

# Transverse Dynamics Lectures

JAI lectures - Michaelmas Term 2022

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# Outline

Introduction

Special Relativity

Lorentz equation

Hill's equation

Weak and Strong focusing

Matrix Formalism

Twiss parameters

Stability condition

Phase Space

Beam emittance and Symplectic Condition

FODO lattice

Dispersion and Chromaticity

# Goals of this course

- ▶ Introduction to one of the core topics in accelerator physics.
- ▶ Explain the basics of the formalism.
- ▶ Give an idea of the related phenomenology.
- ▶ Full derivations are not included in main lectures.
- ▶ Most important thing: learn something and enjoy!

# Some references

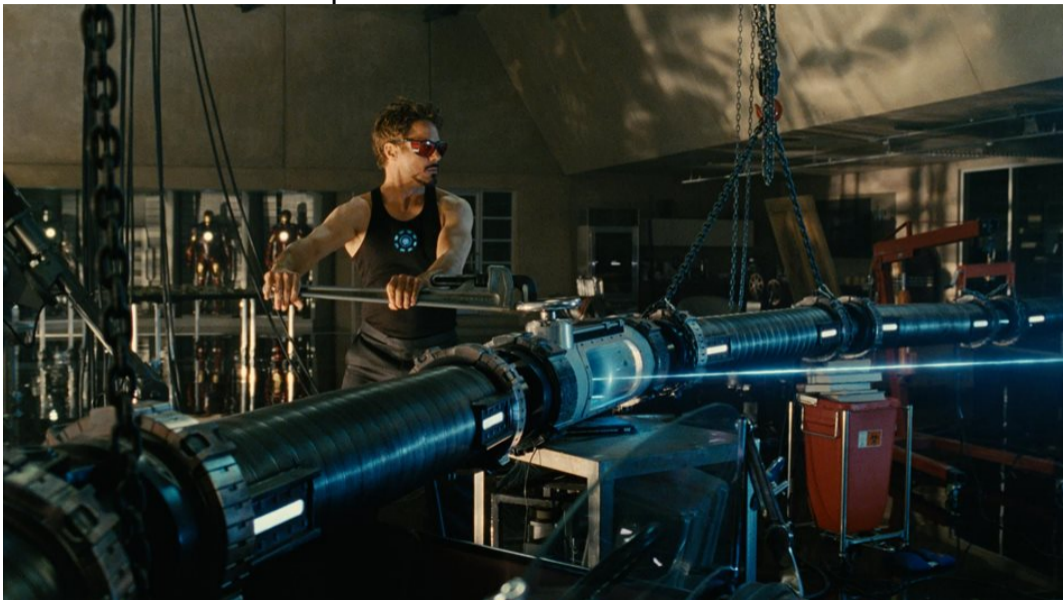
## Books

- ▶ Wilson, Introduction to Particle Accelerators.
- ▶ [Lee, Accelerator Physics.](#)
- ▶ Wiedemann, Particle Accelerator Physics.
- ▶ A. Wolski, Beam Dynamics in High Energy Particle Accelerators.
- ▶ E. Forest, Beam Dynamics: A new attitude framework.
- ▶ A. Chao, Handbook of Accelerator Physics and Engineering.

## Lectures

- ▶ [A. Latina, JUAS Lectures on Transverse Dynamics \(2020\).](#)
- ▶ [H. Garcia, JUAS Lectures on Transverse Dynamics \(2021\).](#)
- ▶ CAS lectures.
- ▶ USPAS lectures.

I did not know how complex an accelerator was...



# Why these lectures?

## What do we want to study?

High energy particles traveling through intense magnetic fields (usually periodic).

## Why transverse dynamics?

- ▶ It covers  $2/3$  of the phase space (4 out of 6 dimensions).
- ▶ Magnets act primarily on the transverse plane.
- ▶ Main accelerator parameters are determined (at first order) by transverse properties:
  - ▶ Luminosity, emittance, brilliance, beam losses, instabilities, tune...

## Special relativity recap.

We need to study the motion of charged particles at (very) high energy.

$$E = \sqrt{p^2 c^2 + (mc^2)^2} \quad (1)$$

where  $m$  is the mass of the particle and  $p$  the particle momentum.

$$\gamma = \frac{1}{\sqrt{1 - \frac{v^2}{c^2}}} \quad (2)$$

Ultra-relativistic approximation  $\gamma \gg 1$ :

$$E = pc \quad (3)$$

What is faster?

1. An electron/positron at LEP ( $E = 100$  GeV).
2. A proton in the LHC ( $E = 7000$  GeV).

# Lorentz Force

The force experienced by a charge  $q$  and speed  $\mathbf{v}$  under the influence of an electric field  $\mathbf{E}$  and a magnetic field  $\mathbf{B}$  is given by the Lorentz equation:

$$\mathbf{F} = q(\mathbf{E} + \mathbf{v} \times \mathbf{B}) \quad (4)$$

- ▶ Electric field  $\mathbf{E}$  for increasing (decreasing) particle speed.
- ▶ Magnetic field  $\mathbf{B}$  for bending particle trajectory.

**Question: Why do we use magnets for bending the trajectory of the beam?**



# Beam rigidity

Lorentz force:

$$F_L = qvB \quad (5)$$

Centripetal force:

$$F_c = m \frac{v^2}{\rho} \quad (6)$$

Null force condition ( $\sum F = 0$ )

$$F_L = F_c \Rightarrow \frac{p}{q} = B\rho \quad (7)$$

Beam rigidity:

$$B\rho \approx 3.33p[\text{GeV}/c] \quad (8)$$

## Applications

- ▶ Given size and magnet technology determines physics reach.
- ▶ Given magnet technology and physics goals determines required size.
- ▶ Given size and physics goal determines technology needed.

## Take home exercise

### **Given current technology ( $B_{\max} \sim 10 \text{ T}$ )**

- ▶ What is the maximum energy of a particle accelerator around the Earth equator?
- ▶ and of an accelerator around the Solar System?

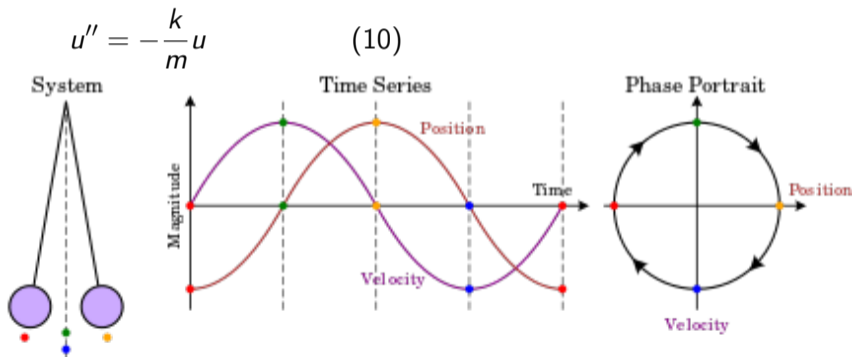
# Harmonic oscillator is back

Restoring force:

$$F = -ku \quad (9) \quad \text{Solution:}$$

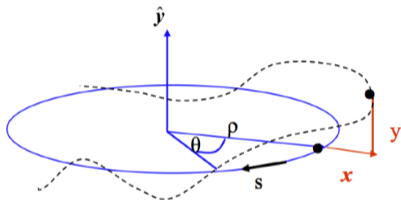
Equation of motion:

$$u = a \cos(\omega t + \phi) \quad (11)$$



# Frenet-Serret reference system

6D phase space:  $(x, x', y, y', z, \delta)$



The coordinates are relative to the reference particle/trajectory.

Coordinate definition:

$$x' = \frac{dx}{ds} = \frac{dx}{dt} \frac{dt}{ds} = \frac{P_x}{P_z} \approx \frac{P_x}{P_0} \quad (12)$$

$$y' = \frac{dy}{ds} = \frac{dy}{dt} \frac{dt}{ds} = \frac{P_y}{P_z} \approx \frac{P_y}{P_0} \quad (13)$$

$$\delta = \frac{\Delta P}{P_0} \quad (14)$$

Pay attention! This is not the set of canonical variables used in Hamilton's equations.

# Multipolar expansion

Any magnetic field can be decomposed in:

$$B_y + iB_x = \sum_{n=1}^{\infty} c_n (x + iy)^{n-1} \quad (15)$$

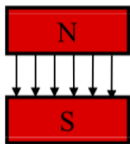
where

$$c_n = b_n + ia_n \quad (16)$$

- ▶  $b_n$  are the normal coefficients.
- ▶  $a_n$  are the skew coefficients.

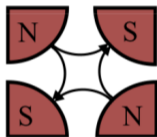
# Magnet types

dipole



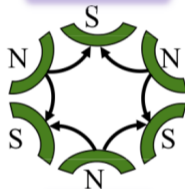
$$n = 1$$

quadrupole



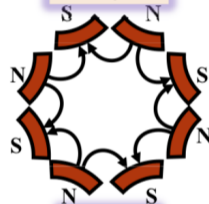
$$n = 2$$

sextupole



$$n = 3$$

octupole



$$n = 4$$

# Magnet types: Dipoles

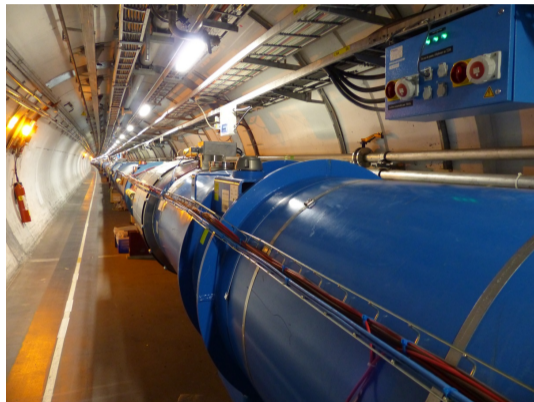
- ▶ Two magnetic poles.
- ▶ Bend particle trajectory.
- ▶ Provide weak focusing.
- ▶ Not required in linear colliders.

Take home exercise: LHC dipoles

The LHC contains 1232 dipole magnets.

Each is 15 m long.

- ▶ **What is the length of the full circumference?**



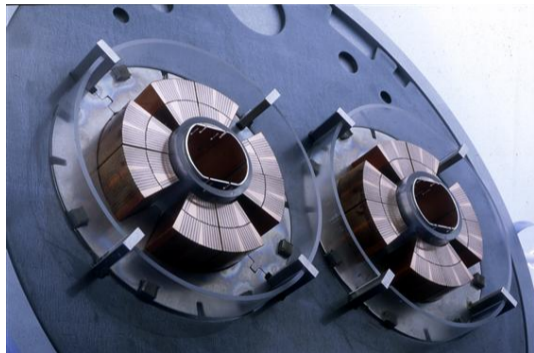
# Magnet types: Quadrupoles

- ▶ Four poles.
- ▶ Focus the beam (horizontally or vertically).

Normalized focusing strength:

$$k = \frac{G}{P/q} [\text{m}^{-2}] \quad (17)$$

$$k [\text{m}^{-2}] \approx 0.3 \frac{G [\text{T/m}]}{P [\text{GeV}/c] / q [e]} \quad (18)$$





## Magnet types: Quadrupoles

The focal length of a quadrupole is:

$$f = \frac{1}{k \cdot L} [\text{m}] \quad (19)$$

where  $L$  is the length of the quadrupole.

Example: Q1 LHC

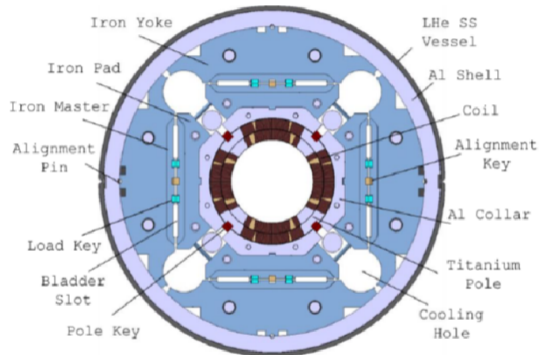
$$L = 6.37\text{m}$$

$$kL = -5.54 \times 10^{-2} \text{m}^{-1}$$



## Magnet types: Quadrupoles

- ▶ The LHC upgrade will require stronger focusing at IP1 and IP5.
- ▶ New quadrupole magnets with stronger gradients are required.
- ▶ Successful tests on short models.

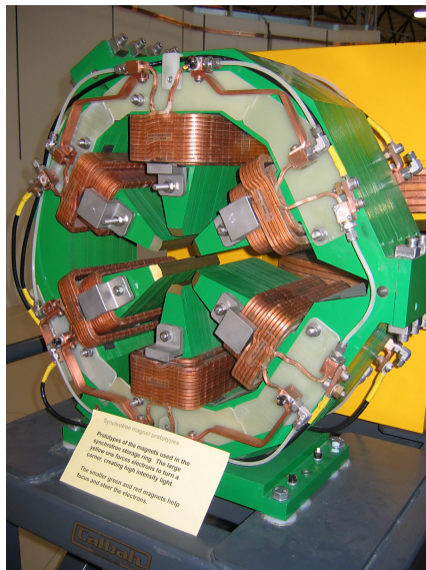


# Magnet types: Sextupoles

- ▶ Six poles.
- ▶ Correct chromatic aberrations.
- ▶ Usually distributed along the arcs.
- ▶ Essential for accelerator performance.

## Other multipoles

- ▶ Octupoles.
- ▶ Decapoles.
- ▶ Dodecapoles.



# Hamiltonian approach

Hamiltonian of a particle with mass  $m$ , charge  $q$  and momentum  $p$  in presence of an electromagnetic field  $(\phi, \mathbf{A})$ :

$$H = c\sqrt{(\mathbf{p} - q\mathbf{A})^2 + m^2c^2} + q\phi \quad (20)$$

Hamilton equation:

$$\frac{dq}{dt} = \frac{\partial H}{\partial p} \quad \frac{dp}{dt} = -\frac{\partial H}{\partial q} \quad (21)$$

Equation (20) will be explained in future lectures including the derivation of the dynamics.

# Hill's equation

- ▶ We expect a solution in the form of a quasi harmonic oscillation: amplitude and phase will depend on the position  $s$  along the ring.
- ▶ The linear motion (dipoles and quadrupoles) can be described by:

$$u'' + K(s)u = 0 \quad (22)$$

where  $K(s) = \left(\frac{1}{\rho^2} + k\right)$  is composed by linear fields only (dipole and quadrupole).

# Hill's equation

$$u'' + K(s)u = 0 \quad (23)$$

## Some remarks

- ▶  $K(s)$  is a non-constant ( $s$ -dependent) restoring force.
- ▶  $K(s)$  is a periodic function with period  $L \Rightarrow K(s + L) = K(s)$
- ▶ Usually in the vertical plane  $1/\rho = 0$ , therefore  $K_y = k_y$ .
- ▶ In a quadrupole  $1/\rho = 0$  and  $K_x = -K_y$  i.e. a horizontal focusing quadrupole defocuses in the vertical plane (and vice versa).
- ▶ In a bending magnet  $k = 0$  so  $K = 1/\rho^2$ .

# Hill's equation: general solution

For  $K(s) = K(s + L)$ :

$$u = \sqrt{2J_u\beta_u(s)} \sin(\phi_u(s) - \phi_{u0}) \quad (24)$$

$$u' = -\frac{\sqrt{2J_u}}{\beta_u(s)} [\cos(\phi_u(s) - \phi_{u0}) + \sin(\phi_u(s) - \phi_{u0})] \quad (25)$$

where  $u = x, y$ .

## Integration constants

- ▶ Action:  $J$  is a constant (related to emittance).
- ▶ Phase constant:  $\phi_0$ .

- ▶ Beta-function:  $\beta(s)$ , periodic function:

$$\beta(s + L) = \beta(s) \quad (26)$$

- ▶ Phase advance:  $\phi(s_0|s) = \int_{s_0}^s \frac{ds'}{\beta(s')}$

# Weak focusing and cyclotrons

In cyclotrons, only dipole magnets are used.  
But still there is some focusing effect.

$$u'' + \left( \frac{1}{\rho^2} + k \right) u = 0 \xrightarrow{k=0} u'' + \frac{1}{\rho^2} u = 0 \quad (27)$$

- ▶ Small and low energy accelerators.
- ▶ Example: mass spectrometer.



Figure: PSI cyclotron (250 MeV protons)



## Strong focusing ( $K > 0$ )

Initial conditions:  $x = x_0, x' = x'_0$  Solution:

$$x(s) = x_0 \cos(\sqrt{K}s) + \frac{x'_0}{\sqrt{K}} \sin(\sqrt{K}s) \quad (28)$$

$$x'(s) = -x_0 \sqrt{K} \sin(\sqrt{K}s) + x'_0 \cos(\sqrt{K}s) \quad (29)$$

Matrix formalism for a focusing quadrupole of length  $L$ :

$$\begin{pmatrix} x \\ x' \end{pmatrix} = \begin{pmatrix} \cos(\sqrt{K}L) & \frac{1}{\sqrt{K}} \sin(\sqrt{K}L) \\ -\sqrt{K} \sin(\sqrt{K}L) & \cos(\sqrt{K}L) \end{pmatrix} \begin{pmatrix} x_0 \\ x'_0 \end{pmatrix} \quad (30)$$

## Strong focusing ( $K < 0$ )

Initial conditions:  $x = x_0, x' = x'_0$  Solution:

$$x(s) = x_0 \cosh(\sqrt{|K|}s) + \frac{x'_0}{\sqrt{|K|}} \sinh(\sqrt{|K|}s) \quad (31)$$

$$x'(s) = -x_0 \sqrt{|K|} \sinh(\sqrt{|K|}s) + x'_0 \cosh(\sqrt{|K|}s) \quad (32)$$

Matrix formalism for a defocusing quadrupole of length  $L$ :

$$\begin{pmatrix} x \\ x' \end{pmatrix} = \begin{pmatrix} \cosh(\sqrt{|K|}L) & \frac{1}{\sqrt{|K|}} \sinh(\sqrt{|K|}L) \\ -\sqrt{|K|} \sinh(\sqrt{|K|}L) & \cosh(\sqrt{|K|}L) \end{pmatrix} \begin{pmatrix} x_0 \\ x'_0 \end{pmatrix} \quad (33)$$

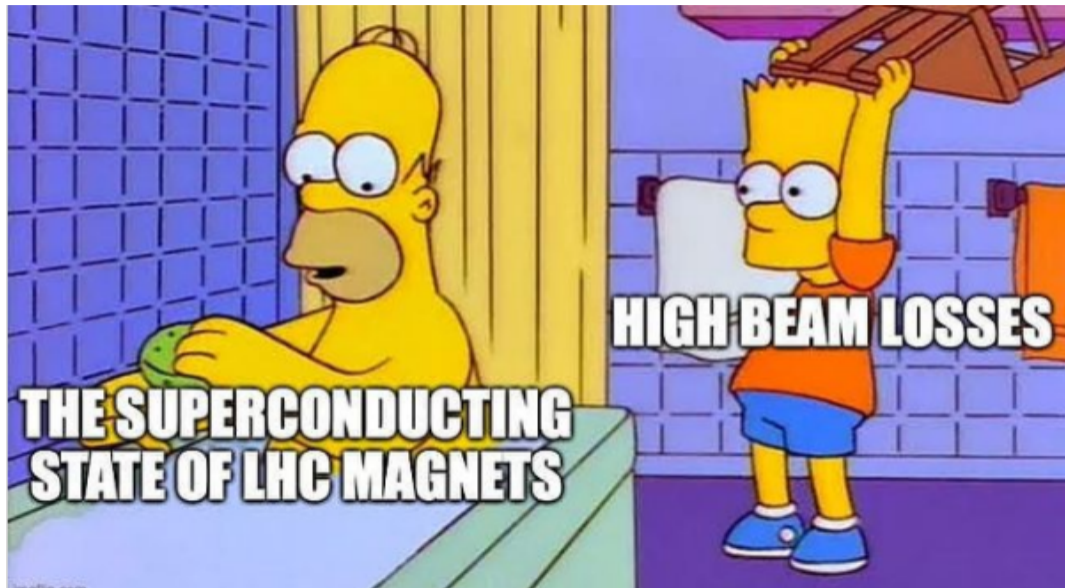
## Recap.

- ▶ Special relativity and magnetic properties.
- ▶ Reference system and Hill's equation (without deviation).
- ▶ Solution of linear homogeneous Hill's equations.
- ▶ Weak and strong focusing.
- ▶ Matrix formulation for dipoles and quadrupoles.

## Next episode

- ▶ Generalization of matrix formalism.
- ▶ Twiss parameters in detail.
- ▶ Phase space.
- ▶ Example: FODO.
- ▶ Dispersion and chromaticity.

End of the section meme



## Part II

## General matrix formalism

The transformation between  $x(s_0)$  and  $x(s)$  can be expressed in a general way:

$$x(s) = M(s|s_0)x(s_0) \quad (34)$$

where the application  $M(s|s_0)$  can be expressed in matrix formalism:

$$\begin{pmatrix} x \\ x' \end{pmatrix} = \begin{pmatrix} C(s|s_0) & S(s|s_0) \\ C'(s|s_0) & S'(s|s_0) \end{pmatrix} \begin{pmatrix} x_0 \\ x'_0 \end{pmatrix} \quad (35)$$

where  $C$  and  $S$  are the cosine-like and sine-like functions and their derivatives  $C'$  and  $S'$  with respect to  $s$ .

## Element concatenation

The transfer matrices for different elements of the lattice can be concatenated to find the full transfer matrix between two locations  $s_0$  and  $s$ ,

$$x(s_n) = M_n(s_n|s_{n-1}) \dots M_2(s_2|s_1)M_1(s_1|s_0)x_0 \quad (36)$$

Remember to multiply matrices in reverse order!

### Lattice design lectures

We will see more about how lattices are designed in practice in MADX.

## Thin lens approximation

When the focal length  $f$  of a quadrupole is much larger than the magnet itself  $L_q$  the transfer matrices can be rewritten as,

$$M_{\text{foc}} = \begin{pmatrix} 1 & 0 \\ -\frac{1}{f} & 1 \end{pmatrix} \quad (37)$$

$$M_{\text{def}} = \begin{pmatrix} 1 & 0 \\ \frac{1}{f} & 1 \end{pmatrix} \quad (38)$$

### Take home exercise

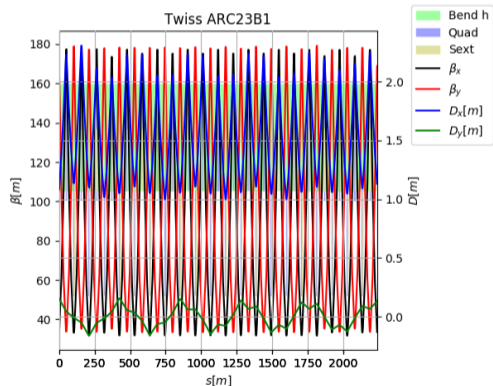
Derive the limits for the thin lens approximation and find the new matrices for quadrupoles in thin lens approximation.



# Twiss parameters

$$u(s) = \sqrt{2J_u\beta_u(s)} \sin(\phi_u(s) - \phi_{u0}) \quad (39)$$

$\beta_u(s)$  is a periodic function given by the periodic properties of the lattice.



$$\phi(s|s_0) = \int_{s_0}^s \frac{ds}{\beta(s')} \quad (40)$$

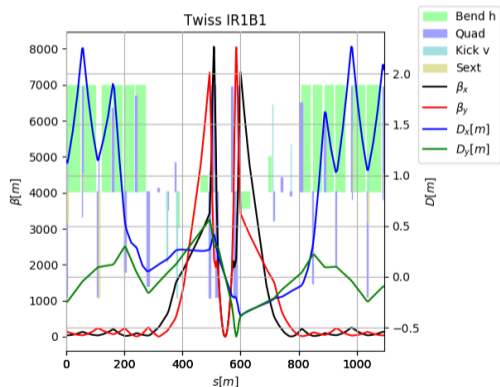
$$\alpha_u(s) = -\frac{1}{2} \frac{d\beta_u}{ds} \quad (41)$$

$$\gamma_u(s) = \frac{1 + \alpha_u^2(s)}{\beta_u(s)} \quad (42)$$

# Twiss parameters

$$u(s) = \sqrt{2J_u\beta_u(s)} \sin(\phi_u(s) - \phi_{u0}) \quad (43)$$

$\beta_u(s)$  is a periodic function given by the periodic properties of the lattice.



$$\phi(s|s_0) = \int_{s_0}^s \frac{ds}{\beta(s')} \quad (44)$$

$$\alpha_u(s) = -\frac{1}{2} \frac{d\beta_u}{ds} \quad (45)$$

$$\gamma_u(s) = \frac{1 + \alpha_u^2(s)}{\beta_u(s)} \quad (46)$$

## Transfer matrix in terms of Twiss parameters

Aim: express  $M$  in terms of the initial and final Twiss parameters (instead of magnetic properties).

Taking  $s(0) = s_0$  and  $\phi(0) = \phi_0$  we can obtain,

$$M = \begin{pmatrix} \sqrt{\frac{\beta_s}{\beta_0}} (\cos \phi_s + \alpha_0 \sin \phi_0) & \sqrt{\beta_s \beta_0} \sin \phi_s \\ \frac{(\alpha_0 - \alpha_s) \cos \phi_s - (1 + \alpha_s \alpha_0) \sin \phi_s}{\sqrt{\beta_s \beta_0}} & \sqrt{\frac{\beta_0}{\beta_s}} (\cos \phi_0 - \alpha_s \sin \phi_s) \end{pmatrix} \quad (47)$$

This expression is very useful when Twiss parameters are known at two different locations.

# How do we measure $\beta$ and $\phi$

## Phase $\phi$

- ▶ Harmonic analysis of oscillations.

## Betatron tune $Q$

- ▶ FFT of transverse beam position over many turns.

## Beta function $\beta$

- ▶  $\beta$  from phase.
- ▶  $\beta$  from amplitude.
- ▶ K-modulation.

## One matrix to rule them all

If we take matrix  $M$  and consider the case for one full turn (i.e.  $\beta_s = \beta_0$  and  $\alpha_s = \alpha_0$ ) the matrix simplifies,

$$\mathcal{M} = \begin{pmatrix} \cos \phi_L + \alpha_0 \sin \phi_L & \beta_0 \sin \phi_L \\ \gamma_0 \sin \phi_L & \cos \phi_0 - \alpha_0 \sin \phi_L \end{pmatrix} \quad (48)$$

The tune  $Q$  is the phase advance of the full ring in  $2\pi$  units.

$$Q = \frac{1}{2\pi} \oint \frac{ds}{\beta(s)} = \frac{\phi_L}{2\pi} \quad (49)$$

then, the one turn matrix  $\mathcal{M}$  can be rewritten,

$$\mathcal{M} = \begin{pmatrix} \cos(2\pi Q) + \alpha_0 \sin(2\pi Q) & \beta_0 \sin(2\pi Q) \\ \gamma_0 \sin(2\pi Q) & \cos(2\pi Q) - \alpha_0 \sin(2\pi Q) \end{pmatrix} \quad (50)$$

# Properties of transfer matrices

1. Phase space area preservation.

$$\det(M) = 1 \quad (51)$$

2. Motion is stable over  $N \rightarrow \infty$

$$|\text{trace}(M)| < 2 \quad (52)$$

## Stability condition (derivation)

Let's consider the transfer matrix  $M$  for a periodic system:

$$M = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \quad (53)$$

we want the motion to be stable over  $N \rightarrow \infty$  turns.

$$x_N = M^N x_0 \quad (54)$$

How can we compute  $M^N$ ?

## Stability condition (derivation)

$$x_N = M^N x_0 \quad (55)$$

- ▶  $\det(M) = ad - bc = 1$
- ▶  $\text{tr}(M) = a + d$

If we diagonalise  $M$ , we can rewrite it as,

$$M = U \cdot \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix} \cdot U^T \quad (56)$$

where  $U$  is some unitary matrix and  $\lambda_1$  and  $\lambda_2$  its eigenvalues.



## Stability condition (derivation)

After  $N$  turns,

$$M^N = U \cdot \begin{pmatrix} \lambda_1^N & 0 \\ 0 & \lambda_2^N \end{pmatrix} \cdot U^T \quad (57)$$

Given that  $\det(M) = 1$ ,

$$\lambda_1 \lambda_2 = 1 \rightarrow \lambda_{1,2} = e^{\pm ix} \quad (58)$$

To have stable motion,  $x \in \mathbb{R}$ .

To find the eigenvalues, use characteristic equation,

$$\det(M - \lambda \mathbb{I}) = \begin{vmatrix} a - \lambda & b \\ c & d - \lambda \end{vmatrix} = 0 \quad (59)$$

Solve it,

$$\lambda^2 - (a + d)\lambda + (ad - bc) = 0$$

$$\lambda^2 + \text{trace}(M)\lambda + 1 = 0$$

$$\text{trace}(M) = \lambda + \frac{1}{\lambda} = e^{ix} + e^{-ix} = 2 \cos x$$

Since  $x \in \mathbb{R}$ ,

$$|\text{trace}(M)| \leq 2$$

# Twiss transport matrix and Twiss parameters evolution

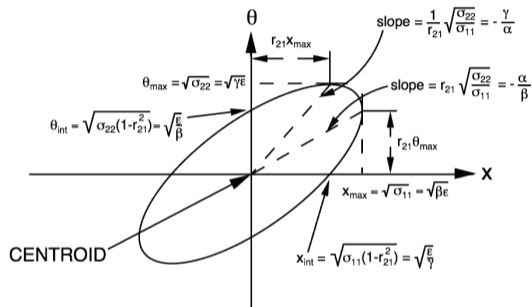
Instead of transporting the coordinates  $x$  and  $x'$  we can transport the Twiss parameters  $(\beta, \alpha, \gamma)$ ,

$$\begin{pmatrix} \beta \\ \alpha \\ \gamma \end{pmatrix}_s = \begin{pmatrix} C^2 & -2CS & S^2 \\ -CC' & CS' + SC' & -SS' \\ C'^2 & -2C'S' & S'^2 \end{pmatrix} \begin{pmatrix} \beta \\ \alpha \\ \gamma \end{pmatrix}_0 \quad (60)$$

- ▶ Given the Twiss parameters at any point in the lattice we can transform them and compute their values at any other point in the ring.
- ▶ The transfer matrix is given by the focusing properties of the lattice elements, the same matrix elements to compute single particle trajectories.

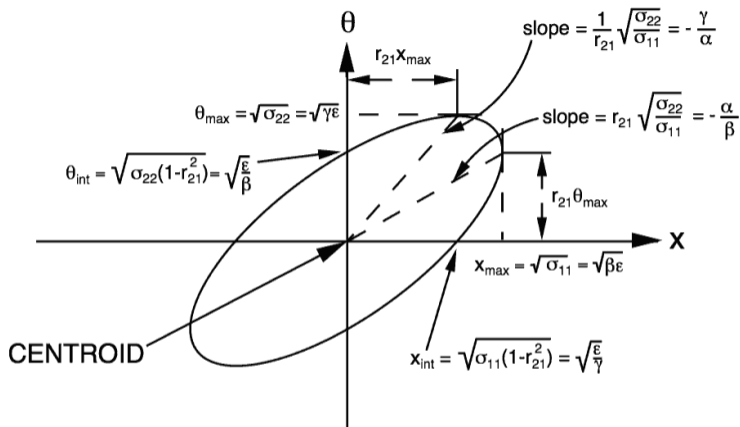
# Phase space properties

- ▶ Area is preserved.
- ▶ Beam size:  $\sigma_u = \sqrt{J_u \beta_u}$ .
- ▶ When  $\sigma_u$  is large  $\sigma_{u'}$  is small.
- ▶ In a  $\beta$  minimum/maximum  $\alpha = 0$  and the ellipse is not tilted.



$$J = \gamma x^2 + 2\alpha x x' + \beta x'^2 \quad (61)$$

# Phase space properties

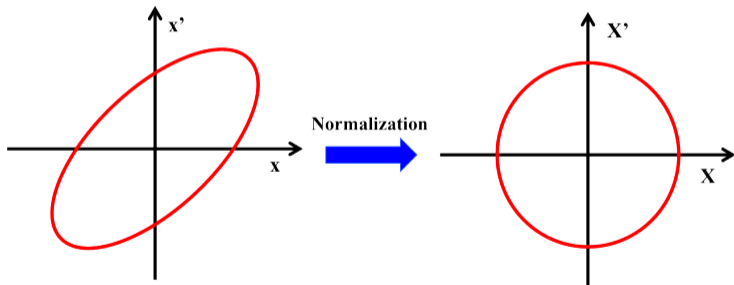


$$J = \gamma x^2 + 2\alpha x x' + \beta x'^2 \quad (62)$$

## Normalized phase space

Can we use another reference frame so it is simpler to describe the system?

$$\mathcal{M} = \begin{pmatrix} \cos \phi & \sin \phi \\ -\sin \phi & \cos \phi \end{pmatrix} \quad (63)$$



For linear systems is fine but it gets much more complex when non-linearities are included (we will see more details in the tutorial).

## Beam emittance: single particle definition

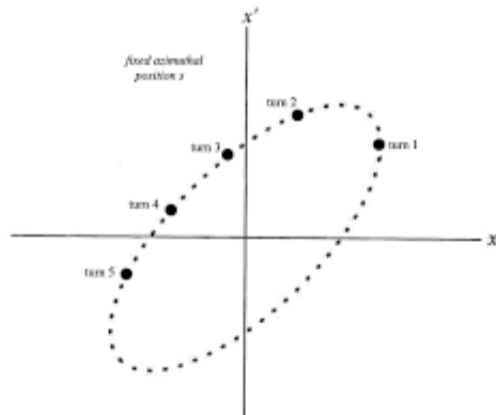
The geometric emittance is a constant of motion only if the beam energy is preserved (conservative system). This quantity is related to the action  $J$  that appeared in the solution of the Hill's equation.

Normalized emittance takes into account beam energy. It is a constant of motion even if energy is not constant:

$$\epsilon_n \equiv \beta_{\text{rel}} \gamma_{\text{rel}} \epsilon \quad (64)$$

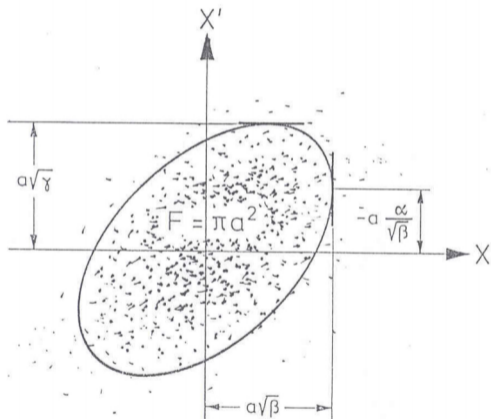
The beam size at any location of the lattice is given by,

$$\sigma = \sqrt{\epsilon \beta} \quad (65)$$



# Beam emittance: statistical definition

The beam is composed of particles distributed in phase space.



Statistical emittance is defined by,

$$\epsilon_{\text{rms}} = \sqrt{\sigma_u^2 \sigma_{u'}^2 + \sigma_{uu'}^2} \quad (66)$$

The rms emittance of a ring in phase space, i.e. particles uniformly distributed in phase  $\phi$  at a fixed action  $J$ , is,

$$\epsilon_{\text{rms}} = \langle J \rangle. \quad (67)$$

If the accelerator is composed of linear elements, and no dissipative forces act  $\epsilon_{\text{rms}}$  is invariant.

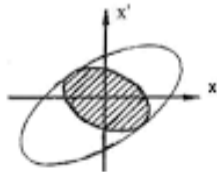
# Beam emittance: phenomenology

## What determines beam emittance

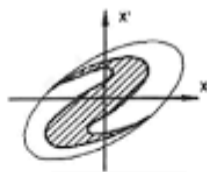
- ▶ Amount of particles.
- ▶ Injector manipulation.
- ▶ Beam transfer efficiency.



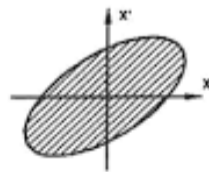
(a) machine phase space



(b) unmatched beam injected



(c) filamenting beam



(d) fully filamented beam

## Sources of emittance growth

- ▶ Intrabeam scattering.
- ▶ Beam-beam interaction.
- ▶ Residual gas scattering.
- ▶ Optics mismatch.
- ▶ Nonlinearities and resonances.
- ▶ Ground motion and PS ripple.



## Liouville's theorem and symplectic condition

Liouville's equation describes the time evolution of the phase space distribution function  $\rho(q, p; t)$ ,

$$\frac{d\rho}{dt} = \frac{\partial\rho}{\partial t} + \sum_{i=1}^N \left( \frac{\partial\rho}{\partial q_i} \dot{q}_i + \frac{\partial\rho}{\partial p_i} \dot{p}_i \right) = 0 \quad (68)$$

where  $(q_i, p_i)$  are the canonical coordinates of the Hamiltonian system.

## Symplectic condition

Liouville's theorem  $\Rightarrow$  invariant volume in phase space. The symplectic condition reads,

$$M^T J M = J \quad (69)$$

where  $J$  is the 6D symplectic matrix

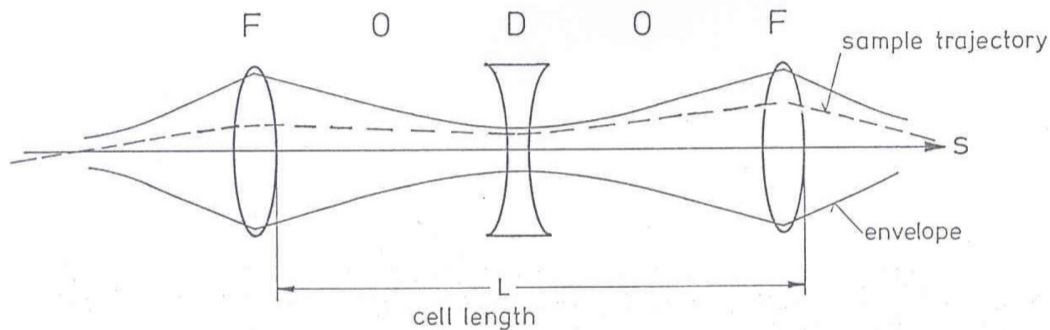
$$J = \begin{pmatrix} 0 & 1 & 0 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & -1 & 0 \end{pmatrix} \quad (70)$$

Take home exercise

Prove that Eq. (69) holds for the matrices described above.

## FODO lattice (The "Hello World" example)

The FODO lattice is a sequence of a Focusing magnet (F), a Drift space (O), a Defocusing magnet (D) and a second drift space.



$$M_{\text{FODO}} = M_0 M_{\text{def}} M_0 M_{\text{foc}} = \begin{pmatrix} 1 + \frac{L}{2f} & L + \frac{L^2}{4f} \\ -\frac{L}{2f^2} & 1 - \frac{L}{2f} - \frac{L^2}{4f^2} \end{pmatrix} \quad (71)$$

## FODO lattice (The "Hello World" example)

### Take-home exercise

Prove that the stability condition for a FODO lattice is given by:

$$f > \frac{L}{4} \quad (72)$$

### What if

We take the FODO lattice and replace drifts by bending magnets?

We will see this in next lectures...

# The end of the ideal world

So far, we have considered ideal linear systems.

While, in the real world...

- ▶ Dispersion.
- ▶ Chromaticity.
- ▶ Misalignment.
- ▶ Magnetic errors.
- ▶ ...

Some of these topics will be covered in next lectures.

# Dispersion

What if particles in a bunch have different momenta?

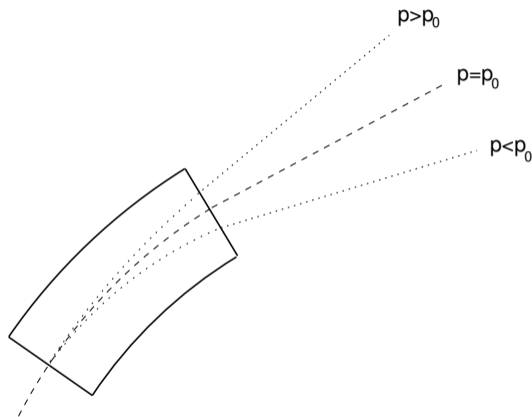
Remember beam rigidity:

$$B\rho = \frac{P}{q} \quad (73)$$

Orbit:

$$x(s) = D(s) \frac{\Delta P}{P_0} \quad (74)$$

where  $D(s)$  is the dispersion function, an intrinsic property of dipole magnets.



# Dispersion

Inhomogeneous Hill's equation:

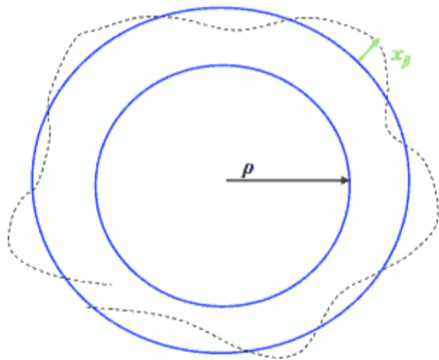
$$u'' + \left( \frac{1}{\rho^2} + k \right) u = \frac{1}{\rho} \frac{\Delta P}{P_0} \quad (75)$$

Particle trajectory:

$$\begin{aligned} u(s) &= u_\beta(s) + u_D(s) = \\ &= u_\beta(s) + D(s) \frac{\Delta P}{P} \end{aligned} \quad (76)$$

where  $D(s)$  is the solution of:

$$D''(s) + K(s)D(s) = \frac{1}{\rho} \quad (77)$$



# Dispersion

Solution:

$$U(s) = C(s)u_0 + S(s)u'_0 + D(s)\frac{\Delta P}{P} \quad (78)$$

this can be added to the transfer matrix representation,

$$M = \begin{pmatrix} C & S & D \\ C' & S' & D' \\ 0 & 0 & 1 \end{pmatrix} \quad (79)$$

Dipole transfer matrix:

$$\begin{pmatrix} \cos\left(\frac{L}{\rho}\right) & \rho \sin\left(\frac{L}{\rho}\right) & \rho\left(1 - \cos\left(\frac{L}{\rho}\right)\right) \\ -\frac{1}{\rho} \sin\left(\frac{L}{\rho}\right) & \cos\left(\frac{L}{\rho}\right) & \sin\left(\frac{L}{\rho}\right) \\ 0 & 0 & 1 \end{pmatrix} \quad (80)$$

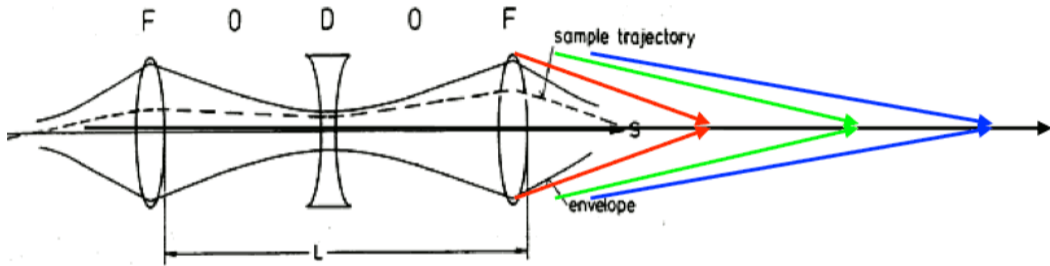
Quadrupole transfer matrix (expanded):

$$\begin{pmatrix} \cos(\sqrt{K}L) & \frac{1}{\sqrt{K}} \sin(\sqrt{K}L) & 0 \\ -\sqrt{K} \sin(\sqrt{K}L) & \cos(\sqrt{K}L) & 0 \\ 0 & 0 & 1 \end{pmatrix} \quad (81)$$



# Chromaticity

All particles do not have the same energy. Therefore, they focalize at different points.

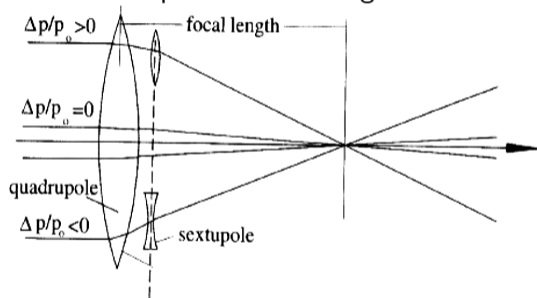


This defines chromaticity,

$$\xi = -\frac{1}{4\pi} \oint \beta(s)k(s)ds \quad (82)$$

# How to correct chromaticity

Sextupoles, through a non-linear magnetic field, correct the effect of energy spread and focuses particles at a single location.



- ▶ Located in dispersive regions.
- ▶ Usually in arcs.
- ▶ Sextupole families.

Now is when the party starts

- ▶ Sextupoles introduce non-linear fields.
- ▶ ...i.e. they induce non-linear motion.
- ▶ resonances, tune shifts, chaotic motion.

# Chromaticity correction

- ▶ Chromatic aberrations must be compensated in both planes.

$$\xi_x = -\frac{1}{4\pi} \oint \beta_x(s)[k(s) - S_F D_x(s) + S_D D_x(S)] ds \quad (83)$$

$$\xi_y = -\frac{1}{4\pi} \oint \beta_y(s)[k(s) + S_F D_x(s) - S_D D_x(S)] ds \quad (84)$$

- ▶ To minimise sextupole strength they must be located near quadrupoles where  $\beta D$  is large.
- ▶ For optimal independent correction  $S_F$  should be located where  $\beta_x/\beta_y$  is large and  $S_D$  where  $\beta_y/\beta_x$  is large.

## Recap.

- ▶ Optics functions and parameters.
- ▶ Phase space and emittance.
- ▶ Example: FODO lattice.
- ▶ Dispersion and chromaticity.

## What do we do with this?

- ▶ We have covered the basic aspects of transverse dynamics.
- ▶ I skipped most of the derivations. You can follow references.
- ▶ In the next two weeks: lattice design and tutorials for a more complete picture.
- ▶ Now you are ready to take the following lectures to become accelerator experts.

Thank you very much!

