Ricci-flat metrics on canonical bundles of Kähler surfaces

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Gravity, Geometry and Symmetry — a celebration for Pietro Fré 70s

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M. Bianchi, U. B., P. Fré, D. Martelli.

Resolution à la Kronheimer of $\mathbb{C}^3/\Gamma$ *singularities and the Monge-Ampère equation for Ricci-flat Kähler metrics in view of D3-brane solutions of supergravity.*

Lett. Math. Phys. **111** (2021), no. 3, Paper no. 79, 79 pages

(Boris A. Dubrovin's memorial volume)

U. B., P. Fré, U. Shazhad (work in progress)

Aim: Construct explicit Ricci-flat metrics on some noncompact Calabi-Yau 3-folds

Motivation: Find D3-brane solutions to 10-dimensional supergravity

Ricci flat metrics

Definition

A Calabi-Yau manifold will be a Kähler manifold with trivial canonical bundle

Calabi's conjecture (1954, 1957) — Yau's theorem (1977, 1978)

If M is a compact Kähler manifold with Kähler metric g and Kähler form ω, and R is any (1,1)-form representing $c_1(M)$, there exists on M a unique Kähler metric $\tilde{g}$ with Kähler form $\tilde{\omega}$ such that

- ω and $\tilde{\omega}$ are in the same class in $H^2(X, \mathbb{R})$
- the Ricci form of $\tilde{\omega}$ is R.

In particular, if $c_1(M) = 0$, then M carries Ricci-flat metrics

Noncompact case

Tian-Yau (1990/1991) proved a similar theorem about the existence of Ricci-flat metrics on the complement of a smooth ample divisor in a smooth projective variety

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Crepant resolutions of finite quotient singularities are another class of noncompact Calabi-Yaus

$$X_0 = \mathbb{C}^n/G, \qquad G \subset SL_n(\mathbb{C}) \text{ a finite subgroup}$$

A resolution of singularities $X \to X_0$ is crepant if $K_X = 0$ (NB $K_{X_0} = 0$)

For $n = 2$ the (unique) crepant resolutions of singularities are Kronheimer's ALE spaces

Ricci-flat metric are constructed by hyperkähler reduction and they are quite explicit, at least in the abelian (A_k) case

Theorem (Joyce 2001)

If X is a crepant resolution of $X_0 = \mathbb{C}^n/G$ $(n \geq 2)$, and the origin is the only fixed locus of the G-action, then in each ALE Kähler class of X there is a unique Ricci-flat ALE Kähler metric.

If the hypothesis on the fixed locus is relaxed (so that the latter is noncompact) the same result holds with the exception that the Ricci-flat metric is QALE

This is just an existence theorem!!

Calabi's trick

A 1979 Calabi's paper gives a hands-on recipe to construct a Ricci-flat metric on the total space of a homolomorphic line bundle $\pi\colon L \to M$ on a Kähler-Einstein manifold M

Reminder

A Kähler metric g on a complex manifold is Kähler-Einstein if

$$\mathrm{Ric}(g) = \lambda\,\omega(g), \qquad \lambda \in \mathbb{R}$$

Kähler potential for a metric on L:

$$\Psi = \Phi \circ \pi + u \circ t$$

- Φ is the Kähler potential of the KE metric on M;
- $t = \sum a_{\mu\bar\lambda}\,\zeta^\mu \bar\zeta^{\bar\lambda}$, where a is an hermitian fibre metric on L
- u is a function of one variable.

The Ricci-flat condition yields an ODE for the function u.

Example: (unique) crepant resolution of $X_0 = \mathbb{C}^3/\mathbb{Z}_3$, with $\mathbb{Z}_3$ acting as

$$(x, y, z) \mapsto \xi\,(x, y, z), \qquad \xi^3 = 1$$

X is the total space of the canonical bundle of $\mathbb{P}^2$

Calabi's trick yields

$$\begin{aligned} u(x) \;&=\; u_0 + \frac{3}{\ell}\left(\sqrt[3]{1+cx} - 1\right) - \frac{1-\xi}{\ell}\log\frac{\sqrt[3]{1+cx}-\xi}{1-\xi} \\ &\quad - \frac{1-\xi^2}{\ell}\log\frac{\sqrt[3]{1+cx}-\xi^2}{1-\xi^2}, \qquad \xi = e^{2\pi i/3} \end{aligned}$$

(ℓ is the curvature of a KE metric on $\mathbb{P}^2$)

Example: a Ricci-flat metric on the total space of $\mathcal{O}(-1)\oplus\mathcal{O}(-1)$ on $\mathbb{P}^1$, which is a (small) resolution of singularities of the conifold

$$(x_1x_2 = x_3x_4) \subset \mathbb{C}^4$$

$$u(x) = -\frac{1}{2}\log(x) + \frac{3}{4}\Bigg[\left(\frac{1}{\sqrt{3}x+\sqrt{3x^2-1}}\right)^{\frac{2}{3}} + \left(\sqrt{3}x+\sqrt{3x^2-1}\right)^{\frac{2}{3}} + \frac{2^{\frac{2}{3}}\,i\left(\sqrt{\sqrt{3}x-1}-\sqrt{\sqrt{3}x+1}\right)}{\left(\sqrt{\sqrt{3}x+1}-\sqrt{\sqrt{3}x-1}\right)^{\frac{1}{3}}}\left({}_2F_1\left[\frac{1}{3},1,\frac{4}{3},\frac{i}{\sqrt{3}x+\sqrt{3x^2-1}}\right] - {}_2F_1\left[\frac{1}{3},1,\frac{4}{3},\frac{-i}{\sqrt{3}x+\sqrt{3x^2-1}}\right]\right) - \frac{1}{\left(\sqrt{3}x+\sqrt{3x^2-1}\right)^{\frac{2}{3}}}\left({}_2F_1\left[\frac{2}{3},1,\frac{5}{3},\frac{i}{\sqrt{3}x+\sqrt{3x^2-1}}\right] + {}_2F_1\left[\frac{2}{3},1,\frac{5}{3},\frac{-i}{\sqrt{3}x+\sqrt{3x^2-1}}\right]\right)\Bigg].$$

This metric was rediscovered in 1990 by Candelas–de la Ossa (but they did not give the explicit form of the Kähler potential)

Why KE?

This does not work in the non-KE case because then the differential equation for u contains terms depending on the coordinates in the base

An interesting case model

$\mathbb{Z}_4$ acting on $\mathbb{C}^3$ as $(x, y, z) \to (ix, iy, -z)$

X is the canonical bundle of $\mathbb{F}_2$, which is not KE

$$\mathbb{F}_2 = \mathbb{P}(\mathcal{O}_{\mathbb{P}^1} \oplus \mathcal{O}_{\mathbb{P}^1}(-2)) = \left\{x_1 y_0^2 = x_2 y_1^2 \subset \mathbb{P}^2 \times \mathbb{P}^1\right\}$$

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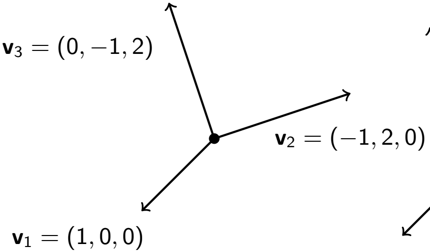
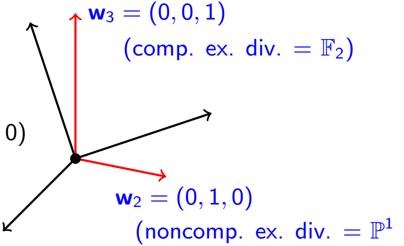
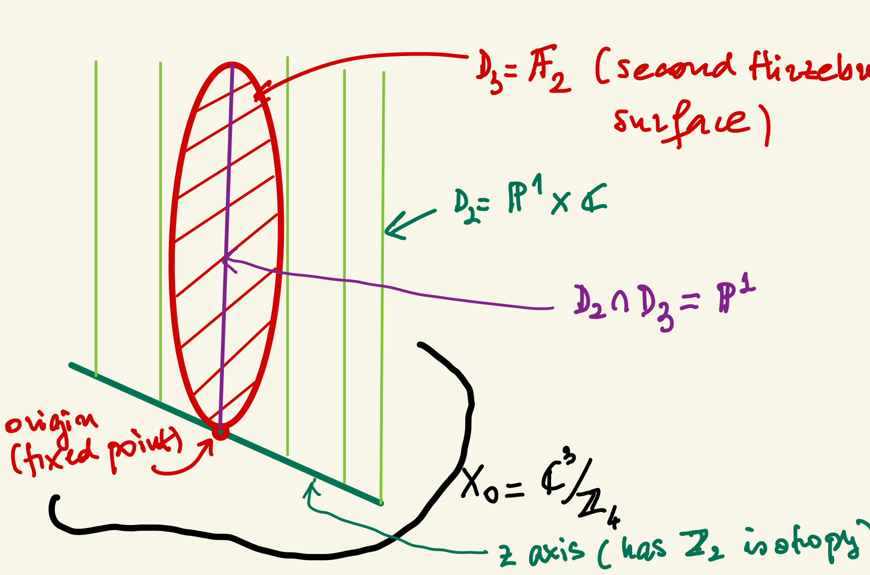
Enters toric geometry



Fan describing $X_0 = \mathbb{C}^3/\mathbb{Z}_4$



Fan describing X


$D_3 = \mathbb{F}_2$ (second Hirzebr
surface)
$D_2 = \mathbb{P}^1 \times \mathbb{C}$
$D_2 \cap D_3 = \mathbb{P}^1$
origin
(fixed point)
$X_0 = \mathbb{C}^3/\mathbb{Z}_4$
z axis (has $\mathbb{Z}_2$ isotropy)

Symplectic toric geometry

Definition

A Kähler toric manifold is a closed connected n-complex dimensional Kähler manifold (M,ω) equipped with an effective holomorphic Hamiltonian action

$$\tau\colon \mathbb{T}^n \to \operatorname{Diff}(M,\omega)$$

$\uparrow$ real n-dimensional torus

The image P of the associated moment map $\mu\colon M \to \mathbb{R}^n$ is a convex polytope (a *Delzant* polytope) and the dual fan Σ_P of P describes M as a toric variety, with an action of the complex torus $\mathbb{T}^n_{\mathbb{C}}$.

The open dense subset $M^\circ \subset M$ where the action of $\mathbb{T}^n$ is free is symplectomorphic to $P^\circ \times \mathbb{T}^n$

$$M^\circ \simeq \mathbb{R}^n \times \mathbb{T}^n = \{(x, y)\}$$

y to be regarded as "positions" and x as "velocities"

A Kälher potential K for ω defined on M° may be used as a "Lagrangian" to pass from complex coordinates to symplectic coordinates. Since ω is $\mathbb{T}^n$-invariant, K only depends on x

$$\omega = \begin{pmatrix} 0 & F \\ -F & 0 \end{pmatrix}, \qquad F = \mathrm{Hess}_x(K)$$

Define the momenta

$$\mu = \frac{\partial K}{\partial x}$$

$$H = \mu\, y - K, \qquad (y,\, \mu) \text{ symplectic coordinates}$$

H is a symplectic potential for the metric

$$J = \begin{pmatrix} 0 & -G^{-1} \\ G & 0 \end{pmatrix}, \qquad G = \mathrm{Hess}_\mu(H)$$

$$g = \begin{pmatrix} G & 0 \\ 0 & G^{-1} \end{pmatrix} \quad \text{Riemannian metric in } (x, y) \text{ coordinates}$$

Ricci-flat condition for the metric g $\Longrightarrow$ Monge-Ampère type equation for H

Symplectic potential

$$\mu = (u, v, w), \qquad H(u, v, w) = H_0(u, v) + \boxed{\bar{H}(v, w)}$$

(decomposition coming from the $SU(2) \times U(1)^2$ isometry)

$$G = \begin{pmatrix} -\frac{u}{u^2 - uv} & \frac{1}{u-2v} & 0 \\ \frac{1}{u-2v} & \frac{1}{u-2v} + \bar{H}^{(2,0)}) & \bar{H}^{(1,1)} \\ 0 & \bar{H}^{(1,1)} & \bar{H}^{(0,2)} \end{pmatrix}$$

Monge-Ampère equation:

$$\boxed{\det G = 1}$$

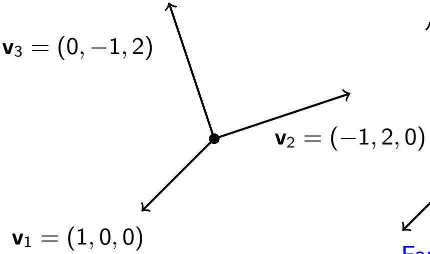
A degenerate, simpler case

Weighted projective space

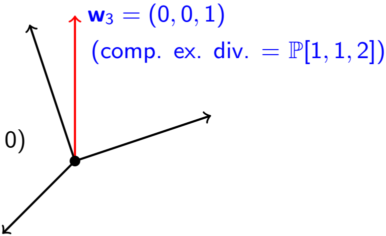
$$\mathbb{P}[1,1,2] = \frac{\mathbb{C}^3 - \{0\}}{\mathbb{C}^*}, \qquad (x,y,z) \to (\lambda x,\ \lambda y,\ \lambda^2 z)$$

$X_3 = \operatorname{tot} K_{\mathbb{P}[1,1,2]}$ a partial resolution of singularities of $\mathbb{C}^3/\mathbb{Z}_4$

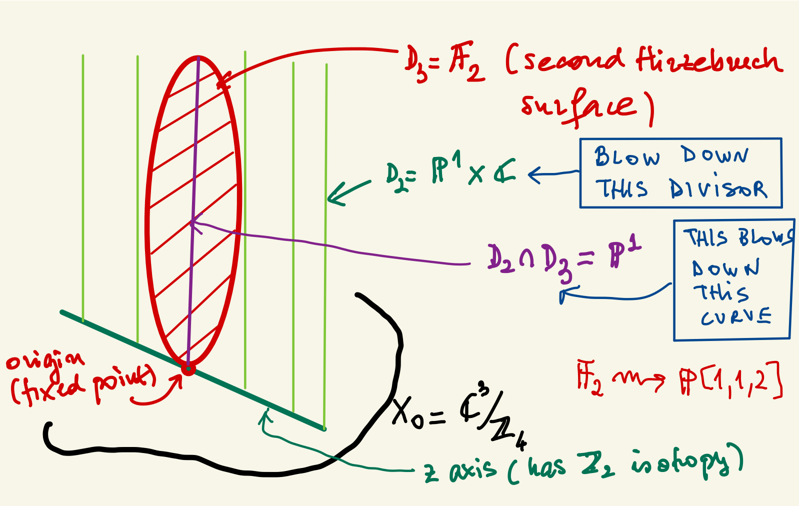
(when only the singularity at the origin is resolved)



Fan describing $X_0 = \mathbb{C}^3/\mathbb{Z}_4$



Fan describing $X_3 = \operatorname{tot} K_{\mathbb{P}[1,1,2]}$


D3 = F2 (second Hirzebruch surface)
D2 = P1 x C
BLOW DOWN THIS DIVISOR
D2 ∩ D3 = P1
THIS BLOWS DOWN THIS CURVE
origin (fixed point)
F2 ⇝ P[1,1,2]
X0 = C3/Z4
z axis (has Z2 isotropy)

$\mathbb{P}[1,1,2]$ is KE, so that Calabi's trick can be applied (albeit $\mathbb{P}[1,1,2]$ is singular).

Another metric can be found by directly solving the MA equation, with boundary condition on the exceptional divisor given by a metric on $\mathbb{P}[1,1,2]$ which we describe in the next slides

Solution:

$$\bar{H}(v,w) = \frac{1}{224}\Big\{7\Big[6(2w-3)\log\Big(-\sqrt{(2v+3w)^2-36v}-2v-3w$$

$$+16v\log\left(\sqrt{(2v+3w)^2-36v}-2v-3w\right)^2$$

$$-2(8v-12w+9)\log\left(\sqrt{(2v+3w)^2-36v}-2v-3w+\frac{9}{2}\right)$$

$$+2(4v+3w)\log\left(\frac{1}{567}\left[4\sqrt{(2v+3w)^2-36v}+8v+12w+9\right]^2+1\right)\Big]$$

$$-4\sqrt{7}(4v-3(w+3))\arctan\left(\frac{4\sqrt{(2v+3w)^2-36v}+8v+12w+9}{9\sqrt{7}}\right)$$

$$-(8v+9)\log\frac{34359738368}{823543}+2\sqrt{7}(8v-27)\arctan\frac{5}{\sqrt{7}}\Big\}$$

The Kähler quotient construction (Sardo Infirri)

$R = \mathbb{C}[G]$ = regular representation of G

$$V = (\mathrm{End}(R) \otimes \mathbb{C}^3)^G$$

Elements of V are represented by triples of matrices

$$\phi(A, B) = \mathrm{Im} \sum_{j=1}^{3} \mathrm{tr}(A_j, B_j^*)$$ Kähler form on V

Moment map: let $\mathfrak{c} = C(\mathfrak{pu}(R)^G)$

$$\mu\colon V \to \mathfrak{c}^* \qquad \mu(A)(\xi) = \frac{1}{2i} \sum_j \xi([A_j, A_j^*]), \quad \xi \in \mathfrak{pu}(R)^G$$

$$N = \{A \in V \,|\, [A, A] = 0 \text{ in } \mathrm{End}(R) \otimes \Lambda^2 \mathbb{C}^3\}$$

If $\zeta \in \mathfrak{c}^*$ is generic, then $\mathbb{P}U(R)^G$ acts freely on $N \cap \mu^{-1}(\zeta)$

$$X_\zeta = (N \cap \mu^{-1}(\zeta))/\mathbb{P}U(R)^G$$

(a principal fiber bundle)

The Kähler metric ϕ of V descends to a Kähler metric ϕ_ζ on X_ζ

There is a natural map $X_\zeta \to X_0$, which (again for ζ generic) is a resolution of singularities

Chamber structure of the space of ζ's!

In our model case $(X_0 = \mathbb{C}^3/\mathbb{Z}_4)$ for ζ generic one gets the unique full crepant resolution of singularities

$$X = \text{tot}\, K_{\mathbb{F}_2}$$

For some values of ζ (generic points of one wall) one gets a partial crepant resolution

$$X_3 = \text{tot}\, K_{\mathbb{P}[1,1,2]}$$

The Kähler metric coming from the quotient construction in not Ricci-flat!!

Idea: write an ansatz as a power series in w and use the restriction of this metric to the compact exceptional divisor as “initial value”. The MA eqns. allows to determine the next terms iteratively.

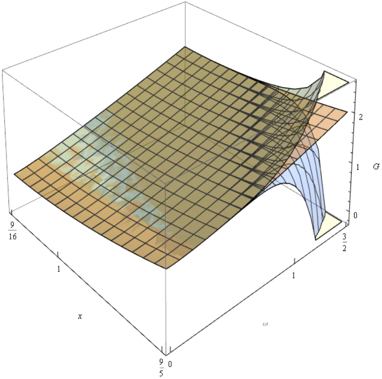
$$x = 2v, \qquad \omega = 3\left(w - \tfrac{3}{2}\right)$$

$$\bar{H} = \frac{1}{8}\omega \log\left(\frac{x(6\Delta - 4x + 9)^2}{2e\left[\Delta^2 + 4x^2 - 6(2\Delta + 3)x\right]}\right) + \frac{1}{8}\omega \log\omega + \sum_{k=1}^{\infty} \frac{N_{k+1}(x,\Delta)}{D_{k+1}(x,\Delta)}\,\omega^{k+1}$$

Δ is a function of the stability parameter ζ

N and D are certain polynomials which can be determined iteratively. However at the moment a recursion formula is not available. What seems to be remarkable is that the polynomials so fare computed have integer coefficients.

To try to understand the behavior of this expansion one can make a series development of the known solution for $\mathbb{P}[1,1,2]$. While the solution is fully regular, the truncations of the series are singular — singularities seem to cancel out when the series is summed



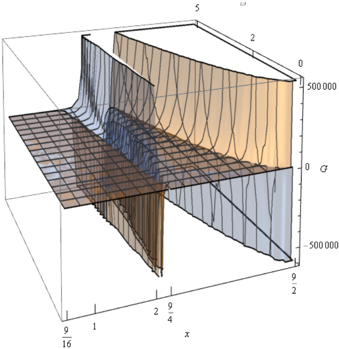


Figure: Plots of the exact symplectic potential H for $\Delta = 0$ ($\mathbb{P}[1,1,2]$) compared to its approximants of order 6 and 7: on the left for small values of ω, on the right extending to large values of ω.

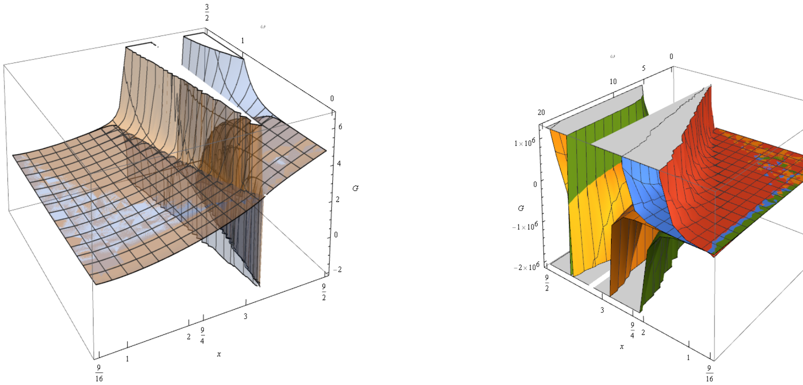


Figure: Approximants of H for $\Delta = \frac{3}{4}$. The plot on the left is for small values of ω and displays two consecutive approximants of order 7 and 8, while the plot on the right extends to large values of ω and displays several approximants

Conjecture: Assume that G is abelian. If

- the resolution of singularities X of $\mathbb{C}^3/G$ has only one compact exceptional divisor E
- X is the total space of the canonical bundle of E
- g_0 is the metric on E given by the Kähler quotient construction

Then the Monge-Ampère equation for the symplectic potential of g is solvable by a series expansion whose leading term yields g_0.

Moreover we expect that g is QALE (ALE when the are no noncompact exceptional divisor)

Epilogue

Some of the things that have been left out

- MacKay correspondence: structure of the resolution of singularities (exceptional divisors) and its cohomology $\Longleftrightarrow$ representation theory of the group G
- Sasaki-Einstein geometry (the resolutions of singularities we are considering are Kähler cones over 5-dimensional Sasaki-Einstein manifolds)

Dear Pietro

Congratulations for your long career

Your contributions to theoretical and mathematical physics

and, most of all, for the almost lifelong friendship!!