[Some comments on] the motion of test bodies in Kerr spacetime

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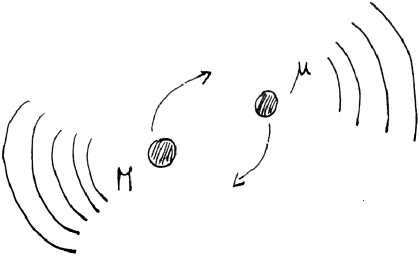

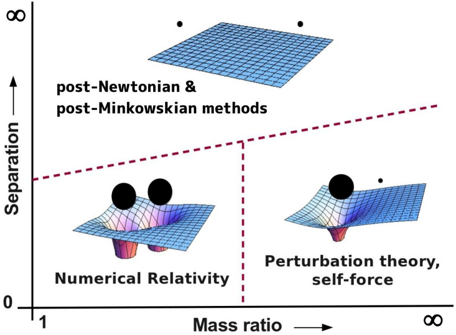

[arXiv:1805.10385]

- Several methods to solve the 2BP in GR
- Large mass ratio: $\epsilon \triangleq \frac{\mu}{M} \ll 1$
  - EMRIs: $10^{-6} < \epsilon < 10^{-4}$
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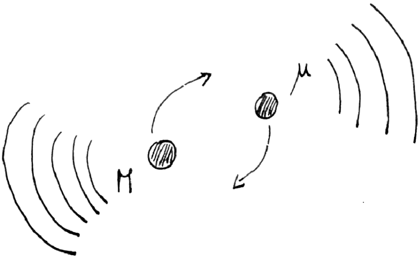

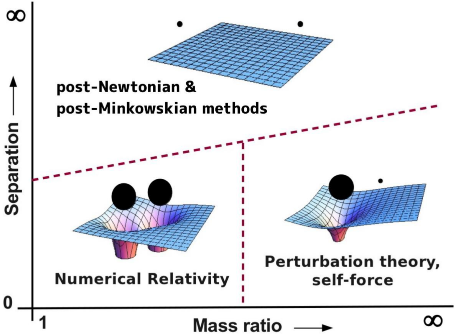

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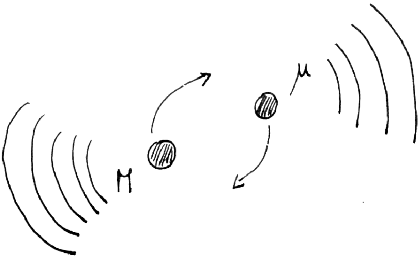

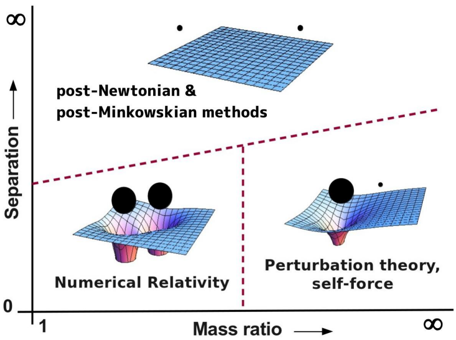

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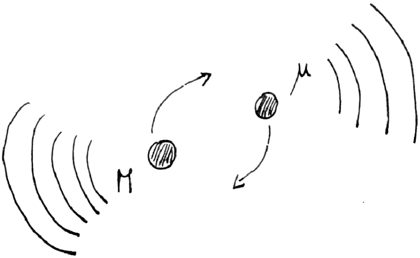

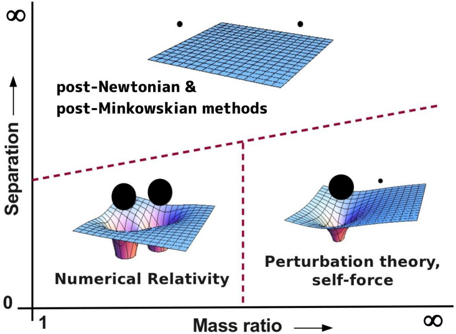

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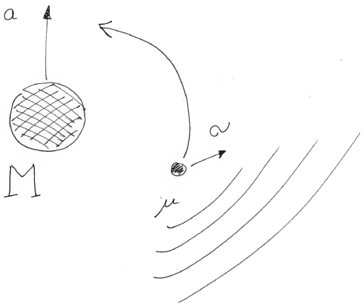
Setup and Challenge(s)

- Motion of a small object in a (Kerr) background $g_{\mu\nu}$: find a pair $(\gamma, \mathfrak{g}_{\mu\nu})$ describing the resultant spacetime *up to some required precision* (*e.g.* LISA: $\Delta\varphi \sim \mathcal{O}(1)$)



- More pragmatic question: what are the gravitational waveforms detectable from the Earth ?
- At the level of EOMs, several corrections arise at $\mathcal{O}(\epsilon)$:

$$\frac{Dz^\mu}{d\tau} = \epsilon\, f^\mu[g_{\alpha\beta}, z^\alpha, T_{\alpha\beta}]$$

$f^\mu = 0$: geodesic motion | $f^\mu = f^\mu_{GSF}$: self-force corrections | $f^\mu = f^\mu_{MP}$: finite-size effects

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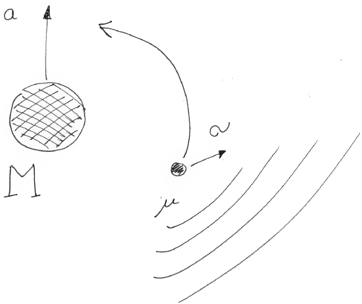

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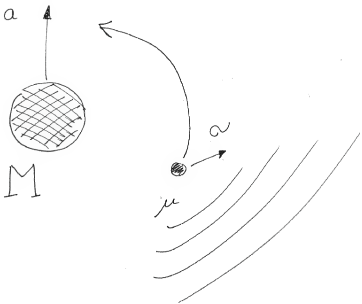

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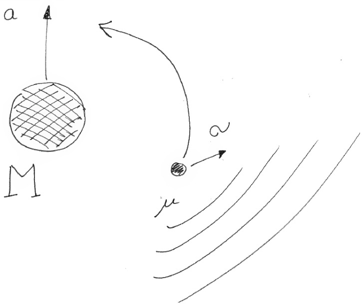

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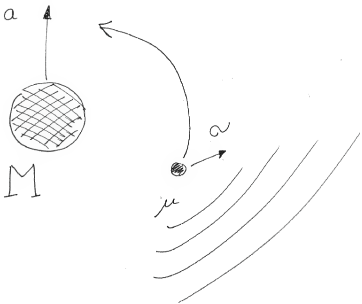

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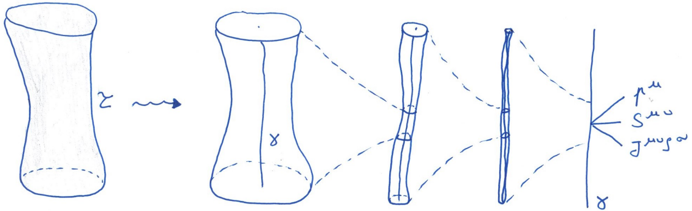

- Test body ≡ finite size structure (spin. . .) **without** backreaction
- Motion of spinning test bodies in curved spacetime: MPD equations

$$\frac{Dp^{\mu}}{d\lambda} = -\frac{1}{2}R^{\mu}{}_{\nu\alpha\beta}v^{\nu}S^{\alpha\beta} + \mathcal{F}^{\mu}, \qquad \frac{DS^{\mu\nu}}{d\lambda} = 2p^{[\mu}v^{\nu]} + \mathcal{L}^{\mu\nu}$$

- Astrophysical objects: perturbative treatment in $\mathcal{S}^2 \triangleq \frac{1}{2}S_{\alpha\beta}S^{\alpha\beta}$
- Here: only spin-induced quadrupole

$$J^{\mu\nu\rho\sigma} = \frac{3(1+\delta\kappa)}{\mu}v^{[\mu}S^{\nu]\lambda}S_{\lambda}{}^{[\rho}v^{\sigma]} = \mathcal{O}\left(\mathcal{S}^2\right)$$

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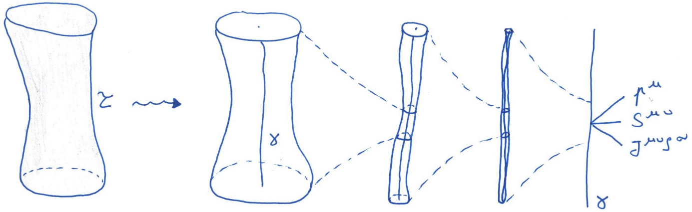

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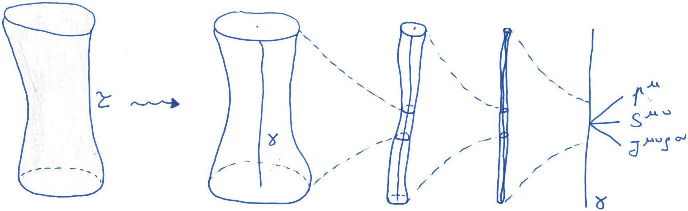

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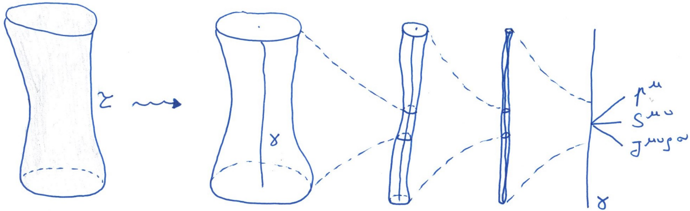

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- Symmetries: 2 Killing vectors + 1 rank-2 Killing tensor
- 4 conserved quantities: μ^2, E, L_z and Q
- Bounded geodesic motion in Kerr is **triperiodic** (in r, θ, ϕ), **separable** and (Liouville) **integrable**
- Allows to turn to **action-angle variables** $(x^\mu, p_\mu) \to (q^\lambda, J_\lambda)$ such that

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Extended bodies

- E, L_z can be deformed and still exactly conserved [Dixon 1979]
- Other **quasi-constants** of the motion investigated by Rüdiger in the 80s @ $\mathcal{O}(\mathcal{S}^1)$ [Rüdiger 1981-83] [Compère and AD 2020]
  - Linear invariant Q_Y (NEW !)
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- For LISA, need to go up to $\mathcal{O}(\epsilon^2)$ and $\mathcal{O}(\mathcal{S}^2)$
- *Multiscale expansion*: separate *slow-time* dynamics ($\sim t_{\text{r-r}}$) from *fast-time* dynamics ($\sim t_o$) [Flanagan and Hinderer 2008]
  - Fast-time equations $\leftrightarrow$ triperiodicity of background geodesics
  - Slow-time equations $\leftrightarrow$ drive the evolution of the constants of motion
- EOMs read

$$\frac{dq_\alpha}{dt} = \omega_\alpha(J_\lambda) + \epsilon\, g^{(1)}_\alpha(q_A, J_\lambda) + \epsilon^2\, g^{(2)}_\alpha(q_A, J_\lambda) + \mathcal{O}(\epsilon^3),$$

$$\frac{dJ_\lambda}{dt} = \epsilon\, G^{(1)}_\lambda(q_A, J_\rho) + \epsilon^2 G^{(2)}_i(q_A, J_\rho) + \mathcal{O}(\epsilon^3)$$

- Still need the fundamental frequencies (possibly with finite-size corrections)
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- Motion of spinning test bodies in curved spacetime: MPD equations

$$\frac{Dp^\mu}{d\lambda} = -\frac{1}{2}R^\mu{}_{\nu\alpha\beta}v^\nu S^{\alpha\beta} + \mathcal{F}^\mu$$
$$\frac{DS^{\mu\nu}}{d\lambda} = 2p^{[\mu}v^{\nu]} + \mathcal{L}^{\mu\nu}.$$

$$p^\mu \triangleq \int_{x^0=cst} d^3x \sqrt{-g}T^{\mu 0}, \qquad S^{\mu\nu} \triangleq \int_{x^0=cst} d^3x \sqrt{-g}\left(\delta x^\mu T^{\nu 0} - \delta x^\nu T^{\mu 0}\right)$$

- Conserved spin magnitude: $\mathcal{S}^2 \triangleq \frac{1}{2}S_{\alpha\beta}S^{\alpha\beta}$
- Quadrupole approximation:

$$\mathcal{F}^\mu = -\frac{1}{6}J^{\alpha\beta\gamma\delta}\nabla^\mu R_{\alpha\beta\gamma\delta}, \qquad \mathcal{L}^{\mu\nu} = \frac{4}{3}R^{[\mu}{}_{\alpha\beta\gamma}J^{\nu]\alpha\beta\gamma}$$

- Spin-induced quadrupole:

$$J^{\mu\nu\rho\sigma} = \frac{3(1+\delta\kappa)}{\mu}v^{[\mu}S^{\nu]\lambda}S_\lambda{}^{[\rho}v^{\sigma]} = \mathcal{O}\left(\mathcal{S}^2\right)$$

$\delta\kappa = 0$ for a Kerr BH ("**BH case**") and $\delta\kappa \neq 0$ otherwise ("**NS case**")

- Tulczyjew spin supplementary condition

$$S^{\mu\nu}p_\mu = 0 \quad \Rightarrow \quad S^{\mu\nu} = -\epsilon^{\mu\nu\alpha\beta}\underbrace{\hat{p}_\alpha}_{\triangleq p_\alpha/\mu}S_\beta, \quad v^\mu = v^\mu(p^\alpha, S^\alpha)$$

THE SETUP: MATHISSON-PAPAPETROU-DIXON EQUATIONS

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$$p^\mu \triangleq \int_{x^0=cst} d^3x \sqrt{-g} T^{\mu 0}, \qquad S^{\mu\nu} \triangleq \int_{x^0=cst} d^3x \sqrt{-g} \left(\delta x^\mu T^{\nu 0} - \delta x^\nu T^{\mu 0}\right)$$

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- Motion of spinning test bodies in curved spacetime: MPD equations

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The setup: Mathisson-Papapetrou-Dixon equations

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CONSERVED QUANTITIES [Rüdiger 1981-83]

- Basic idea: enforce directly the (quasi-)conservation

$$\dot{Q}^{(A)} = v^\lambda \nabla_\lambda Q^{(A)} \stackrel{!}{=} \mathcal{O}\left(\mathcal{S}^k\right)$$

- Historically: homogeneous Ansätze in the number of p's and S's

$$Q^{(1)} = A_\mu p^\mu + B_{\alpha\beta} S^{\alpha\beta},$$
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- Relaxed spin vector: $S^\alpha = \left(\delta^\alpha_\beta + \hat{p}^\alpha \hat{p}_\beta\right) s^\beta$
- All the terms of the constraints take the form

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- All the Kerr spacetime quantities can be expressed in terms of $(g_{\mu\nu}, \epsilon_{\mu\nu\rho\sigma}, \mathcal{R}, \xi_{\alpha}, N_{\alpha\beta})$ and their conjugated. Examples:

$$Y_{\alpha\beta} = -\operatorname{Re}\left(\mathcal{R}N_{\alpha\beta}\right), \qquad R_{\alpha\beta\gamma\delta} = M\operatorname{Re}\left(\frac{3N_{\alpha\beta}N_{\gamma\delta} - G_{\alpha\beta\gamma\delta}}{\mathcal{R}^3}\right).$$

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"SCALAR" CONSTRAINT EQUATION

- Generating set of contractions for the constraint equations:

$$\mathcal{S}^2 \triangleq s_\alpha s^\alpha, \quad \mathcal{P}^2 \triangleq -\hat{p}_\alpha \hat{p}^\alpha, \quad \mathcal{A} \triangleq s_\alpha \hat{p}^\alpha, \quad A \triangleq N_{\lambda\mu}\xi^\lambda \hat{p}^\mu,$$
$$B \triangleq N_{\alpha\mu} s^\alpha \hat{p}^\mu, \quad C \triangleq N_{\lambda\alpha}\xi^\lambda s^\alpha, \quad D \triangleq h_{\lambda\alpha}\xi^\lambda s^\alpha, \quad E \triangleq -\xi_\alpha \hat{p}^\alpha,$$
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- Powers of $\mathcal{R}$ introduced through

$$\alpha_K^{(n,p)} \triangleq \mathrm{Re}\left(\frac{K\bar{\mathcal{R}}^n}{\mathcal{R}^p}\right), \quad \omega_K^{(n,p)} \triangleq \mathrm{Im}\left(\frac{K\bar{\mathcal{R}}^n}{\mathcal{R}^p}\right).$$

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$$\hat{\nabla} T \triangleq \hat{p}^\lambda \nabla_\lambda (T_{\mu_1 \ldots \mu_k}) \ell^{\mu_1} \ldots \ell^{\mu_k},$$

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- **In this formulation, the constraints become purely algebraic relations!**

"Scalar" constraint equation

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The linear constraint

$$\boxed{Q^{(1)} \triangleq X_\mu p^\mu + W_{\mu\nu} S^{\mu\nu}, \qquad \dot{Q}^{(1)} \overset{!}{=} \mathcal{O}\left(\mathcal{S}^3\right)}$$

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- Non-trivial conserved quantity at linear order: Rüdiger quadratic invariant [Rüdiger 1983]

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- Holds in any RF spacetime admitting a KY tensor & unique in Kerr [Compère and AD 2021]

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SOME INSIGHTS ABOUT NEUTRON STARS ($\delta\kappa \neq 0$ CASE)

- Adding a deformation

$$Q = Q_{BH} \to Q_{BH} + \delta\kappa Q_{NS}$$

allows to separate NS constraint from BH one $\Rightarrow$ independent treatment

- **Linear invariant:** must be supplemented by a $[2, 3]$ piece $Q_{NS}^{(1)} = \delta\kappa M_{\alpha\beta\mu\gamma\delta} S^{\alpha\beta} S^{\gamma\delta} p^{\mu}$ such that $(N \triangleq {}^*M^*)$

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