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# BOSE CONDENSATION IN DENSE QCD

## From the dilute Bose gas to speedy Goldstone bosons

XQCD 2023 Coimbra, Portugal

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July 26, 2023

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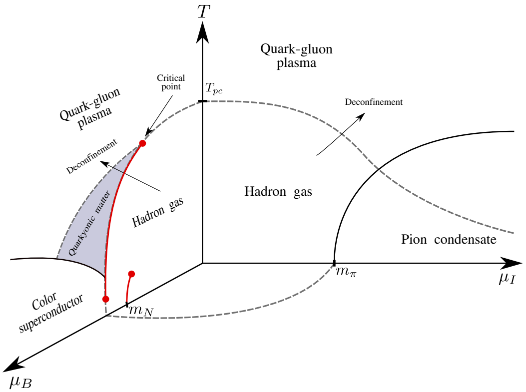
References: Eur. Phys. J. C 80 (2020), Eur. Phys. J.C 81 (2021), 2206.04291, 2306.14472

# Introduction

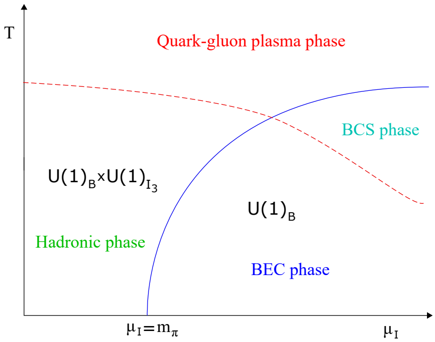
- QCD phase diagram




- Strong magnetic field $B$ - no sign problem
- Finite isospin chemical potential $\mu_I$ - no sign problem (see talks by Bandyopadhyay and Mannarelli)

# Introduction

- QCD phase diagram




- This talk: Finite $\mu_I$ and finite $\mu_S$ at $T = 0$

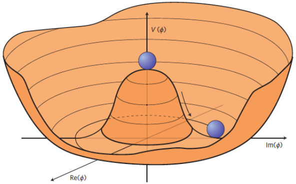
# From linear to nonlinear sigma model

- Linear sigma model med $SU(2)_L \times SU(2)_R \sim O(4)$-symmetry

$$\mathcal{L} = \frac{1}{2}\partial_\mu \sigma \partial^\mu \sigma + \frac{1}{2}\partial_\mu \pi \partial^\mu \pi - \frac{1}{2}m^2(\sigma^2 + \pi^2) + H\sigma - \frac{\lambda}{4}(\sigma^2 + \pi^2)^2$$

- Polar coordinates

$$(v + S)\Sigma = (v + S)e^{i\tau_a \phi_a / v}$$




- Lagrangian

$$\mathcal{L} = \frac{1}{4}v^2\left(1 + \frac{S}{v}\right)^2 \langle \partial_\mu \Sigma^\dagger \partial^\mu \Sigma \rangle + \frac{1}{2}\partial_\mu S \partial^\mu S + H(v + S)\Sigma \to \frac{1}{4}f^2 \langle \partial_\mu \Sigma^\dagger \partial^\mu \Sigma \rangle + \frac{1}{4}f^2 m^2 \langle \Sigma^\dagger + \Sigma \rangle$$

# Chiral perturbation theory

- Chiral perturbation theory is a low-energy theory for QCD based on its global symmetries and degrees of freedom [2]
- $SU(2)_L \times SU(2)_R$ for two flavors (pions). $SU(3)_L \times SU(3)_R$ for three flavors (pions, kaons, and eta)

$$
\begin{aligned}
\mathcal{L}_2 &= \frac{1}{4} f^2 \langle \nabla^\mu \Sigma^\dagger \nabla_\mu \Sigma \rangle + \frac{1}{4} f^2 m^2 \langle \Sigma + \Sigma^\dagger \rangle \,, \\
\Sigma &= e^{i\tau_a \phi_a / f} \,, \qquad \nabla_\mu \Sigma \equiv \partial_\mu \Sigma - i \left[ v_\mu, \Sigma \right] \,, \\
v_\mu &= \delta_{\mu,0} \text{diag}(\mu_u, \mu_d) = \delta_{\mu,0} (\tfrac{1}{3}\mu_B + \tfrac{1}{2}\mu_I, \tfrac{1}{3}\mu_B - \tfrac{1}{2}\mu_I) \,, \\
f &\sim f_\pi \qquad\qquad m \sim m_\pi
\end{aligned}
$$

- Results independent of $\mu_B$

[2] Gasser and Leutwyler ’84 and ’85

# Rotated ground state and pion condensation

- Ansatz for ground state [3]

$$\Sigma_\alpha = \cos\alpha\mathbb{1} + i\tau_1\sin\alpha = e^{i\tau_1\alpha}$$

- Static Hamiltonian

$$\begin{aligned}\mathcal{H}_{\rm static} &= -\frac{1}{4}f^2\langle[v_\mu,\Sigma^\dagger][v^\mu,\Sigma]\rangle - \frac{1}{4}f^2m^2\langle\Sigma^\dagger+\Sigma\rangle \\ &= -f^2m^2\cos\alpha - \frac{1}{2}f^2\mu_I^2\sin^2\alpha\ .\end{aligned}$$

- Competition between the two terms

$$\cos\alpha_0 = \frac{m^2}{\mu_I^2}\ , \quad \mu_I^2 \geq m^2\ ,$$
$$\alpha_0 = 0\ , \quad \mu_I^2 \leq m^2\ .$$

[3] Son and Stephanov 01

# Second-order transition and Silver-Blaze property

- Landau-Ginzburg functional

$$\Omega_0(\alpha) \quad = \quad -f^2m^2\cos\alpha - \frac{1}{2}f^2\mu_I^2\sin^2\alpha = \frac{1}{2}f^2[\mu_I^2 - m^2]\alpha^2 + \frac{1}{24}f^2[4\mu_I^2 - m^2]\alpha^4 + \cdots \;.$$

- Transition to a Bose-condensed phase is second order at $\mu_I = m$
- Thermodynamic functions are independent of $\mu_I$ up to the transition point

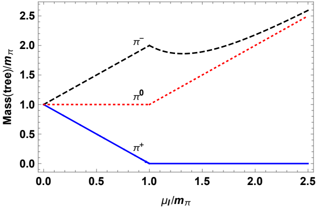
$$\begin{aligned} p &= -\Omega_0\Big|_{\alpha=\alpha_0} = \frac{1}{2}f^2\mu_I^2\left[1-\frac{m^2}{\mu_I^2}\right]^2 \;, \\ n_I &= -\frac{\partial\Omega_0}{\partial\alpha}\Big|_{\alpha=\alpha_0} = f^2\mu_I\left[1-\frac{m^4}{\mu_I^4}\right] \;, \\ \varepsilon &= -p + n_I\mu_I = -p + \sqrt{p(p+2f^2m^2)} \;. \end{aligned}$$

# Quadratic Lagrangian and spectrum

- Quadratic Lagrangian

$$
\begin{aligned}
\mathcal{L}_2^{\text{quadratic}} &= \frac{1}{2}\partial^\mu\phi_a\partial_\mu\phi_a + \frac{1}{2}m_{12}(\phi_1\partial_0\phi_2 - \phi_2\partial_0\phi_1) - \frac{1}{2}m_a^2\phi_a^2\ , \\
m_1^2 &= m^2\cos\alpha - \mu_I^2\cos^2\alpha\ , \\
m_2^2 &= m^2\cos\alpha - \mu_I^2\cos 2\alpha\ , \\
m_3^2 &= m^2\cos\alpha + \mu_I^2\sin^2\alpha\ , \\
m_{12} &= 2\mu_I\cos\alpha\ .
\end{aligned}
$$

- Mass spectrum



# NLO Lagrangian and NLO pressure

- NLO Lagrangian

$$
\begin{aligned}
\mathcal{L}_4 &= \frac{l_1}{4}\langle\nabla_\mu\Sigma^\dagger\nabla^\mu\Sigma\rangle^2+\frac{l_2}{4}\langle\nabla_\mu\Sigma^\dagger\nabla_\nu\Sigma\rangle\langle\nabla^\mu\Sigma^\dagger\nabla^\nu\Sigma\rangle+\frac{(l_3+l_4)m^4}{16}\langle\Sigma+\Sigma^\dagger\rangle^2 \\
&+\frac{l_4m^2}{8}\langle\nabla_\mu\Sigma^\dagger\nabla^\mu\Sigma\rangle\langle\Sigma+\Sigma^\dagger\rangle+h_1\langle m^4\rangle\ ,
\end{aligned}
$$

- NLO contributions to thermodynamic potential

$$
\begin{aligned}
\Omega_1^{\rm static} &= (l_1+l_2)\mu_I^4\sin^4\alpha+l_4m^2\mu_I^2\cos\alpha\sin^2\alpha+(l_3+l_4)m^4\cos^2\alpha\ , \\
\Omega_1^{\rm one-loop} &= \frac{1}{2}\int_P\log\left[(P^2+m_1^2)(P^2+m_2^2)+p_0^2m_{12}^2\right]+\frac{1}{2}\int_P\log\left[P^2+m_3^2\right]\ .
\end{aligned}
$$

# NLO pressure

- Expansion of $\Omega_{0+1}(\alpha)$ shows second order transition at physical pion mass $m_\pi$ and not $m$.
- Silver-Blaze property
- Pressure

$$
\begin{aligned}
p &= -\Omega_1(\alpha_0) - \Omega_0(\alpha_0) \\
&= \frac{1}{2} f^2 \mu_I^2 \left[1 - \frac{m^2}{\mu_I^2}\right]^2 - \frac{m^4}{3(4\pi)^2} \left[\bar{l}_1 + 2\bar{l}_2 - 3\bar{l}_4\right] \\
&\quad - \frac{5m^8}{24(4\pi)^2 \mu_I^4} \left[1 - \frac{4}{5}\bar{l}_1 - \frac{8}{5}\bar{l}_2 + \frac{6}{5}\bar{l}_3 - \frac{6}{5}\log\frac{M^2\mu_I^2}{\mu_I^4 - m^4}\right] + \frac{\mu_I^4}{4(4\pi)^2}\left[1 + \frac{2}{3}\bar{l}_1 + \frac{4}{3}\bar{l}_2 + \log\frac{M^4}{\mu_I^4 - m^4}\right] \\
&\quad - \frac{5m^{12}}{12(4\pi)^2(\mu_I^4 - m^4)\mu_I^4}\, {}_3F_2\left[\begin{matrix} 1, & 1, & \frac{7}{2} \\ & 4, \ 5 & \end{matrix} \middle| -\frac{m_{12}^2}{m_2^2}\right] .
\end{aligned}
$$

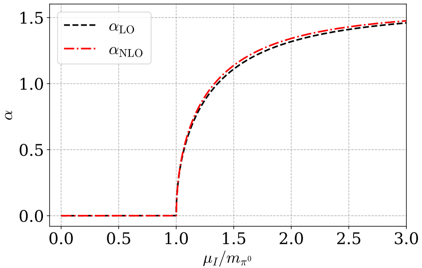
# Parameter fixing

- Physical masses

$$m_\pi^2 = m^2 \left[1 - \frac{m^2}{2(4\pi)^2 f^2}\bar{l}_3\right] \ , \quad f_\pi^2 = f^2 \left[1 + \frac{2m^2}{(4\pi)^2 f^2}\bar{l}_4\right] \ .$$

- $\bar{l}_i$ measured in experiment
- Bare and physical quantitities

$$m_\pi = 131\text{MeV} \ , \qquad f_\pi = 90.5\text{MeV} \ ,$$
$$m = 132.5\text{MeV} \ , \qquad f = 84.9\text{MeV} \ ,$$

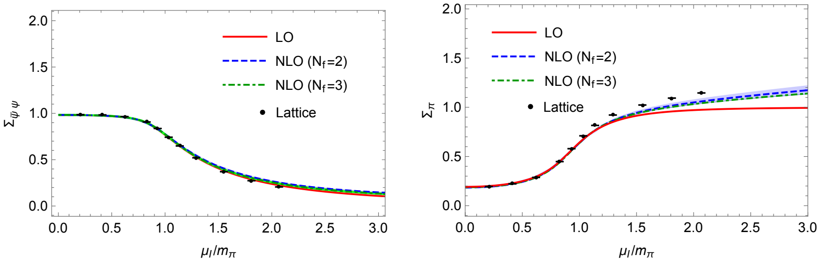


# Quark and pion condensates

- Calculate $\Omega$ in presence of pionic source (in addition to quark mass $m_q$).

$$\langle\bar{\psi}\psi\rangle = \frac{\partial\Omega}{\partial m_q} = \langle\bar{\psi}\psi\rangle_0 \cos\alpha + \ldots$$

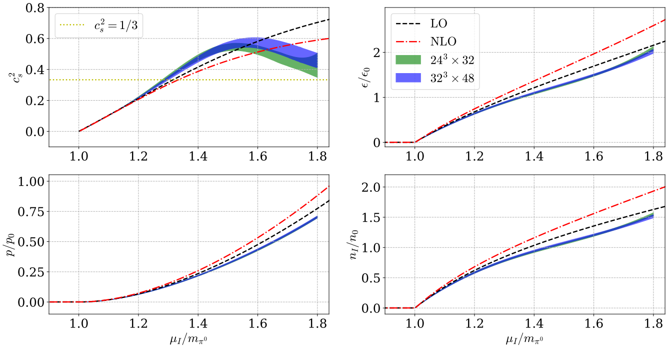
$$\langle\pi^+\rangle = \frac{\partial\Omega}{\partial j} = \langle\bar{\psi}\psi\rangle_0 \sin\alpha + \ldots$$




- Lattice results by Brandt et al ’18-’22 (here with finite source $j$)

# LO and NLO thermodynamics

- Speed of sound, energy density, pressure, and isospin density [4]




- BEC-BCS crossover(?) not within $\chi$PT [5]

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[4] Lattice data Brandt et al ’18-’22

[5] Kojo ’23

# Adding electromagnetic interactions

- Extra terms since only $U(1)_Q$ symmetry

$$\mathcal{L} \quad = \quad -\frac{1}{4}F_{\mu\nu}F^{\mu\nu} + \frac{1}{4}f^2\langle\nabla_\mu\Sigma\nabla^\mu\Sigma\rangle + \frac{1}{4}f^2\langle\Sigma^\dagger\chi + \chi^\dagger\Sigma\rangle + C\langle Q\Sigma Q\Sigma^\dagger\rangle + \mathcal{L}_{\text{gf}} + \mathcal{L}_{\text{ghost}}$$

- Meson masses

$$m^2_{\pi^\pm} \quad = \quad m^2_{\pi^0} - \frac{2Ce^2}{f^2}\ ,$$
$$m^2_{K^\pm} \quad = \quad m^2_{K^0} - \frac{2Ce^2}{f^2}\ .$$

- Higgs phase - massive photons and no propagating photon
- $C$ determined by mass difference between charged and neutral mesons

# Phase diagram three flavors

- Chemical potentials

$$\mu_B = \frac{3}{2}(\mu_u + \mu_d)\ , \mu_I = \frac{1}{2}(\mu_u - \mu_d)\ , \mu_S = (\mu_u + \mu_d - 2\mu_s)$$

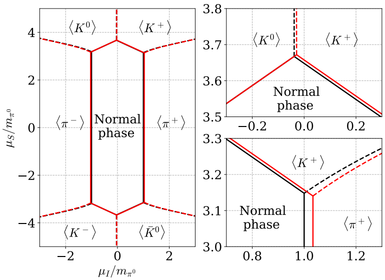
$$v_0 = \frac{1}{3}(\mu_B - \mu_S)\mathbb{1} + \frac{1}{2}\mu_I\lambda_3 + \frac{1}{\sqrt{3}}\mu_S\lambda_8 = \tfrac{1}{3}(\mu_B - \mu_S)\mathbb{1} + \frac{1}{2}\mu_{K^\pm}\lambda_Q + \frac{1}{2}\mu_{K^0}\lambda_K\ ,$$

$$\mu_{K^\pm} = \mu_I + \frac{1}{\sqrt{3}}\mu_S\ , \mu_{K^0} = -\mu_I + \frac{1}{\sqrt{3}}\mu_S\ ,$$

- Ansaẗze for ground state

$$\begin{aligned} \Sigma_\alpha &= e^{i\lambda_3\alpha} \\ \Sigma_\alpha &= e^{i\lambda_5\alpha} \\ \Sigma_\alpha &= e^{i\lambda_7\alpha} \end{aligned}$$

# Phase diagram three flavors [6]




- Thermodynamic potential independent of $\mu_i$ in the normal phase (Silver-Blaze property) etc
- Second-order phase transition from the vacuum for $\mu_I = m_{\pi^\pm}$, $\mu_{K^\pm} = \frac{1}{2}\mu_I + \mu_S = m_{K^\pm}$ etc
- First-order transitions between the condensed phases

[6] Kogut and Toublan '01

# Dilute Bose gas

- Lagrangian

$$\mathcal{L} \quad = \quad \psi^\dagger(i\partial_0 + \mu_{\rm NR})\psi - \frac{1}{2m}\nabla\psi^\dagger\cdot\nabla\psi - \frac{g}{4}(\psi^\dagger\psi)^2 - \frac{g_3}{36}(\psi^\dagger\psi)^3 + \cdots \ ,$$

- Phase symmetry $\psi \to e^{i\phi}\psi$ and Galilean symmetry
- Non-renormalizable Lagrangian
- $g$ is related to the $s$-wave scattering length and $g_3$ encodes $3 \to 3$ scattering.
- Perturbative expansion in gas parameter $\sqrt{na^3}$ [7]

$$\varepsilon(n) \quad = \quad \frac{2\pi a n^2}{m}\left[1 + \frac{128}{15\sqrt{\pi}}\sqrt{na^3} + \left(\frac{32\pi - 24\sqrt{3}}{3}\ln(na^3) + C\right) na^3\right] \ .$$

[7] Bogoliubov '47, Lee, Huang, and Yang '57, Wu, Hugenholz and Pines, Sawada '59,. Braaten and Nieto '97.

# Nonrelativistic limit

- Nonrelativistic chemical potential $\mu_I = m_\pi + \mu_{\rm NR}$ and expansion of pressure etc

$$
\begin{aligned}
p &= \frac{m_\pi}{8\pi a}\mu_{\rm NR}^2\left[1-\frac{32}{15\pi}\sqrt{4m_\pi\mu_{\rm NR}a^2}\right] \;, \\
n_I &= \frac{m_\pi}{4\pi a}\mu_{\rm NR}\left[1-\frac{8}{3\pi}\sqrt{4m_\pi\mu_{\rm NR}a^2}\right] \;, \\
\mu_{\rm NR} &= \frac{4\pi a n}{m_\pi}\left[1+\frac{32}{3\sqrt{\pi}}\sqrt{n_I a^3}\right] \;.
\end{aligned}
$$

- Energy density [8]

$$
\varepsilon = -p + n_I\mu_I = m_\pi n_I + \frac{2\pi a n_I^2}{m_\pi}\left[1+\frac{128}{15\sqrt{\pi}}\sqrt{n_I a^3}\right] \;.
$$

- $a$ includes one-loop corrections to the $s$-wave scattering length from $\chi$PT

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[8] Bogoliubov '48, Lee, Huang, and Yang '57,

# Low-energy effective theory for the Goldstone boson

- Gaps and Goldstone mode at low $p$

$$
\begin{aligned}
E_-(p) &= \sqrt{\frac{\mu_I^4 - m^4}{\mu_I^2 + 3m^2}}\,p + \mathcal{O}(p^2)\ , \\
E_+(p) &= \mu_I\sqrt{1 + \frac{3m^2}{\mu_I^2}} + \mathcal{O}(p^2)\ , \\
E_3(p) &= \sqrt{\mu_I^2 + p^2}\ .
\end{aligned}
$$

- Massive modes should decouple at low energies $p \ll \mu_I$
- Construction of effective theory $\mathcal{L}_{\rm eff}$ using symmetries and matching

# Son's effective theory [9]

- Effective Lagrangian given in terms of pressure $p(\mu)$

$$\mathcal{L}_{\rm eff} = -p(\sqrt{(\partial_0\phi-\mu_I)^2+\partial_i\phi\partial^i\phi}$$

- Valid below the gap, coherence length...
- Derivative expansion

$$\begin{aligned}
\mathcal{L}_{\rm eff} &= \frac{1}{2}\partial_0\phi\partial_0\phi - \frac{1}{2}c_s^2(\nabla\phi)^2 + c_1\partial_0\phi(\partial_\mu\phi\partial^\mu\phi) + c_2(\partial_\mu\phi\partial^\mu\phi)^2 + c_3(\partial_i\phi\partial^i\phi)^2 + \ldots \\
c_s &= \sqrt{\frac{\mu_I^4-m^4}{3m^4+\mu_I^4}}\,, \\
c_1 &= \frac{2m^4}{f\mu_I}\frac{1}{3m^4+\mu_I^4}\,,
\end{aligned}$$

- Free ultrarelativistic theory in chiral limit up to renormalization effects

[9] Son '02

# Son's effective theory

- Same approach in CFL phase of QCD [10]

$$\mathcal{L}_{\text{eff}} = \frac{1}{2}\partial_0\phi\partial_0\phi - \frac{1}{2}c_s^2(\nabla\phi)^2 + c_1\partial_0\phi(\partial_\mu\phi\partial^\mu\phi) + ..$$

$$c_s = \sqrt{\frac{1}{3} - \frac{m_s^2}{9\mu_B^2}},$$

$$c_1 = \frac{\pi}{9\mu_B^2}\left(1 + \frac{m_s^2}{4\mu_B^2}\right)$$

- Damping rate goes as $O(p^5)$ for small $p$ [11]

[10] Manuel et al '05, '07, and '10

[11] Beliaev '58

# Conclusions and outlook

- Conclusions
  1. First calculations of quark and pion condensates at NLO. Good agreement with lattice
  2. First calculations including LO electromagnetic effects.
  3. Reduces to the dilute Bose gas in nonrelativistic limit
  4. Phonon damping rate $\mathcal{O}(p^5)$ for small $p$
  5. BCS phase not described within $\chi$PT
- Outlook
  1. NLO electromagnetic effects?
  2. Finite-temperature calculations for two flavors only valid for very low $T$. Three flavors?
  3. Systematic calculations of relativistic corrections. Effective Lagrangian [12]

[12] Braaten, Hammer, and Hermans '03