k \cdot \cos(\theta)$ $\therefore \dfrac{\Delta a}{a} = \dfrac{\ell \beta k}{2}\sin(2\theta)$

Sin(θ)Cos(θ) = 1/2 Sin (2θ)

- Each turn θ advances by 2πQ
- On the n[th] turn θ = θ + 2nπQ
- Over many turns: $$\frac{\Delta a}{a} = \frac{\ell \beta k}{2}\sum_{n=1}^{\infty}\sin\left(2\left(\theta + 2n\pi Q\right)\right)$$

$\therefore \mathbf{\Delta a \rightarrow 0}$

This term will be 'zero' as it decomposes in Sin and Cos terms and will give a series of + and – that cancel out in all cases where the fractional tune q ≠ 0.5

- For q = 0.5 the phase term, 2(θ + 2nπQ) is constant:

$$\sum_{n=1}^{\infty}\sin\left(2\left(\theta + 2n\pi Q\right)\right) = \infty$$

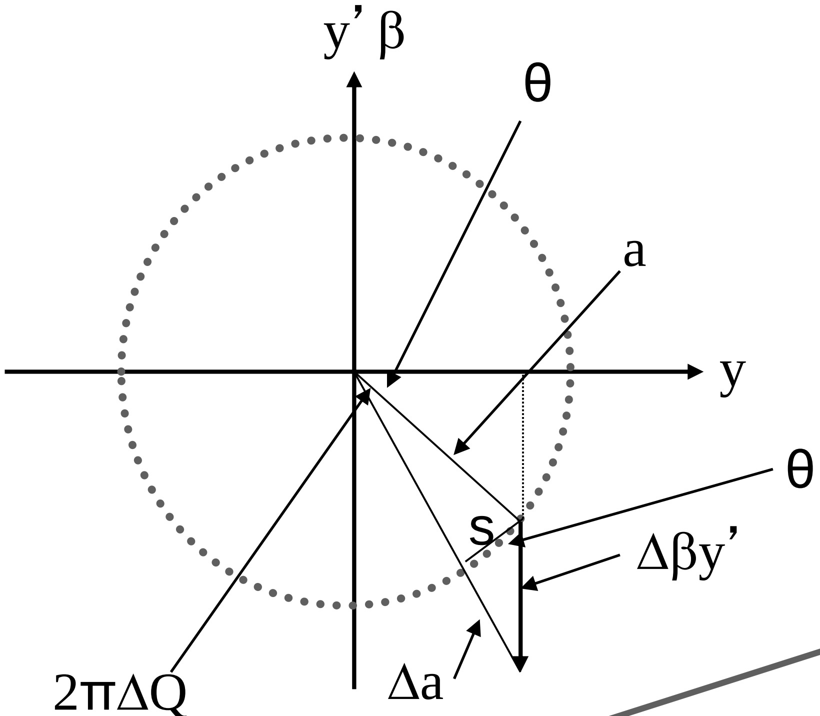
and thus: $$\frac{\Delta a}{a} = \infty$$

Resonant Condition - Quadrupole

- In this case the amplitude will grow continuously until the particle is lost.
- Therefore, conclude as before that: **quadrupoles excite 2^{nd} order resonances for q=0.5**
  - Namely, for Q = 0.5, 1.5, 2.5, 3.5,…etc……

Resonant Condition - Quadrupole

- Study phase θ:



2πQ = phase angle over 1 turn = θ
Δβy' = kick
a = old amplitude
Δa = change in amplitude
2πΔQ = change in phase

y does not change at the kick

y = a cos(θ)

In a quadrupole **Δy' = lky**

$$s = \Delta(\beta y')\cos\theta$$

$$2\pi\Delta Q = \frac{\Delta(\beta y')\cos\theta}{a} \rightarrow \Delta Q = \frac{1}{2\pi}\cdot\frac{\beta\cdot\cos(\theta)\cdot l\cdot a\cdot k\cdot\cos(\theta)}{a}$$

2πΔQ << Therefore Sin(2πΔQ) ≈ 2πΔQ

Resonant Condition - Quadrupole

- So have: $$\Delta Q = \frac{1}{2\pi} \cdot \frac{\beta \cdot \cos(\theta) \cdot l \cdot a \cdot k \cdot \cos(\theta)}{a}$$
- Since: $Cos^2(\theta) = \frac{1}{2} Cos(2\theta) + \frac{1}{2}$ can rewrite this as:

$$\Delta Q = \frac{1}{4\pi} \cdot l \cdot \beta \cdot k \cdot (\cos(2\theta) + 1)$$, which is correct for the 1st turn

- Each turn θ advances by 2πQ
- On the n^{th} turn θ = θ + 2nπQ
- Over many turns: $$\Delta Q = \frac{1}{4\pi} \ell \beta k \left[\sum_{n=1}^{\infty} \underbrace{\cos\left(2\left(\theta + 2\pi n Q\right)\right)}_{\text{'zero'}} + 1 \right]$$
- Averaging over many turns: $$\Delta Q = \frac{1}{4\pi} \beta.k.ds$$

Resonant Condition - Sextupole

- Can apply the same arguments for a sextupole:
- For a sextupole $\boxed{\Delta y' = \ell k y^2}$ and thus $\boxed{\Delta y' = \ell k a^2 \cos^2\theta}$
- Get : $\boxed{\frac{\Delta a}{a} = \ell \beta k a \sin\theta \cos^2\theta = \frac{\ell \beta k a}{2}\left[\cos 3\theta + \cos\theta\right]}$
- Summing over many turns gives:

$$\boxed{\frac{\Delta a}{a} = \frac{\ell \beta k a}{2} \sum_{n=1}^{\infty} \underbrace{\cos 3(\theta + 2\pi n Q)}_{\text{3rd order resonance term}} + \underbrace{\cos(\theta + 2\pi n Q)}_{\text{1st order resonance term}}}$$

3rd order resonance term

1st order resonance term

- **Sextupoles** excite **1st** and **3rd** **order resonance**

q = 0

q = 0.33

Resonant Condition - Octupole

- Can apply the same arguments for an octupole:
- For an octupole $\Delta y' = \ell k y^3$ and thus $\Delta y' = \ell k a^3 \cos^3 \theta$
- We get : $\dfrac{\Delta a}{a} = \ell \beta k a^2 \sin\theta \cos^3\theta$
- Summing over many turns gives:

$$\frac{\Delta a}{a} \propto a^2(\cos 4(\theta+2\pi nQ) + \cos 2(\theta+2\pi nQ))$$

4th order resonance term

2nd order resonance term

Amplitude squared

q = 0.5

q = 0.25

- **Octupole** errors excite **2nd** and **4th** **order resonance** and are very important for larger amplitude particles.

Can restrict dynamic aperture

Stopband

- The tune does not stay constant in the machine. This leads to a variation of Q for each turn.

- This variation can go up and down, giving a range of possible values for Q, which we can call ΔQ.

- This range of values has a width, which is called the stopband of the resonance.

- Not only do you want to avoid the resonances, but you want to avoid being in the stopband of a resonance as well, as it may pull you into the resonance itself.

Stopband

- $\Delta Q = \dfrac{1}{4\pi}\beta.k.ds$ which is the expression for the change in **Q** due to a quadrupole… (fortunately !!!)

- But note that Q changes slightly on each turn

Related to Q

$$\Delta Q = \frac{1}{4\pi} l \cdot \beta \cdot k(\underline{\cos(2\theta) + 1})$$

Max variation 0 to 2

- Q has a range of values varying by: $\dfrac{\ell \beta k}{2\pi}$

- This width is called the **stopband** of the resonance.

- So even if q is not exactly 0.5, it must not be too close, or at some point it will find itself at exactly 0.5 and 'lock on' to the resonant condition.

Intermediate Summary

- **Quadrupoles** excite **2nd** order resonances.
- **Sextupoles** excite **1st** and **3rd** order resonances.
- **Octupoles** excite **2nd** and **4th** order resonances.

- This is true for small amplitude particles and low strength excitations.
- However, for stronger excitations, sextupoles will excite higher order resonances (non-linear).

Coupling

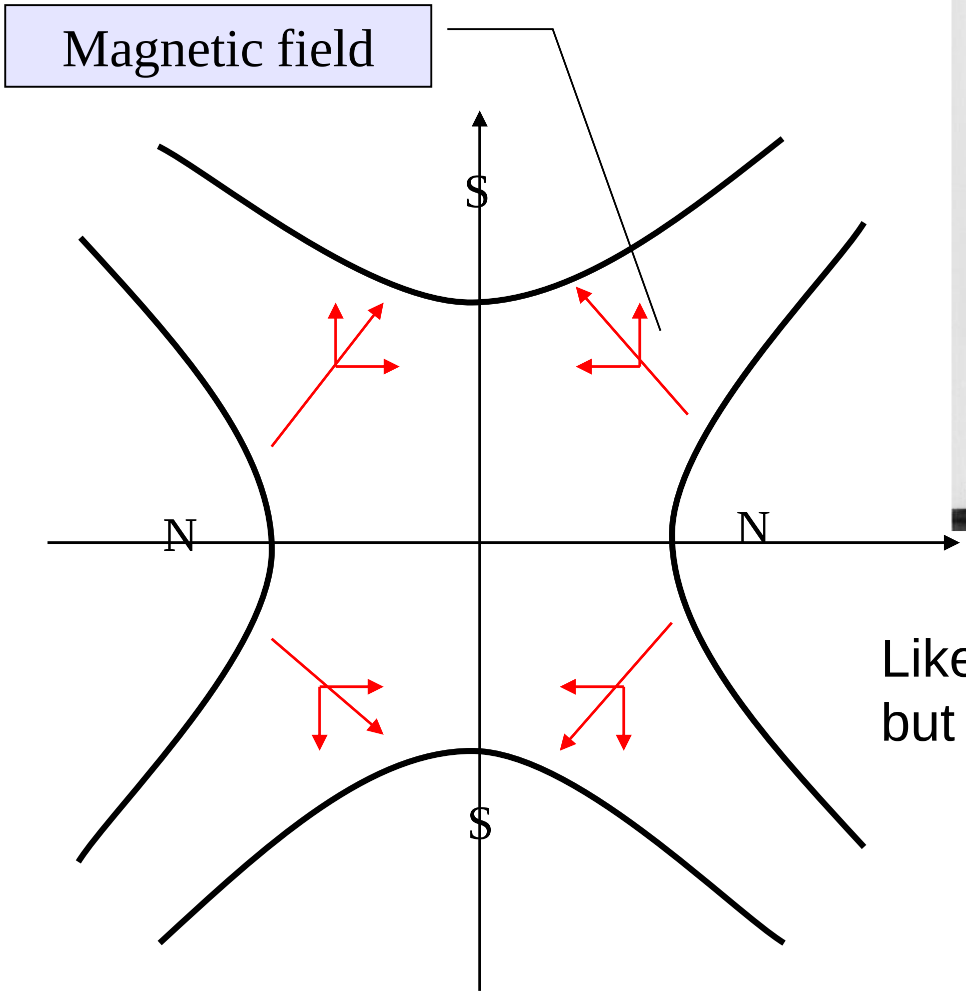
- Coupling is a phenomena that converts betatron motion in one plane (horizontal or vertical) into motion in the other plane.

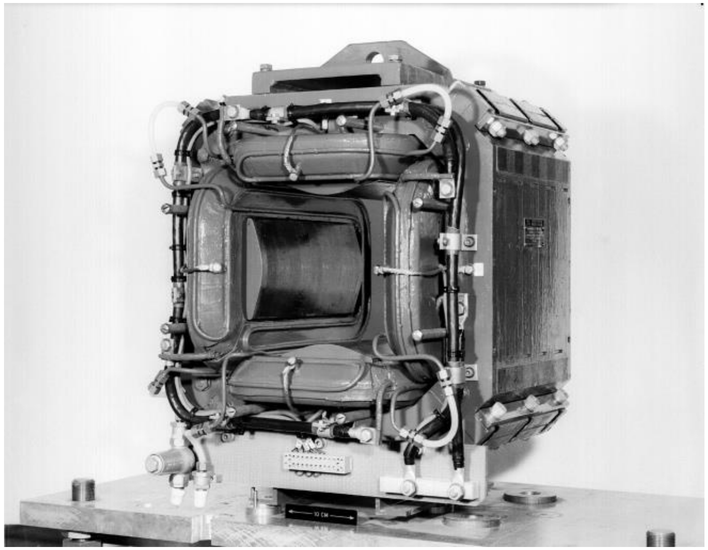
- Fields that will excite coupling are:

  - Skew quadrupoles, which are normal quadrupoles, but tilted by 45º about their longitudinal axis.

  - Solenoidal (longitudinal magnetic field).

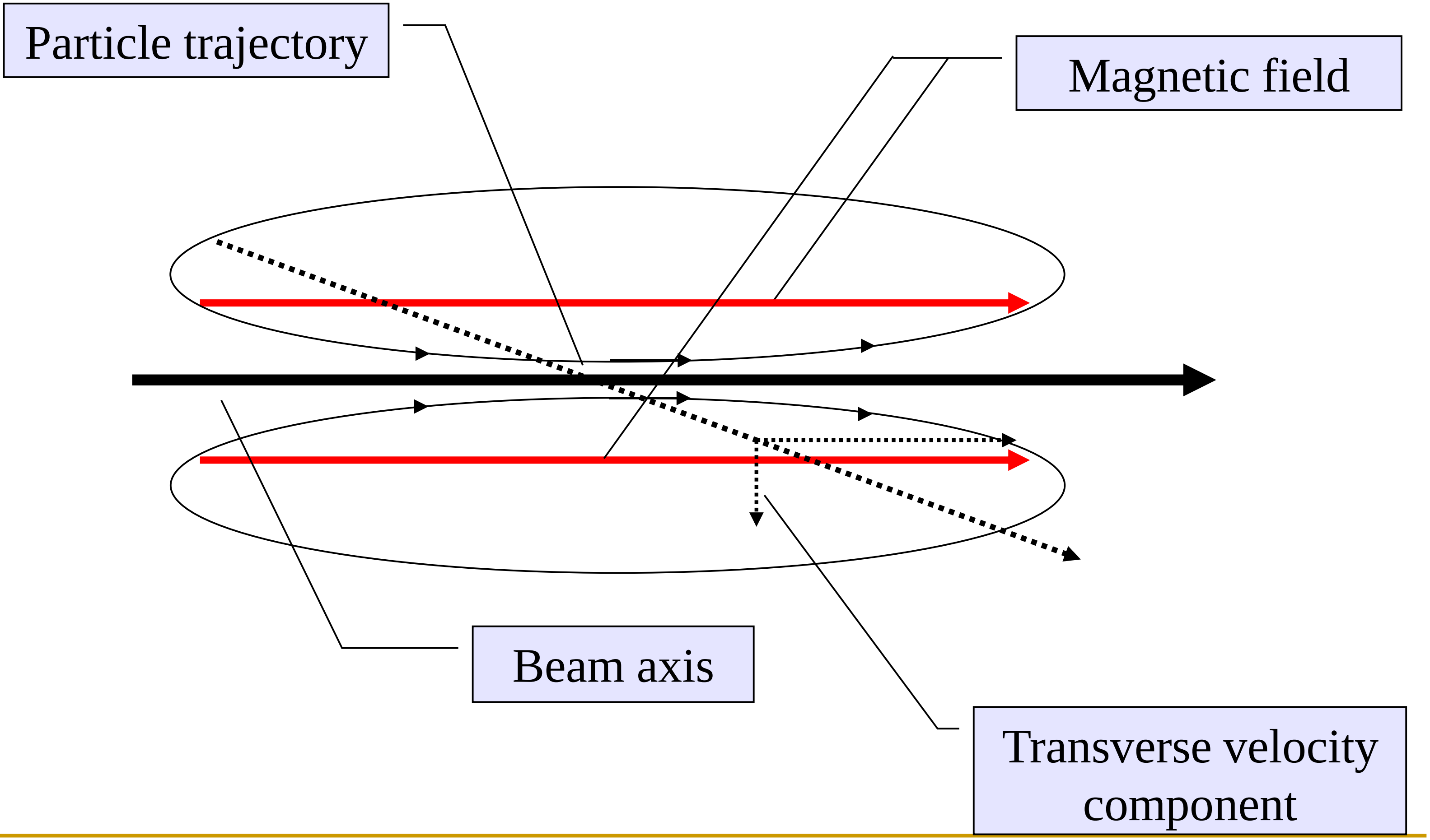
Skew Quadrupole



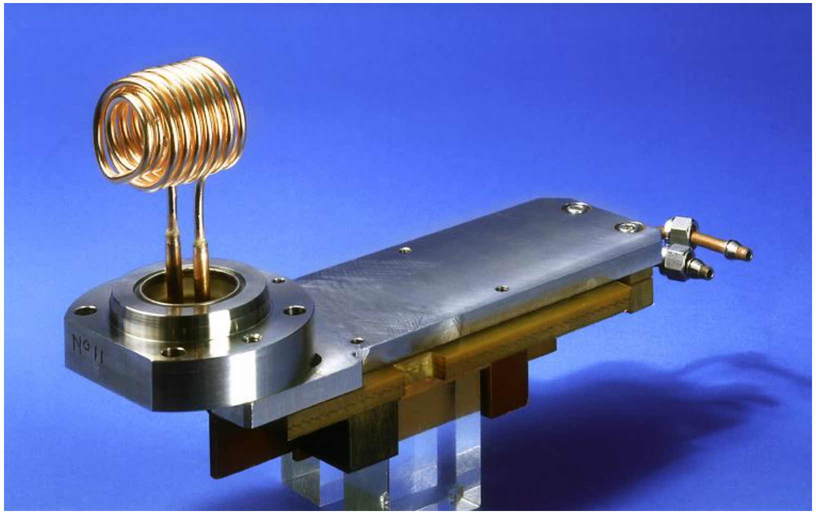


Like a normal quadrupole, but tilted by 45º

Solenoid - Longitudinal Field (1)

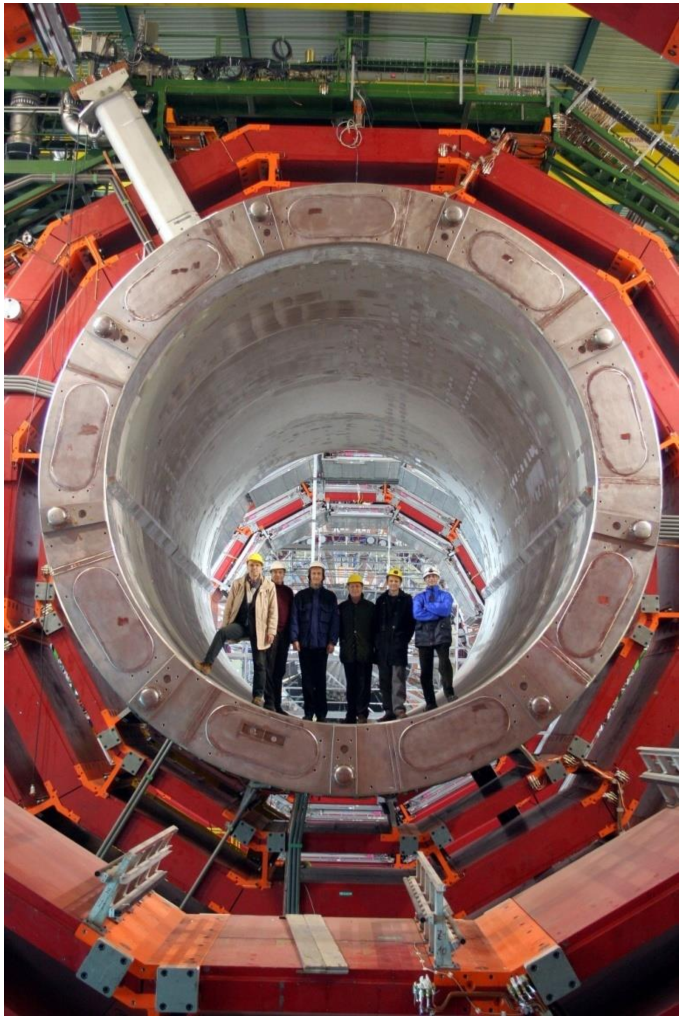


Solenoid - Longitudinal Field (2)



Above:
The LPI solenoid that was used for the initial focusing of the positrons.
It was pulsed with a current of 6 kA for some 7 μs, it produced a longitudinal magnetic field of 1.5 T.

At right:
the somewhat bigger CMS solenoid



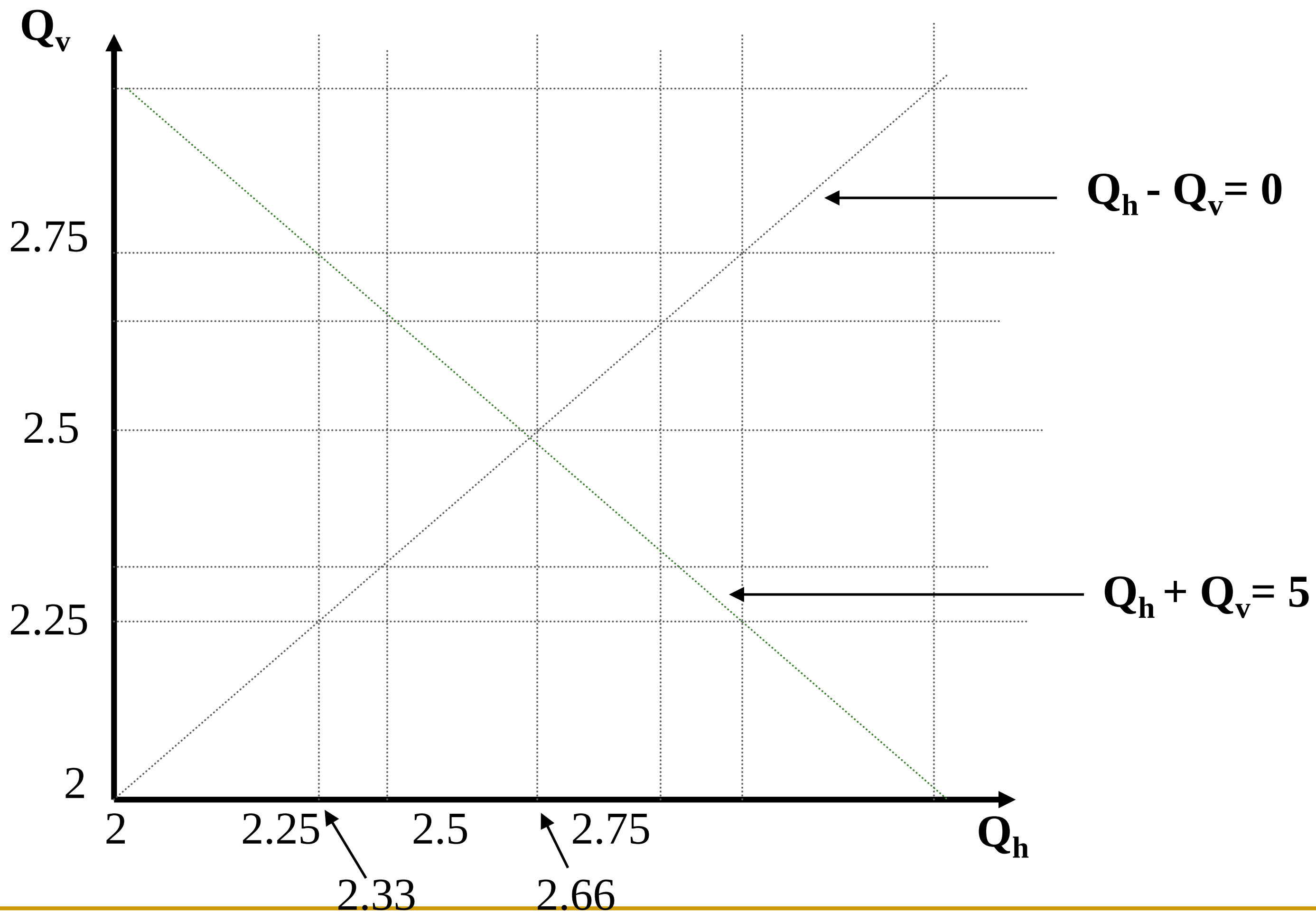
Coupling and Resonance

- This coupling means that one can transfer oscillation energy from one transverse plane to the other.
- Exactly as for linear resonances (single particle) there are resonant conditions.

$$nQ_h \pm mQ_v = \text{integer}$$

- If meet one of these conditions, the transverse oscillation amplitude will again grow in an uncontrolled way.

General Tune Diagramme



Resonant Conditions

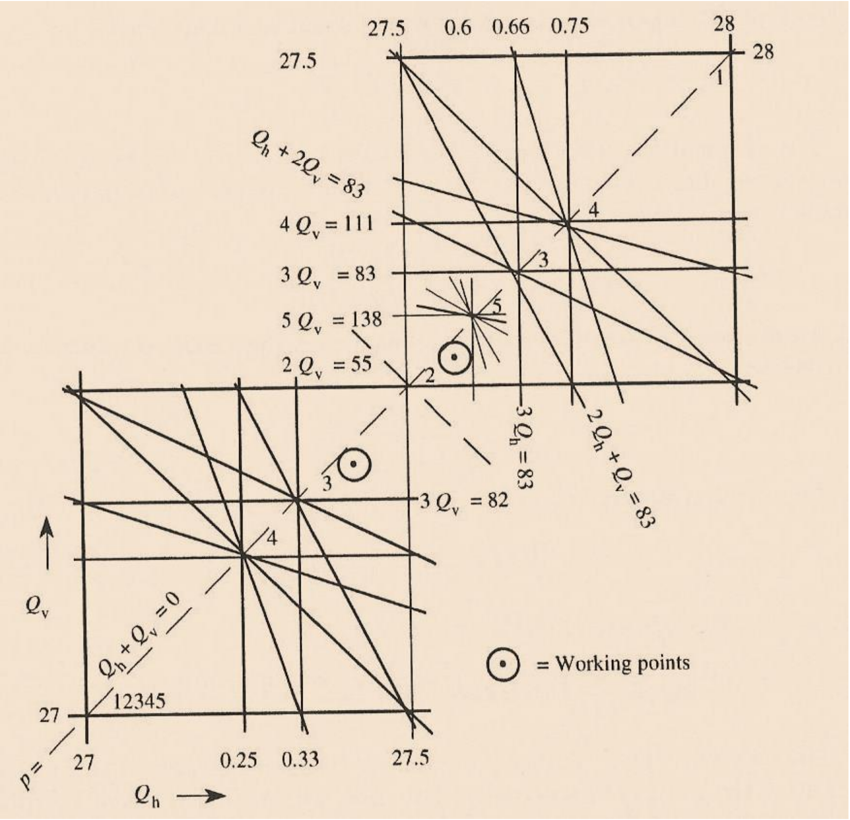
- Change in tune or phase advance resulting from errors.
  - Steer Q away from certain fractional values which can cause motion to resonate and result in beam loss.
- Resonance takes over and walks proton out of the beam for:

$$\boxed{lQ_h + mQ_v = p}$$

where

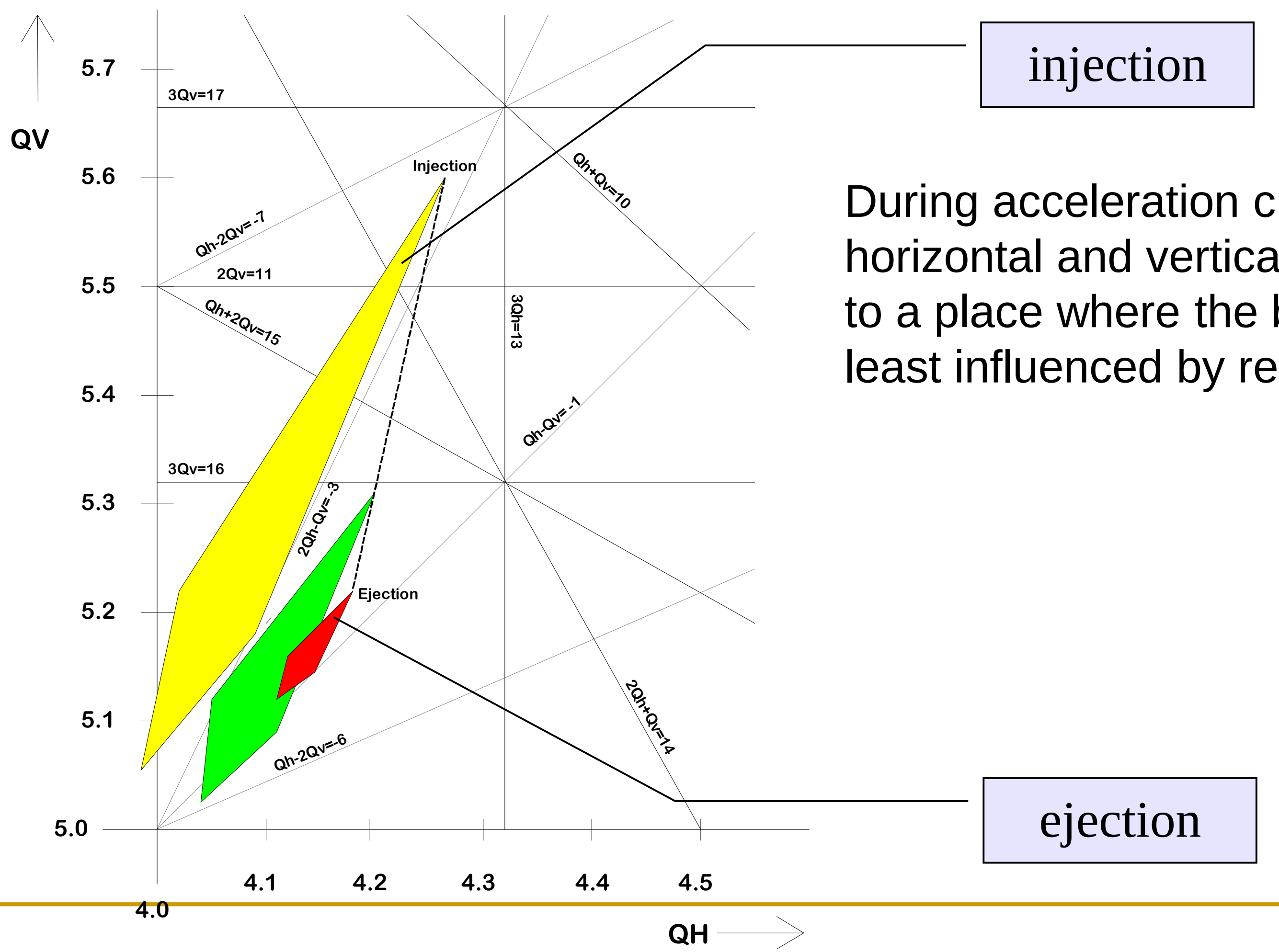
$$\boxed{|l| + |m|}$$

is resonance order and *p* is azimuthal frequency that drives it.



SPS Working Diagramme

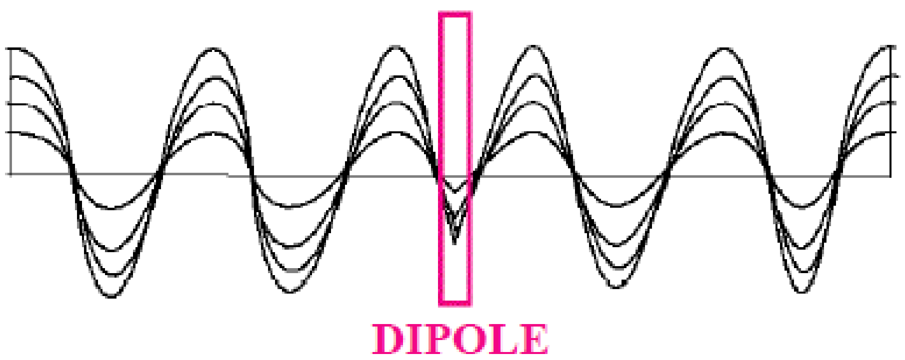
PS Booster Tune Diagramme



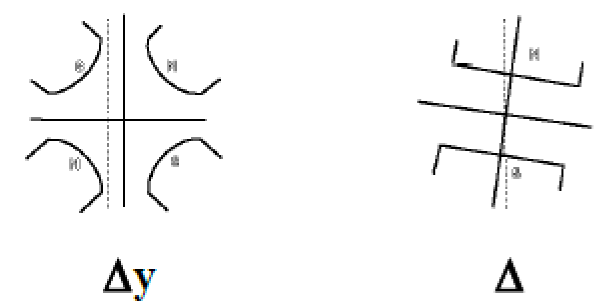
During acceleration change the horizontal and vertical tune to a place where the beam is least influenced by resonances.

Imperfection: Closed-orbit Distortion

- As current is slowly raised in dipole:
  - The zero-amplitude betatron particle follows distorted orbit.
  - Distorted orbit is closed.
  - Particle still obeys Hill's Equation.
  - Except at the kink (dipole) it follows a betatron oscillation.
  - Other particles with finite amplitudes oscillate about this new closed orbit.



Sources of Closed-orbit Distortion



Type of element	Source of kick	r.m.s. value	$\langle \Delta Bl/(B\rho)\rangle_{\mathrm{rms}}$	Plane
Gradient magnet	Displacement	$\langle \Delta y \rangle$	$k_i l_i \langle \Delta y \rangle$	x, z
Bending magnet (bending angle $= \theta_i$)	Tilt	$\langle \Delta \rangle$	$\theta_i \langle \Delta \rangle$	z
Bending magnet	Field error	$\langle \Delta B/B \rangle$	$\theta_i \langle \Delta B/B \rangle$	x
Straight sections (length $= d_i$)	Stray field	$\langle \Delta B_{\mathrm{s}} \rangle$	$d_i \langle \Delta B_{\mathrm{s}} \rangle / (B\rho)_{\mathrm{inj}}$	x, z

Imperfection: Chromaticity

- The focusing in a machine (and thus tune) depends on the momentum.
- The variation of the tune with momentum offset ($\delta \stackrel{\text{def}}{=} {}^{\Delta p}/_{p_0}$) is called chromaticity.
  - Inserting a momentum perturbation is akin to adding a bit of extra focusing to the one-turn matrix which depends on the unperturbed focusing, K_0.

$$M_{\text{one turn}}(\delta) = \begin{pmatrix} 1 & 0 \\ K_0\delta ds & 1 \end{pmatrix} \begin{pmatrix} \cos(2\pi Q) + \alpha\sin(2\pi Q) & \beta\sin(2\pi Q) \\ -\gamma\sin(2\pi Q) & \cos(2\pi Q) - \alpha\sin(2\pi Q) \end{pmatrix}$$

$$M_{\text{one turn}}(\delta) = \begin{pmatrix} \cos(2\pi Q) + \alpha\sin(2\pi Q) & \beta\sin(2\pi Q) \\ -\gamma\sin(2\pi Q) + K_0\delta[\cos(2\pi Q) + \alpha\sin(2\pi Q)]ds & \cos(2\pi Q) - \alpha\sin(2\pi Q) + K_0\delta\beta\sin(2\pi Q)ds \end{pmatrix}$$

- The trace is related to the new tune:

$$\cos(2\pi Q_{\text{new}}) = \frac{1}{2}\text{Tr}\, M = \cos(2\pi Q) + \frac{K_0\delta}{2}\beta\sin(2\pi Q)ds$$

Chromaticity and Tune

- Going through a bit of math:

$$\cos(2\pi Q_{\text{new}}) = \frac{1}{2}\text{Tr } M = \cos(2\pi Q) + \frac{K_0\delta}{2}\beta\sin(2\pi Q)ds$$

$$\cos(2\pi Q_{\text{new}}) = \cos(2\pi(Q+dQ)) \approx \cos(2\pi Q) - 2\pi\sin(2\pi Q)dQ$$

- Last two terms must be equal, therefore

$$dQ = -\frac{K(s)\delta}{4\pi}\beta(s)ds$$

Integrate around ring

$$\Delta Q = -\frac{\delta}{4\pi}\oint K(s)\beta(s)\,ds$$

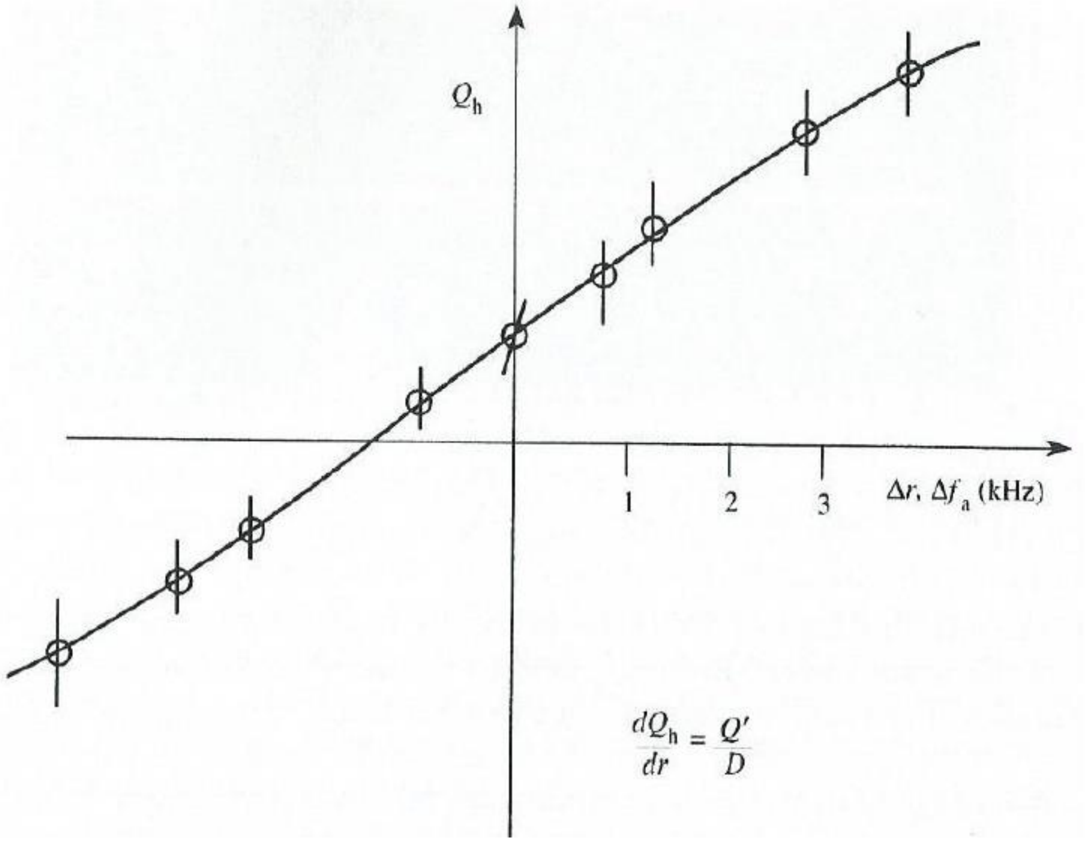
Total change in tune

- The tune will always have a bit of a spread due to the momentum spread. You can define the natural chromaticity

$$\xi_N \equiv \left(\frac{\Delta Q}{Q}\right) / \left(\frac{\Delta p}{p_0}\right) = -\frac{1}{4\pi Q}\oint K(s)\beta(s)\,ds \approx -1.3\,Q$$

Measurement of Chromaticity

- Steering the beam to a new mean radius, and adjusting the RF frequency to vary the momentum, can measure the Q.



Chromaticity Correction

- Need a way to connect the momentum offset, δ, to focusing.
- We can do this using sextupoles, which give nonlinear focusing (dependent on position) and dispersion (momentum-dependent position).

Dispersion (1)

- Dispersion, D(s), is defined as the change in particle position with fractional momentum offset, δ.
  - Originates from momentum dependence of dipole bends.

- Add explicit momentum dependence to EOM: $x'' + K(s)x = \frac{\delta}{\rho(s)}$

$$x(s) = C(s)x_0 + S(s)x'_0 + D(s)\delta_0$$
$$x'(s) = C'(s)x_0 + S'(s)x'_0 + D'(s)\delta_0$$

$$D(s) = S(s)\int_0^s \frac{C(\tau)}{\rho(\tau)}d\tau - C(s)\int_0^s \frac{S(\tau)}{\rho(\tau)}d\tau$$

Particular sol'n inhomog. DE w/ periodic ρ(s).

- The trajectory has two parts: $x(s) = \text{betatron} + \eta_x(s)\delta \qquad \eta_x(s) \equiv \frac{dx}{d\delta}$

$$\begin{pmatrix} x(s) \\ x'(s) \\ \delta(s) \end{pmatrix} = \begin{pmatrix} C(s) & S(s) & D(s) \\ C'(s) & S'(s) & D'(s) \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} x_0 \\ x'_0 \\ \delta_0 \end{pmatrix}$$

Dispersion (2)

- Noting that dispersion is periodic $\eta_x(s+C)=\eta_x(s)$

$$\begin{pmatrix}\eta_x(s)\\ \eta_x'(s)\\ \delta(s)\end{pmatrix}=\begin{pmatrix}C(s) & S(s) & D(s)\\ C'(s) & S'(s) & D'(s)\\ 0 & 0 & 1\end{pmatrix}\begin{pmatrix}\eta_x(s)\\ \eta_x'(s)\\ \delta_0\end{pmatrix}$$

- In an achromat, D = D' = 0. If we let $\delta_0=0$ we can simplify the process and solve to find

$$\begin{pmatrix}\eta_x(s)\\ \eta_x'(s)\end{pmatrix}=\begin{pmatrix}C(s) & S(s)\\ C'(s) & S'(S)\end{pmatrix}\begin{pmatrix}\eta_x(s)\\ \eta_x'(s)\end{pmatrix}+\begin{pmatrix}D(s)\\ D'(s)\end{pmatrix}=M\begin{pmatrix}\eta_x(s)\\ \eta_x'(s)\end{pmatrix}+\begin{pmatrix}D(s)\\ D'(s)\end{pmatrix}$$

$$(I-M)\begin{pmatrix}\eta_x(s)\\ \eta_x'(s)\end{pmatrix}=\begin{pmatrix}D(s)\\ D'(s)\end{pmatrix}\quad\Rightarrow\quad\boxed{\begin{pmatrix}\eta_x(s)\\ \eta_x'(s)\end{pmatrix}=(I-M)^{-1}\begin{pmatrix}D(s)\\ D'(s)\end{pmatrix}}$$

  - Solving gives

$$\boxed{\begin{aligned}\eta(s)&=\frac{[1-S'(s)]D(s)+S(s)D'(s)}{2(1-\cos\Delta\phi)}\\ \eta'(s)&=\frac{[1-C(s)]D'(s)+C'(s)D(s)}{2(1-\cos\Delta\phi)}\end{aligned}}$$

Chromaticity Correction

- Recall that we define the natural chromaticity as

$$\xi_N \equiv \left(\frac{\Delta Q}{Q}\right) / \left(\frac{\Delta p}{p_0}\right) = -\frac{1}{4\pi Q}\oint K(s)\beta(s)\,ds$$

- And that the trajectory goes as

$$x(s) = x_{\mathrm{betatron}}(s) + \eta_x(s)\delta$$

- If we describe the sextupole B field as $B_y = b_2 x^2$, we can then break it down as

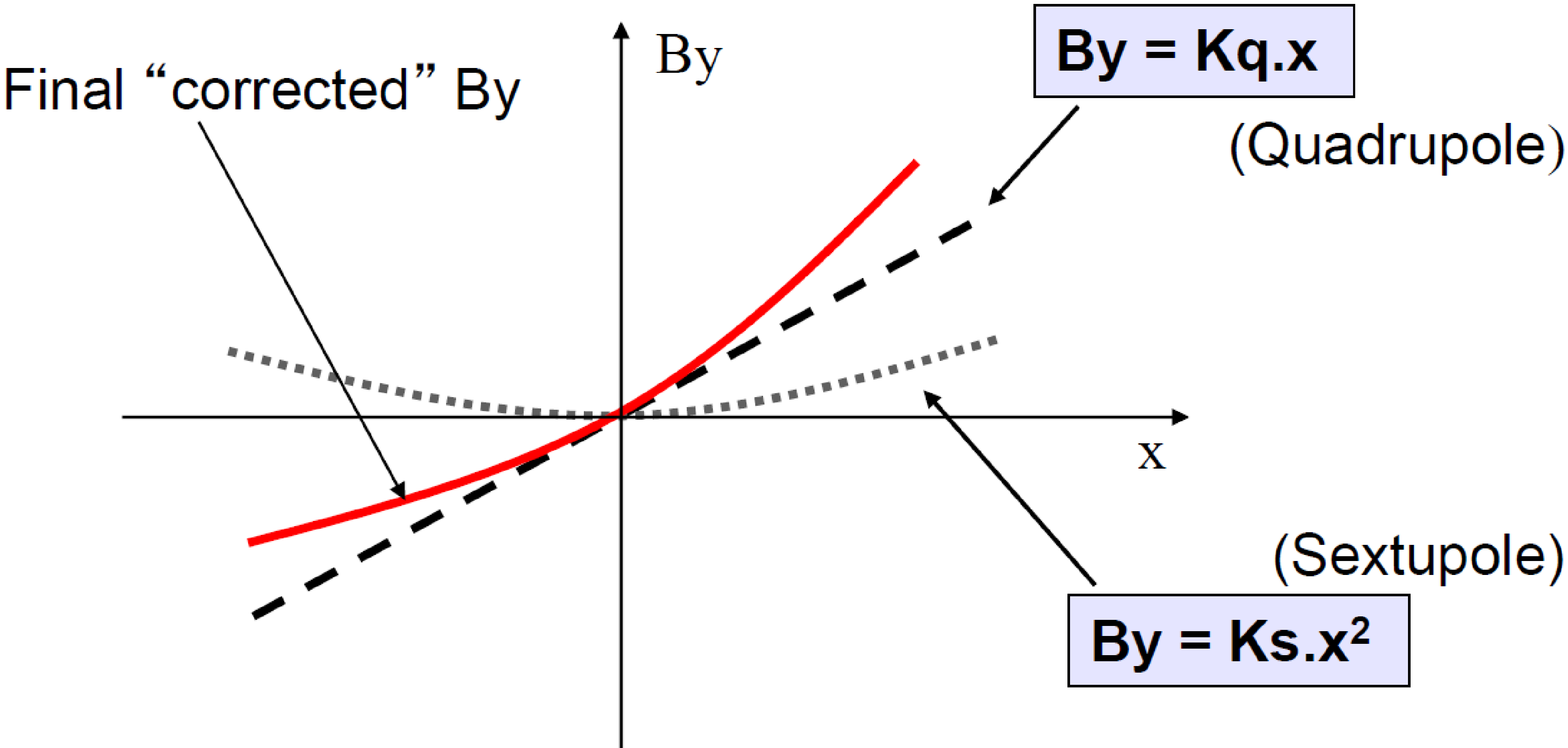
$$B_y(\mathrm{sext}) = b_2[x_{\mathrm{betatron}}(s) + \eta_x(s)\delta]^2 \approx \underbrace{b_2 x^2_{\mathrm{betatron}}}_{\text{Nonlinear}} + \underbrace{2b_2 x_{\mathrm{betatron}}(s)\eta_x(s)\delta}_{\text{Like quad: K(s)}}$$

- You end up getting a total chromaticity from all sources as

$$\xi = -\frac{1}{4\pi Q}\oint [K(s) - b_2(s)\eta_x(s)]ds$$

Notice that this means strong focusing (large K) requires large sextupoles!

Chromaticity Correction



- Since dispersion describes how momentum changes radial position of particles, sextupoles alter focusing field seen by particles as a function of momentum.
- Sextupole field acts to increase quadrupole magnetic field for particles that have positive displacement & decrease field for particles with negative displacements.

Sextupoles & Chromaticity

- There are two chromaticities ξ_h, ξ_v
- However, the effect of a sextupole depends on $\beta(s)$ and this varies around the machine.
- Two types of sextupoles are used to correct the chromaticity.
- One (SF) is placed near QF quadrupoles where β_h is large and β_v is small, this will have a large effect on ξ_h
- Another (SD) placed near QD quadrupoles, where β_v is large and β_h is small, will correct ξ_v
- Sextupoles should be placed where D(s) is large, in order to increase their effect, since Δk is proportional to D(s).

Bibliography

- M. H. Blewett, *Theoretical Aspects of the Behaviour of Beams in Accelerators and Storage Rings* (CERN Yellow Report, 1977)
- S. Y. Lee – *Accelerator Physics* (World Scientific, 2011)
- H. Wiedemann – *Particle Accelerator Physics* (Springer-Verlag, 2007)
- E. Wilson – *An Introduction to Particle Accelerators* (Oxford University Press, 2001)