Self-Similar Solutions in Dark-Fluid Cosmology

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Motivation

- The properties and existence of dark matter is one of the most fascinating questions in cosmology.
- The scale-free nature of gravitational interaction in both Newtonian gravity and the general theory of relativity gives rise to the concept of self-similarity
- This implies that the governing partial differential equations are invariant under scale transformation if we consider appropriate matter fields.
- Self-similar solutions (SSs) have a wide range of applications in astrophysics
- We studied different kinds of dark fluid models with self-similar solutions.

Self-similarity in General Relativity

- In GR, the concept of SSs is not quite straightforward because GR has general covariance against coordinate transformation.
- Can be seen in two ways: Properties of the space-time, and properties of the matter fields [Cahill and Taub (1971)]
- Self-similarity of the space-time $\Rightarrow$ **Homothetic vector fields (HVF):**

$$\mathcal{L}_\xi g_{\mu\nu} = 2\alpha g_{\mu\nu}$$

- A special type of Killing vector fields: If $\alpha = 0 \Rightarrow \xi$ Killing vector.
- If $\alpha \neq 0 \Rightarrow \xi$ can be rescaled, that α is unity
- If we are interested in a vacuum solution can be reduced to

$$\mathcal{L}_\xi R_{\mu\nu} = \mathcal{L}_\xi G_{\mu\nu} = 0$$

Self-Similarity of Matter Fields

If we consider a non-vacuum solution for the EFE's:

$$G_{\mu\nu} = 8\pi G T_{\mu\nu} \Rightarrow \mathcal{L}_\xi T_{\mu\nu} = 0$$

The kinematic self-similar solution can be defined via a kinematic self-similar vector ξ (KSS). The KSS vector satisfies the following identities:

$$\mathcal{L}_\xi h_{\mu\nu} = 2\delta h_{\mu\nu} \quad (1)$$

$$\mathcal{L}_\xi u_\mu = \alpha u_\mu \quad (2)$$

The definition of the $h_{\mu\nu} = g_{\mu\nu} + u_\mu u_\nu$ projection tensor.

Parameters of the KSS solution

- The α and δ are dimensionless constants.
- If $\delta \neq 0 \Rightarrow \alpha/\delta$ similarity index
- The similarity index characterizes the scale-independent transformations
- If $\alpha/\delta = 1 \Rightarrow$ **KSS** vector fields, becomes **HVF** $\Rightarrow 1^{st}$ kind of SS

	$\alpha = 0$	$\alpha \neq 0$
$\delta = 0$	Killing vector	2^{nd} kind
$\delta \neq 0$	0^{th} kind	Infinite kind

Metric of the Space-time

The line element of the general symmetric spacetimes is given by:

$$ds^2 = -e^{2\Phi(t,r)}dt^2 + e^{2\Psi(t,r)}dr^2 + R(t,r)^2[d\theta^2 + \Sigma(k,\theta)^2 d\phi^2] \qquad (3)$$

where,

$$\Sigma(k,\theta) = \begin{cases} \sin(\theta), & k = 1 \\ \theta, & k = 0 \\ \sinh(\theta), & k = -1 \end{cases}$$

We adopt comoving frames:

$$u_\mu = (e^{-\Phi}, 0, 0, 0)$$

Kinematic Self-similar Solutions

The general KSS vector can be written in the comoving form:

$$\boldsymbol{\xi}(r,t) = h_1(r,t)\frac{\partial}{\partial t} + h_2(r,t)\frac{\partial}{\partial r}$$

Solutions can be parallel, orthogonal, or tilted:

- If $h_1(r,t) = 0 \Rightarrow$ **orthogonal** to the fluid flow
- If $h_2(r,t) = 0 \Rightarrow$ **parallel** to the fluid flow
- If $h_1, h_2 \neq 0 \Rightarrow$ **tilted**.

In the most general tilted case, the KSS VF can be rewritten:

$$\boldsymbol{\xi} = (\alpha + \beta t)\frac{\partial}{\partial t} + r\frac{\partial}{\partial r}$$

Tilted case

- $\alpha = 1$ and $\beta = 0 \Rightarrow$ **KSS** of the first kind ($\xi = r/t$)
- $\alpha = 0$ and $\beta = 1 \Rightarrow$ **KSS** of the zeroth kind ($\xi = r/e^t$)
- $\alpha \neq 0, 1$ and $\beta = 0 \Rightarrow$ **KSS** of second kind ($\xi = r/(\alpha t)^{1/\alpha}$)

The corresponding line element:

$$ds^2 = -e^{2\Phi(\xi)}dt^2 + e^{2\Psi(\xi)}dr^2 + R^2(\xi)d\Omega^2$$

- $\alpha \neq 0, 1$ and $\beta \neq = 0 \Rightarrow$ **KSS** of infinite kind ($\xi = r/t$)

The corresponding line element and **KSS VF**:

$$ds^2 = -e^{2\Phi(\xi)}dt^2 + \frac{e^{2\Psi(\xi)}}{r^2}dr^2 + R^2(\xi)d\Omega^2 \quad (4)$$

$$\boldsymbol{\xi} = t\frac{\partial}{\partial t} + r\frac{\partial}{\partial r} \quad (5)$$

Parallel and Orthogonal case

- Parallel and not infinite kind:

$$ds^2 = -t^{2(\alpha-1)}e^{\Phi(r)}dt^2 + t^2dr^2 + t^2R(r)^2d\Omega^2$$

- Parallel and infinite kind:

$$ds^2 = -e^{\Phi(r)}dt^2 + t^2dr^2 + R(r)^2d\Omega^2$$

- Orthogonal and not infinite kind:

$$ds^2 = -r^{2\alpha}dt^2 + e^{2\Phi(t)}dr^2 + r^2R(t)^2d\Omega^2$$

- Orthogonal and infinite kind:

$$ds^2 = -r^2dt^2 + \frac{e^{2\Phi(t)}}{r^2}dr^2 + R(t)^2d\Omega^2$$

Equation of State

We introduce and focus on the following equations of state:

- Perfect fluid (EoS 1):

$$p(r,t) = K\rho^n(r,t)$$

  where $K \neq 0$ and $n = 1$

- Generalized Chaplygin gas (EoS 2) [Li and Xu (2014)]:

$$p(r,t) = -\frac{A}{\rho^n}$$

  where A positive const, and $0 < n \leq 1$

Solutions

The complete list of kinematic self-similar solutions with a perfect fluid in the spherically symmetric spacetime for a perfect fluid and EOS2 and EOS3

Matter Field	Kind	Solution
Vacuum [Cahill and Taub]	first, parallel	Minkowski
	first, orthogonal	Minkowski
	second, tilted	Minkowski
	zeroth, tilted	Minkowski
	zeroth, orthogonal	Minkowski
	infinite, parallel	Schwarzschild

Solution II.

Matter Field	Kind	Solution
EoS I.	zeroth, orthogonal	Flat FRWL
	infinite, parallel	All static solution
EoS II.	second, parallel	Flat FRWL
	second, tilted	KSS LTB
	zeroth, orthogonal	Flat FRWL
	infinite, tilted	Flat FRWL
	infinite, parallel	All static solution

Summary

- Self-similarity as a concept has been applied to many parts of physics.
- In Newtonian gravity the application of self-similarity is straightforward and widely studied
- The self-similarity in general relativity is derived from the concept of homothety.
- However some EoS can't fit into this framework therefore it is natural to introduce the concept of incomplete similarity, called kinematic self-similarity.
- Future question: Kinematic self-similarity of Einstein-Maxwell equations (Axion motivated)