Status and Outlook for $B$-mixing parameters on the lattice

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Outline

1. Introduction
2. Challenges in $b$-physics on the lattice
3. Status: Recent results
4. Outlook: Ongoing Work by RBC/UKQCD/JLQCD
5. Conclusion
Flavour Physics: The Cabibbo-Kobayashi-Maskawa matrix

\[ \text{experiment} \approx \text{CKM-factors} \times \text{non-perturbative inputs} \times \text{known terms} \]

**CKM Matrix**
- parameterises transitions up-type ↔ down-type
- fundamental SM parameters
- unitary matrix in the SM:
  \[ V V^\dagger = 1_3 \]

**Unitarity Triangle**

yellow & orange bands = exp. measurements of \( \Delta m_d(s) + \text{SM} \) predictions for mixing parameters
⇒ Tests CKM unitarity!

Non-unitarity of CKM ⇔ New Physics Beyond the SM
Neutral $B_{(s)}$ meson mixing - background

Neutral mesons oscillate:

$$|B_{L,H}\rangle = p |B_q^0\rangle \pm q |\bar{B}_q^0\rangle$$

⇒ splittings in mass eigenstates:

- mass splitting $\Delta m_q \equiv m_H - m_L$
- width splitting $\Delta \Gamma_q \equiv \Gamma_L - \Gamma_H$

Time dependence:

$$|B_q^0(t)\rangle = g_+(t) |B_q^0\rangle + \frac{q}{p} g_-(t) |\bar{B}_q^0\rangle$$

$$|\bar{B}_q^0(t)\rangle = \frac{p}{q} g_-(t) |B_q^0\rangle + g_+(t) |\bar{B}_q^0\rangle$$

where $q = d, s$

Occurs at loop level in SM ⇒ sensitive probe of new physics!
Neutral $B_{(s)}$ meson mixing - experiment

$$|g_\pm(t)|^2 = \frac{e^{-\Gamma_q t}}{2} \left[ \cosh \left( \frac{\Delta \Gamma_q t}{2} \right) \pm \cos(\Delta m_q t) \right]$$

$\Delta m$ experimentally accessible as a frequency!

$B_d^0$: Many results

$B_s^0$: “Only” CDF, CMS and LHCb

$\Delta m_d = 0.5065(19) \text{ps}^{-1}$

$\Delta m_s = 17.765(06) \text{ps}^{-1}$

Well below per cent level!

[HFLAV 2206.07501]
Neutral $B_{(s)}$ meson mixing - theory

Short distance dominated $\Rightarrow$ described by $\mathcal{H}^{\Delta b=2}$ eff. weak Hamiltonian. OPE factorises this into

- **Perturbative model-dependent Wilson coefficients** $C_i(\mu)$
- **Non-perturbative model-independent matrix elements**

$$\langle B_{(s)}^0 | \mathcal{H}^{\Delta b=2} | B_{(s)}^0 \rangle = \sum_i C_i(\mu) \langle B_{(s)}^0 | \mathcal{O}_i^{\Delta b=2}(\mu) | B_{(s)}^0 \rangle$$

- 5 independent (parity even) operators $\mathcal{O}_i$, only $\mathcal{O}_1$ relevant for $\Delta m$:

$$\mathcal{O}_1 = (\bar{b}_a \gamma_\mu (1 - \gamma_5) q_a) (\bar{b}_b \gamma_\mu (1 - \gamma_5) q_b) = \mathcal{O}_{VV+AA}$$

- Define bag parameters: $\hat{B}_{B_q}^{(i)} = \langle \bar{B}_q^0 | \mathcal{O}_i | B_q^0 \rangle / \langle \bar{B}_q^0 | \mathcal{O}_i | B_q^0 \rangle_{VSA}$

$$\Delta m_q = |V^*_{tb} V_{tq}|^2 \times f_{B_q}^2 \hat{B}_{B_q}^{(1)} \times m_{B_q} \mathcal{K}$$

$\Rightarrow$ **Non-perturbative matrix elements calculable on the lattice**
Extracting CKM matrix elements

We write $\Delta m_q$ in terms of the Renormalisation Group Independent (RGI) bag parameter $\hat{B}_{B_q}$:

$$\Delta m_q = |V_{tb}^* V_{tq}|^2 \frac{G_F^2 m_W^2 m_{B_q}^2}{6\pi^2} S_0(x_t) \eta_2 B f_{B_q}^2 \hat{B}_{B_q}^{(1)}$$

- $\Delta m_d$ and $\Delta m_s$ are known experimentally to $\ll 1\%$ accuracy
- Combined other inputs are known to 0.4%
- Typical precision for $f_{B_q} \sqrt{\hat{B}_{B_q}^{(1)}}$ is currently a few percent.
- Part of statistic and systematic errors cancel in $SU(3)$ breaking ratios:

$$\xi^2 \equiv \frac{f_{B_s}^2 \hat{B}_{B_s}^{(1)}}{f_{B_d}^2 \hat{B}_{B_d}^{(1)}} = \left| \frac{V_{td}}{V_{ts}} \right|^2 \frac{\Delta m_s}{\Delta m_d} \frac{m_B}{m_{B_s}} \Rightarrow \text{access to } |V_{td}/V_{ts}|$$
Lattice QCD in a nutshell

Based on the **Path Integral** formulation.

\[
\langle O \rangle_M = \frac{1}{Z} \int \mathcal{D}[\psi, \bar{\psi}, U] O[\psi, \bar{\psi}, U] e^{iS[\psi, \bar{\psi}, U]}
\]

**Minkowski:** Highly oscillatory, infinite dimensional integral.

\Rightarrow Wick rotate to Euclidean (i.e. imaginary) time \((t \rightarrow i\tau)\).

\[
\langle O \rangle_E = \frac{1}{Z} \int \mathcal{D}[\psi, \bar{\psi}, U] O_E[\psi, \bar{\psi}, U] e^{-S_E[\psi, \bar{\psi}, U]}
\]

**Euclidean:** Exponentially decaying, infinite dimensional integral.

\Rightarrow Discretise space-time and **interpret** as a probability distribution.

- Lattice spacing \(a\) (UV regulator)
- Box of length \(L\) (IR regulator)

**Lattice:** Exponentially decaying and finite dimensional
Multiple scale problem on the lattice: back of the envelope

Control effects of IR (finite volume) and UV (discretisation) regulators:

\[ m_\pi L \gtrsim 4 \quad a^{-1} \gg \text{Mass scale of interest} \]

\[
\begin{array}{cccccccc}
10^0 & 10^1 & 10^2 & 10^3 & 10^4 & 10^5 & 10^6 \\
 m_q [\text{MeV}] \\
\end{array}
\]

For \( m_\pi = m_\pi^{\text{phys}} \sim 140 \text{ MeV} \) and \( \overline{m}_b(m_b) \approx 4.2 \text{ GeV} \):

\[ L \gtrsim 5.6 \text{ fm} \quad a^{-1} \gg 4.2 \text{ GeV} \approx (0.05 \text{ fm})^{-1} \]

Requires \( N \equiv L/a \gg 120 \Rightarrow N^3 \times (2N) \gg 4 \times 10^8 \) lattice sites.

**VERY EXPENSIVE** to satisfy both constraints simultaneously...

... needs to be repeated for different values of \( a \).
How to simulate the $b$-quark?

For now choose between:

**Effective action for $b$**
- Can tune to $m_b$
- comes with systematic errors which are hard to estimate/reduce

**Relativistic action for $b$**
- Theoretically cleaner and systematically improvable
- Need to control extrapolation in heavy quark mass

**Different properties:**
- computational cost
- chirality
- tuning errors
- systematic errors
- cut off effects
- renormalisation

**BUT SOON:**
Huge efforts in the community to produce very fine lattice spacings:
⇒ Direct simulation of $\approx m_b^{\text{phys}}$ will become possible!
Fewer results than in the light sector, but very complementary results from

- different gauge field configurations
- different valence light actions
- different valence heavy actions
- different methodologies

Includes computations with $m_{\pi}^{\text{phys}}$!
Going beyond $\hat{B}^{(1)}_{Bq}$

HPQCD19, FNAL/MILC16, HPQCD09, RBC/UKQCD18

$B_{(n)}^{(1)} / B_{B_d}^{(1)} |_{\text{avg}} = 1.01(2)$

$B_{(n)}^{(2)} / B_{B_d}^{(2)} |_{\text{avg}} = 1.06(2)$

$B_{(n)}^{(3)} / B_{B_d}^{(3)} |_{\text{avg}} = 1.08(3)$

$B_{(n)}^{(4)} / B_{B_d}^{(4)} |_{\text{avg}} = 0.96(2)$

$B_{(n)}^{(5)} / B_{B_d}^{(5)} |_{\text{avg}} = 0.97(2)$

HPQCD19, FNAL/MILC16, HPQCD09, ETM13

$B_{B_s}^{(1)} |_{\text{avg}} = 0.84(3)$

$B_{B_s}^{(2)} |_{\text{avg}} = 0.83(4)$

$B_{B_s}^{(3)} |_{\text{avg}} = 0.85(5)$

$B_{B_s}^{(4)} |_{\text{avg}} = 1.03(4)$

$B_{B_s}^{(5)} |_{\text{avg}} = 0.94(3)$

[Plots from HPQCD’19: 1907.01025]

Mostly in agreement, but some spread in $B_{B_s}^{(i)}$ parameters.
CKM matrix elements

|                  | $f_B(\hat{B}_B)^{1/2}$ [GeV] | $f_B(\hat{B}_B)^{1/2}$ [GeV] | $\xi$ | $|V_{td}|$ | $|V_{ts}|$ | $|V_{ts}/V_{td}|$ |
|------------------|-----------------------------|-----------------------------|------|---------|---------|-----------------|
| HPQCD19          | 0.2106(55)                  | 0.2561(57)                  | 1.216(16) | 0.00867(23) | 0.04189(93) | 0.2071(27) |
| FNAL/MILC16      | 0.2277(98)                  | 0.2746(88)                  | 1.206(18) | 0.00800(35) | 0.0390(13) | 0.2052(33) |
| RBC/UKQCD18      |                            |                            | 1.194(+12) |            |          |                 |

- Reasonable agreement between lattice results, but some spread
- Tree-only fit somewhat differs
- Uncertainty dominated by theory

⇒ Further theory work required!

Current lattice uncertainties dominated by
- Renormalisation/matching
- Heavy quarks (discretisation/extrapolation)

⇒ we are working on improving both!
• Computation of $SU(3)$ breaking ratios $f_{D_s}/f_D$, $f_{B_s}/f_B$, $B_{B_s}/B_{B_d}$ and $\xi$
• Renormalisation constants cancel
• Extrapolation from $m_h < m_b$ to $m_b$
• Simultaneous fit to $m_{\pi}^2$, $a^2$, $\frac{1}{m_H}$

\[ \begin{align*}
0.1 & \quad 0.2 & \quad 0.3 & \quad 0.4 & \quad 0.5 & \quad 0.6 \\
0.10 & \quad 0.15 & \quad 0.20 & \quad 0.25 & \quad 0.30 & \quad 0.35 \\
\end{align*} \]

\[ \begin{align*}
0.00 & \quad 0.05 & \quad 0.10 & \quad 0.15 & \quad 0.20 & \quad 0.25 & \quad 0.30 & \quad 0.35 \\
1.05 & \quad 1.10 & \quad 1.15 & \quad 1.20 & \quad 1.25 & \quad 1.30 \\
\end{align*} \]

\[ \begin{align*}
0.00 & \quad 0.02 & \quad 0.04 & \quad 0.06 & \quad 0.08 & \quad 0.10 & \quad 0.12 \\
1.08 & \quad 1.10 & \quad 1.12 & \quad 1.14 & \quad 1.16 & \quad 1.18 \\
\end{align*} \]

\[ \begin{align*}
0.00 & \quad 0.02 & \quad 0.04 & \quad 0.06 & \quad 0.08 & \quad 0.10 & \quad 0.12 \\
1.17 & \quad 1.18 & \quad 1.19 & \quad 1.20 & \quad 1.21 & \quad 1.22 \\
\end{align*} \]
RBC/UKQCD/JLQCD: ongoing work

RBC/UKQCD’18 dominated by
- chiral-continuum fit
  ⇒ heavy quark extrapolation
  ⇒ estimates of higher order $1/m_H$ terms

- Supplement existing dataset with finer JLQCD ensembles
  ⇒ reduce extrapolation in $m_H$ significantly
- all domain wall fermion set-up
  ⇒ only physical block-diagonal renormalisation pattern
- all 5 operators $\hat{B}_{B_d}^{(i)}$ and $\hat{B}_{B_s}^{(i)}$
- (3+3) lattice spacings, 2 ensembles with physical pion mass
  ⇒ good control over all required limits
- new correlator fitting strategy
Huge data-set

- 15 ensembles with varying $a, L, m_l(\rightarrow M_\pi), m_s$.
- Wide range of heavy-quark masses (4-6) per ensemble $m_{H_s}^{\text{max}} \sim 75\% m^{\text{phys}}_{B_s}$.
- 6 lattice spacings (2 different discretisations).
- 2 $m^{\text{phys}}_\pi$ ensembles.
- 2 pairs of ensembles which only differ in $m_s$.
- 1 pair which differs in $V$.

Fitting strategy & extrapolations

- Two independent correlation function analyses. (✓)
- RI/sMOM renormalisation analysis. (✓, preliminary)
- Fit to perform (ongoing)
  - $a \rightarrow 0, L \rightarrow \infty$
  - $M_\pi \rightarrow M^{\text{phys}}_\pi, m_s \rightarrow m^{\text{phys}}_s$
  - $M_{H_s} \rightarrow M_{B_s}$

  for $\hat{B}^{(i)}_{B_q}, f_{B_q}, f_{B_q} \sqrt{\hat{B}^{(i)}_{B_q}}, \xi$, ratios.
- Assess robustness of fit and all systematic uncertainties.

**GOAL:** Controlled prediction of all observables and their correlations
Mild behaviour with $1/M_{H_s}$

Data a lot closer to physical $b$-quark mass

Precision on individual datapoints $\lesssim O(1\%)$.

Starting to do global fits
renormalised bag parameters $B_1 \ (VV+AA)$

- $a^{-1}, M_\pi, m_\ell, V$ effects not taken into account yet...

- ... but seem to be mild

- strong $1/M_{H_s}$ behaviour but similar for all ensembles
Mild discretisation effects
Seemingly mild behaviour with \( a, M_\pi, m_s, V \)
strong \( 1/M_{H_s} \) dependence

Large discretisation effects
More noticeable chiral effects
mild \( 1/M_{H_s} \) dependence
Summary and Outlook

Literature Summary
- Complementary lattice results:
  - ensembles
  - light quark actions
  - heavy quark treatment
  - renormalisation
- Physical pion mass results
- Results for full operator basis
- First result (ratios only) w/o effective action for the $b$-quark

RBC/UKQCD/JLQCD
- Huge data set (15 ensembles)
- Target all $5+5$ bag parameters, decay constants, ratios
- Good HQ reach $\Rightarrow$ no effective action prescription for $b$-quark
- $\chi$-symmetry $\Rightarrow$ continuum-like block-diagonal renormalisation
- Control all systematics (data)

CKM Status and Future
- $|V_{td}|$ and $|V_{ts}|$ known at $\approx 2.5\%$, $|V_{td}/V_{ts}|$ at $\approx 1.5\%$
- Uncertainty theory dominated - work is ongoing
- RBC/UKQCD/JLQCD: Aim to reduce uncertainties with all DWF approach (data generated using Grid and Hadrons [Grid, Hadrons])
ADDITIONAL SLIDES
Neutral $B_{(s)}$ meson mixing - theory

$$
\begin{align*}
\langle B_q^0 | H_{\text{eff}} | \bar{B}_q^0 \rangle &\propto \langle B_q^0 | H_{\Delta b=2} | \bar{B}_q^0 \rangle + \sum_n \frac{\langle B_q^0 | H_{\Delta b=1} | n \rangle \langle n | H_{\Delta b=1} | \bar{B}_q^0 \rangle}{E_n - M_{B_q}} \\
\text{Short distance} &\quad \text{Long distance}
\end{align*}
$$

Short distance $\propto \left| \sum_{q'=u,c,t} \frac{m_{q'}^2}{M_W^2} V_{q'b} V_{q'q}^* \right|^2 \approx \frac{m_t^4}{M_W^4} |V_{tb} V_{tq}^*|^2$

SD: Top enhanced: $m_t^2 V_{tb} V_{tq}^* \gg m_c^2 V_{cb} V_{cq}^* \gg m_u^2 V_{ub} V_{uq}^*$

LD: Only $m_c, m_u$ in intermediate states: no top + CKM suppressed

$\Rightarrow$ Short distance dominated.
\( N_f = 2 + 1 \) flavours of staggered quarks (asqtad) in the sea

- 4 lattice spacings, pion masses from 177 – 555 MeV
- Valence light & strange: asqtad
- Fermilab method for the \( b \)-quark
- *mostly non-perturbative* 1-loop lattice perturbation theory

Computation of \( f_{B_q} \sqrt{\hat{B}_{B_q}} \) and \( \xi \)

- \( f_{B_q} \) taken from the PDG average to access to \( \hat{B}_{B_q} \)
- All 5 operators for \( B_d \) and \( B_s \)

\[
\begin{align*}
\sqrt{\hat{B}_{B_d}} f_{B_d} &= 227.7(9.5) \text{ MeV} \\
\sqrt{\hat{B}_{B_s}} f_{B_s} &= 274.6(8.4) \text{ MeV} \\
\xi &= 1.206(18)
\end{align*}
\]
| \langle \mathcal{O}_1^d \rangle | \langle \mathcal{O}_2^d \rangle | \langle \mathcal{O}_3^d \rangle | \langle \mathcal{O}_4^d \rangle | \langle \mathcal{O}_5^d \rangle | \langle \mathcal{O}_1^s \rangle | \langle \mathcal{O}_2^s \rangle | \langle \mathcal{O}_3^s \rangle | \langle \mathcal{O}_4^s \rangle | \langle \mathcal{O}_5^s \rangle | \xi | \rho_n | \omega \nu | \mu | \tau | \kappa | \nu | \tau | 
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| statistics | inputs | \kappa | tuning | matching | chiral | LQ disc | HQ disc | fit | total |
| 4.2 | 0.4 | 2.1 | 3.2 | 2.3 | 0.6 | 4.6 | 7.7 |
| 4.6 | 0.3 | 1.1 | 3.7 | 2.6 | 0.6 | 4.6 | 8.0 |
| 8.7 | 0.2 | 2.1 | 12.6 | 4.8 | 1.2 | 9.9 | 19.0 |
| 3.7 | 0.4 | 1.7 | 2.2 | 1.9 | 0.5 | 3.9 | 6.4 |
| 4.7 | 0.5 | 2.5 | 4.7 | 2.7 | 0.8 | 4.9 | 9.1 |
| 2.9 | 0.4 | 1.5 | 2.1 | 1.6 | 0.4 | 3.2 | 5.4 |
| 3.1 | 0.3 | 0.8 | 2.5 | 1.6 | 0.4 | 3.1 | 5.5 |
| 5.9 | 0.3 | 1.4 | 8.6 | 3.0 | 0.7 | 6.9 | 13.0 |
| 2.7 | 0.4 | 1.2 | 1.6 | 1.3 | 0.3 | 2.9 | 4.8 |
| 3.4 | 0.4 | 1.8 | 3.4 | 1.9 | 0.5 | 3.6 | 6.7 |
| 0.8 | 0.4 | 0.3 | 0.5 | 0.4 | 0.1 | 0.7 | 1.4 |

Uncertainty dominated by chiral-continuum limit fit, in particular

- statistical
- heavy quark discretisation errors
- matching
• $N_f = 2 + 1 + 1$ flavours of staggered quarks (HISQ) in sea light quarks using HISQ in the valence
• 3 lattice spacings, 2 physical pion mass ensembles
• improved nonrelativistic QCD action for the $b$
• all 5 operators for $B_d$ and $B_s$
• blinded analysis
• Computation of the $\hat{B}_{B_q}^{(i)}$
• $\xi$ and $f_{B_q}\sqrt{\hat{B}_{B_q}}$ accessed by using decay constants taken from a different computation

\[
\hat{B}_{B_d}^{(1)} = 1.222(61) \\
\hat{B}_{B_s}^{(1)} = 1.232(53) \\
\hat{B}_{B_s}^{(1)}/\hat{B}_{B_d}^{(1)} = 1.008(25)
\]
Uncertainty dominated by matching terms $\alpha_s^2$ and $\alpha_s \Lambda_{QCD}/m_b$. 

<table>
<thead>
<tr>
<th></th>
<th>$B_{B_s}^{(1)}$</th>
<th>$B_{B_s}^{(2)}$</th>
<th>$B_{B_s}^{(3)}$</th>
<th>$B_{B_s}^{(4)}$</th>
<th>$B_{B_s}^{(5)}$</th>
<th>$B_{B_s}^{(1)}/B_{B_d}^{(1)}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>lattice</td>
<td>1.4</td>
<td>1.4</td>
<td>1.5</td>
<td>1.6</td>
<td>1.5</td>
<td>1.5</td>
</tr>
<tr>
<td>$\eta_q^q$</td>
<td>0.0</td>
<td>2.3</td>
<td>2.3</td>
<td>2.1</td>
<td>1.2</td>
<td>0.0</td>
</tr>
<tr>
<td>$\alpha_s^2$ terms</td>
<td>2.1</td>
<td>2.9</td>
<td>5.2</td>
<td>1.9</td>
<td>1.5</td>
<td>0.1</td>
</tr>
<tr>
<td>$\alpha_s \Lambda_{QCD}/m_b$ terms</td>
<td>2.9</td>
<td>2.8</td>
<td>2.9</td>
<td>2.8</td>
<td>2.7</td>
<td>0.0</td>
</tr>
<tr>
<td>$(a\Lambda_{QCD})^{2n}$ terms</td>
<td>1.8</td>
<td>1.9</td>
<td>2.3</td>
<td>1.5</td>
<td>1.8</td>
<td>0.1</td>
</tr>
<tr>
<td>$m_l$ extrapolation</td>
<td>0.4</td>
<td>0.4</td>
<td>0.7</td>
<td>0.5</td>
<td>0.4</td>
<td>1.9</td>
</tr>
<tr>
<td>Total</td>
<td>4.3</td>
<td>5.3</td>
<td>7.0</td>
<td>4.6</td>
<td>4.1</td>
<td>2.5</td>
</tr>
</tbody>
</table>

Status and Outlook for $B$-mixing parameters on the lattice
**Light and strange**
- Unitary light quark mass
- Physical strange quark mass
- DWF parameters same between sea and valence

**Heavy: charm and b(eyond)**
- Möbius DWF
- $M_5 = 1.0$, $L_s = 12$
- Stout smeared (3 hits, $\rho = 0.1$)
- Many source-sink separations
- Range of heavy quark masses $m_c \lesssim m_h \lesssim m_b/2$

**Ensembles**
- $N_f = 2 + 1$ flavours, 3 lattice spacings, 2 physical pion mass ensembles

$\Rightarrow$ All Domain Wall Fermion mixed action set-up
• Computation of $SU(3)$ breaking ratios $f_{D_s}/f_D$, $f_{B_s}/f_B$, $B_{B_s}/B_{B_d}$ and $\xi$
• Renormalisation constants cancel
• Benign behaviour with $1/m_H$
• Simultaneous fit to $m_\pi^2$, $a^2$, $1/m_H$

\[ \begin{align*}
\xi_{\text{phys}} & (m_{B_s}, a = 0) \\
(m_\pi^2, m_{B_s}) & \text{ fit}
\end{align*} \]
## RBC/UKQCD: error budget [%]

### Table 1: Error Budget

<table>
<thead>
<tr>
<th>Source</th>
<th>$f_{D_s}/f_D$</th>
<th>$f_{B_s}/f_B$</th>
<th>$\xi$</th>
<th>$B_{B_s}/B_{B_d}$</th>
</tr>
</thead>
<tbody>
<tr>
<td><strong>central</strong></td>
<td>1.1740</td>
<td>1.1949</td>
<td>1.1939</td>
<td>0.9984</td>
</tr>
<tr>
<td>stat</td>
<td>0.43%</td>
<td>0.50%</td>
<td>0.56%</td>
<td>0.45%</td>
</tr>
<tr>
<td>fit chiral-CL</td>
<td>+0.31%</td>
<td>+0.34%</td>
<td>+0.38%</td>
<td>+0.42%</td>
</tr>
<tr>
<td>fit heavy mass</td>
<td>-0.32%</td>
<td>-0.54%</td>
<td>-0.45%</td>
<td>-0.01%</td>
</tr>
<tr>
<td>H.O. heavy</td>
<td>0.00%</td>
<td>0.47%</td>
<td>0.35%</td>
<td>0.21%</td>
</tr>
<tr>
<td>H.O. disc.</td>
<td>0.01%</td>
<td>0.01%</td>
<td>0.12%</td>
<td>0.17%</td>
</tr>
<tr>
<td>$m_u \neq m_d$</td>
<td>0.08%</td>
<td>0.07%</td>
<td>0.08%</td>
<td>0.01%</td>
</tr>
<tr>
<td>finite size</td>
<td>0.18%</td>
<td>0.18%</td>
<td>0.18%</td>
<td>0.18%</td>
</tr>
<tr>
<td>total systematic</td>
<td>+0.38%</td>
<td>+0.61%</td>
<td>+0.56%</td>
<td>+0.66%</td>
</tr>
<tr>
<td>total sys+stat</td>
<td>+0.58%</td>
<td>+0.79%</td>
<td>+0.80%</td>
<td>+0.80%</td>
</tr>
</tbody>
</table>

### Notes:
- Uncertainty dominated by:
  - chiral-continuum fit
  - heavy quark extrapolation
  - estimates of higher order $1/m_H$ terms

### Results:
- $f_{D_s}/f_D = 1.1740(51)^{+68}_{-68}$
- $f_{B_s}/f_B = 1.1949(60)^{+95}_{-175}$
- $B_{B_s}/B_{B_d} = 0.9984(45)^{+80}_{-63}$
- $\xi = 1.1939(67)^{+95}_{-177}$
Global fits all correlated with satisfying $p$-values.

- sys error: includes chiral-CL (left), heavy mass (right), H.O. terms, $m_u \neq m_d$ and FV.

![Graphs showing chiral-continuum and heavy mass stability](image)
Lattice strategy to simulate meson mixing

\[ C_{3}^{ab,O}(t; \Delta T) \sim \sum_{i} \sum_{j} \frac{Z_{a}^{i}Z_{b}^{j}}{4E(i)E(j)} \langle P(j) | O | \bar{P}(i) \rangle e^{-E(j)t}e^{-E(i)(\Delta T-t)} \]

**\( \Delta T \)** separation between source and sink

**\( t \)** position of operator \( O \) between source and sink
From correlation functions to matrix elements

\[ C_{3}^{ab,O}(t; \Delta T) \sim \sum_{i} \sum_{j} \frac{Z_{a}^{i}Z_{b}^{j}}{4E(i)E(j)} \langle P(j)\left| O \right| \bar{P}(i) \rangle e^{-E(j)t}e^{-E(i)(\Delta T-t)} \]

- Simultaneous fits of 2-point functions and 3-point functions for multiple \( \Delta T \)'s to extract \( \langle P_{gr}\left| O_{(i)} \right| \bar{P}_{gr} \rangle \) with highest possible confidence and precision.
- Two separate independent analyses.
- Finalising investigation of stability of fits.

expl: heavy-light \( O_{2} \) for \( a^{-1} \sim 3.5 \text{ GeV} \) →
We want

\[ \text{Amplitude} = C^{\overline{\text{MS}}} (\mu) \langle O \rangle^{\overline{\text{MS}}} (\mu) \]

- Wilson coefficients (typically) computed in $\overline{\text{MS}}$ at some scale $\mu$.
- Operators $\langle O \rangle^{\text{bare}} (a)$ computed with lattice regulator $a^{-1}$.
- Renormalise $\langle O \rangle^{\text{bare}} (a)$ at scale $\mu$ in regularisation independent (RI) scheme, by computing a non-pert. renormalisation factor $Z^{RI}(\mu, a)$.

\[ \langle O \rangle^{RI} (\mu) = \lim_{a^2 \to 0} Z^{RI}(\mu, a) \langle O \rangle^{\text{bare}} (a) \]

- Match to preferred scheme (e.g. $\overline{\text{MS}}$) using P.T. at $\mu$: $R^{\overline{\text{MS}}\leftarrow RI}(\mu)$
- If the operators mix: $C$ and $\langle O \rangle$ become vectors, $R$ and $Z$ matrices.

\[ \text{Amplitude} = C_i^{\overline{\text{MS}}} (\mu) R_{ij}^{\overline{\text{MS}}\leftarrow RI}(\mu) \lim_{a \to 0} Z_{jk}^{RI}(\mu, a) \langle O_k \rangle^{\text{bare}} (a) \]

- Chirally symmetric fermions $\Rightarrow$ $R$ and $Z$ are block diagonal.
NPR for Neutral Meson Mixing operators

- Analogous to the neutral kaon mixing case \([1708.03552, 1812.04981]\)
- 5 operators \(O_1 = O_{VV+AA}, \ O_2 = O_{VV-AA}, \ O_3 = O_{SS-PP}, \ O_4 = O_{SS+PP}, \ O_5 = O_{TT}\)
- Block diagonal renormalisation matrix (up to \(O(\alpha m_{\text{res}})\))

\[
Z_{ij}^{RI}(\mu, a) = \begin{pmatrix}
Z_{11} & 0 & 0 & 0 & 0 \\
0 & Z_{22} & Z_{23} & 0 & 0 \\
0 & Z_{32} & Z_{33} & 0 & 0 \\
0 & 0 & 0 & Z_{44} & Z_{45} \\
0 & 0 & 0 & Z_{54} & Z_{55}
\end{pmatrix}
\]

- Generalise idea from 1701.02644 for fully non-perturbative mixed action renormalisation to four quark operators

\[
\frac{\mathcal{P}[\Lambda_A](ll)\mathcal{P}[\Lambda_A](hh)}{(\mathcal{P}[\Lambda_A](lh))^2} = \frac{(Z_A^{lh})^2}{Z_A^{ll}Z_A^{hh}}
\]

- Data production of the mixed action NPR in progress.