How baryons appear in low-energy QCD : Domain-wall Skyrmion phase in strong magnetic field

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Outline

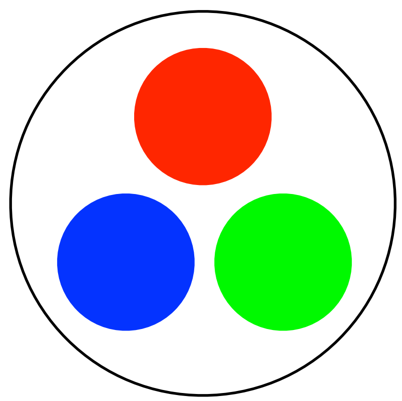
- Introduction

- Chiral soliton lattice   Son and Stephanov (2008); Brauner and Yamamoto (2017)

- Domain wall Skyrmion   Eto, KN and Nitta (2023)

Baryons and mesons

- Baryon = Particle composed of quarks



$$\simeq \quad \epsilon^{a_1 a_2 a_3} q^{f_1}_{a_1} q^{f_2}_{a_2} q^{f_3}_{a_3}$$

- Pions = Nambu-Goldstone (NG) bosons :

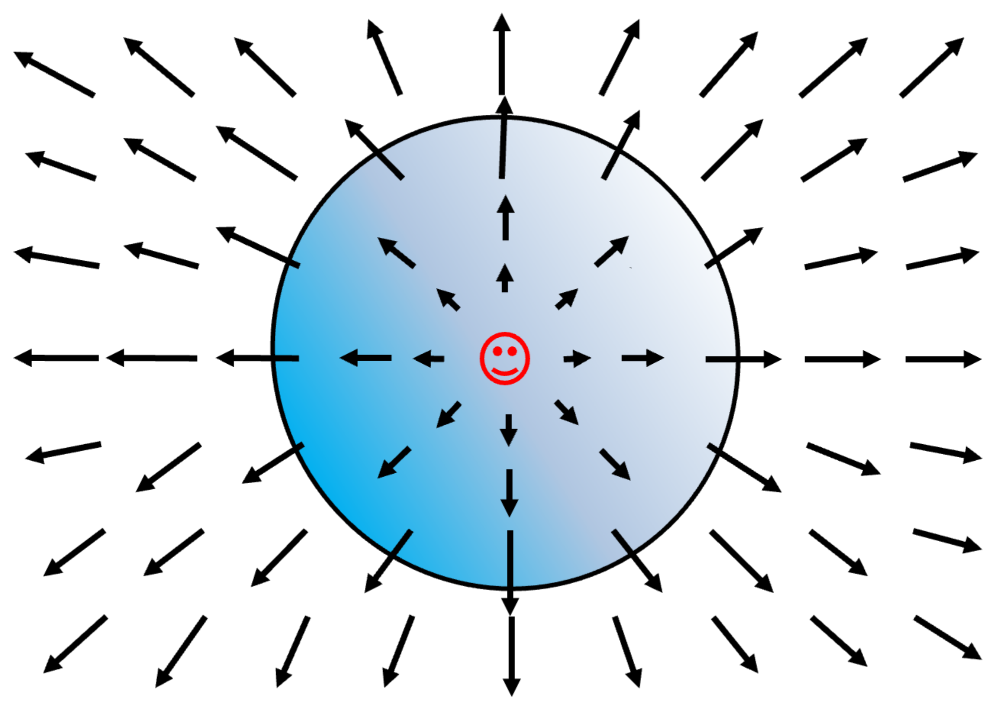
- Degrees of freedom of the low-energy dynamics :

$$\Sigma(x) = \exp\left(\frac{\mathrm{i}\pi_a \tau_a}{f_\pi}\right)$$

Skyrmion

- Can the baryons be made by pions (rather than quarks)?

- Baryon as soliton = Skyrmion Skyrme (1961)



Hedgehog

- Topological charge :

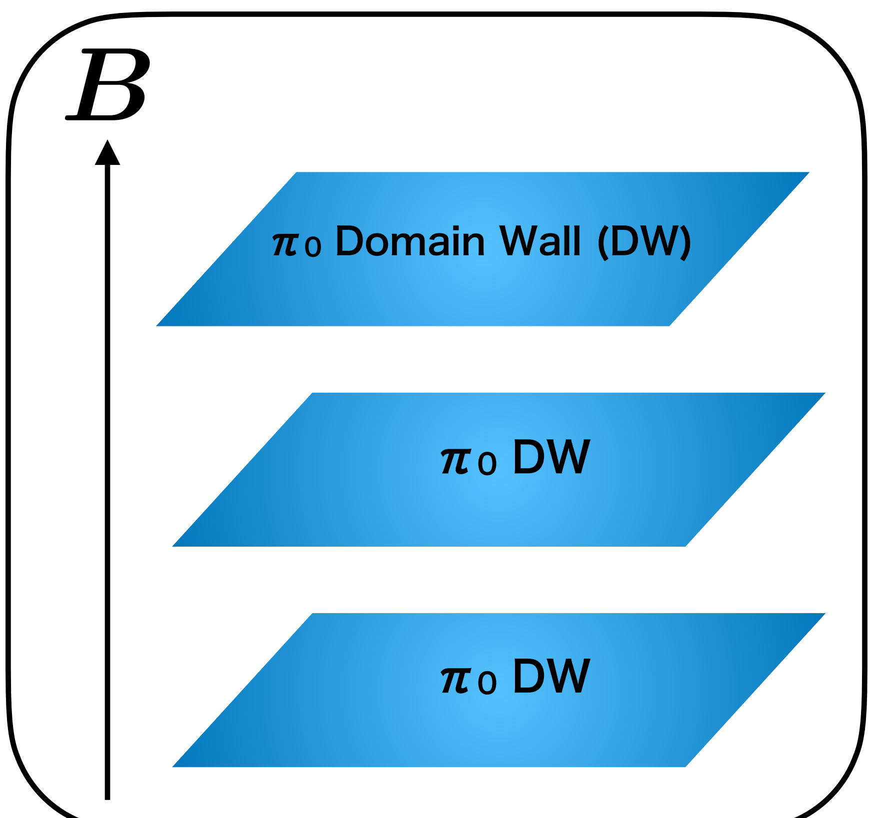
$$j^\mu_{\rm B} = \frac{\epsilon^{\mu\nu\alpha\beta}}{24\pi^2}\mathrm{tr}(\Sigma\partial_\nu\Sigma^\dagger\Sigma\partial_\alpha\Sigma^\dagger\Sigma\partial_\beta\Sigma^\dagger)$$

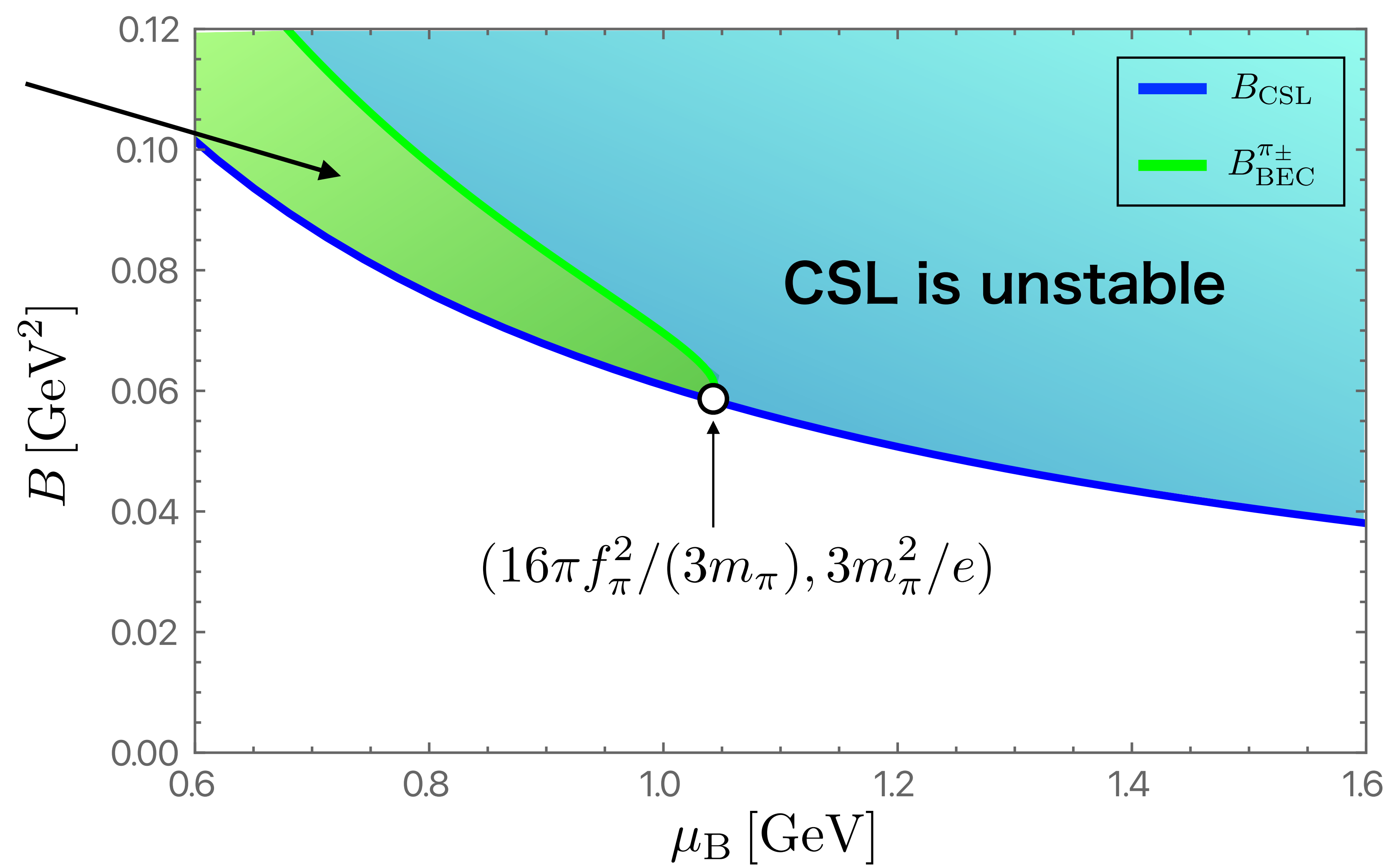
- R^3 surrounds the configuration space of the pions S^3 : $\pi_3(S^3)$

Solitonic phase in QCD

- **Solitons appear in the ground state of dense QCD at B.**
  - **Based on the analysis with the chiral perturbation theory.**

Chiral Soliton Lattice (CSL)





Son and Stephanov (2008); Brauner and Yamamoto (2017)

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ChPT w/ topological terms

- **Normal terms :** $\mathcal{L}_{\rm ChPT} = \frac{f_\pi^2}{4}{\rm tr}\left(D_\mu \Sigma D^\mu \Sigma^\dagger\right) - \frac{f_\pi^2 m_\pi^2}{4}(2 - \Sigma - \Sigma^\dagger)$

- **Baryon current in terms of mesons = $\pi_3(S^3)$ topological charge**

$$j_{\rm B}^\mu = -\frac{\epsilon^{\mu\nu\alpha\beta}}{24\pi^2}{\rm tr}\{L_\nu L_\alpha L_\beta - 3ie\partial_\nu\left[A_\alpha Q(L_\beta + R_\beta)\right]\}$$

Skyrimon charge

U(1)$_{\rm em}$ gauged part

$$L_\mu \equiv \Sigma\partial_\mu\Sigma^\dagger\,,\ R_\mu \equiv \partial_\mu\Sigma^\dagger\Sigma\,,\ Q = {\rm diag}(2/3, -1/3)$$

- "trial and error" U(1)$_{\rm em}$ gauging while preserving the baryon number

Son and Stephanov (2008); Goldstone and Wilczek (1981)

- **Effective Lagrangian :** $\mathcal{L}_{\rm B} = -A_{\rm B}^\mu j_{{\rm B}\mu}\,, \quad A_{\rm B}^\mu = (\mu_{\rm B}, \mathbf{0})$

Son and Stephanov (2008)

sine-Gordon theory with the topological term

- Ignoring charged pions : $\Sigma = \mathrm{e}^{\mathrm{i}\phi_3\tau_3}$
- Reduced Hamiltonian (B is oriented in z-direction) :

$$\mathcal{H} = \frac{f_\pi^2}{2}(\partial_z\phi_3)^2 + f_\pi^2 m_\pi^2(1-\cos\phi_3) - \frac{e\mu_\mathrm{B}}{4\pi^2}B\partial_z\phi_3$$

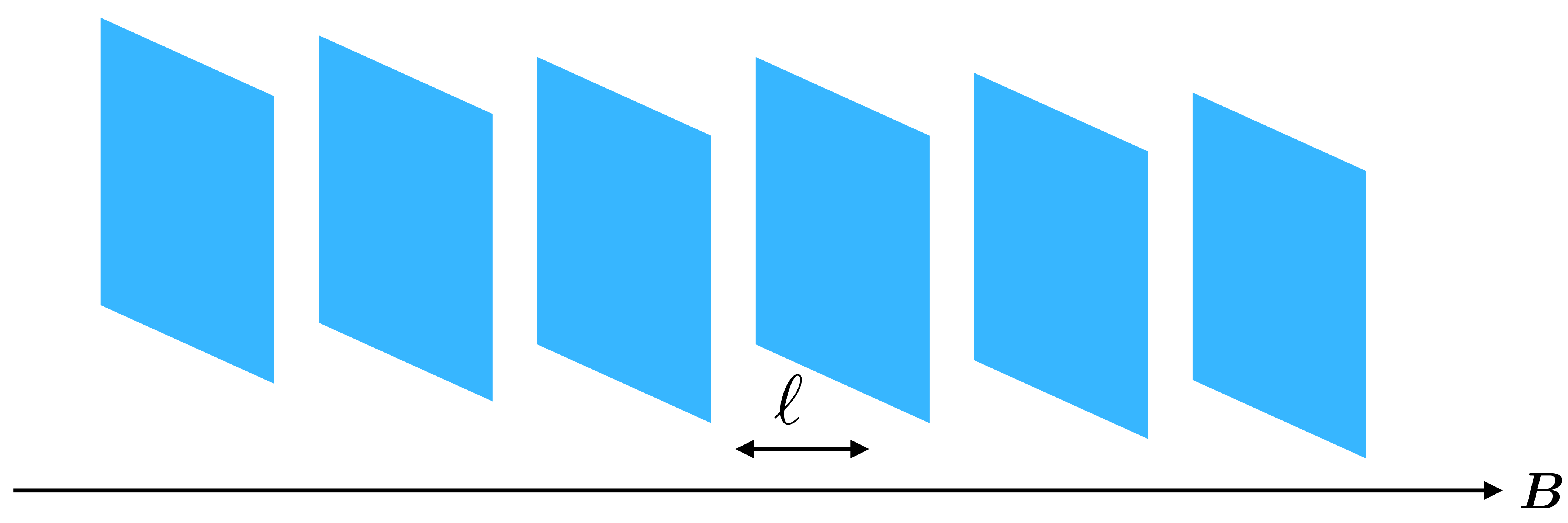
- The last term stems from the 2nd term of the skyrmion term.

$$\mathcal{L}_\mathrm{B} = -\mu_\mathrm{B}\frac{\epsilon^{0ijk}}{24\pi^2}\mathrm{tr}\{L_iL_jL_k - 3\mathrm{i}e\partial_i\left[A_jQ(L_k+R_k)\right]\}$$

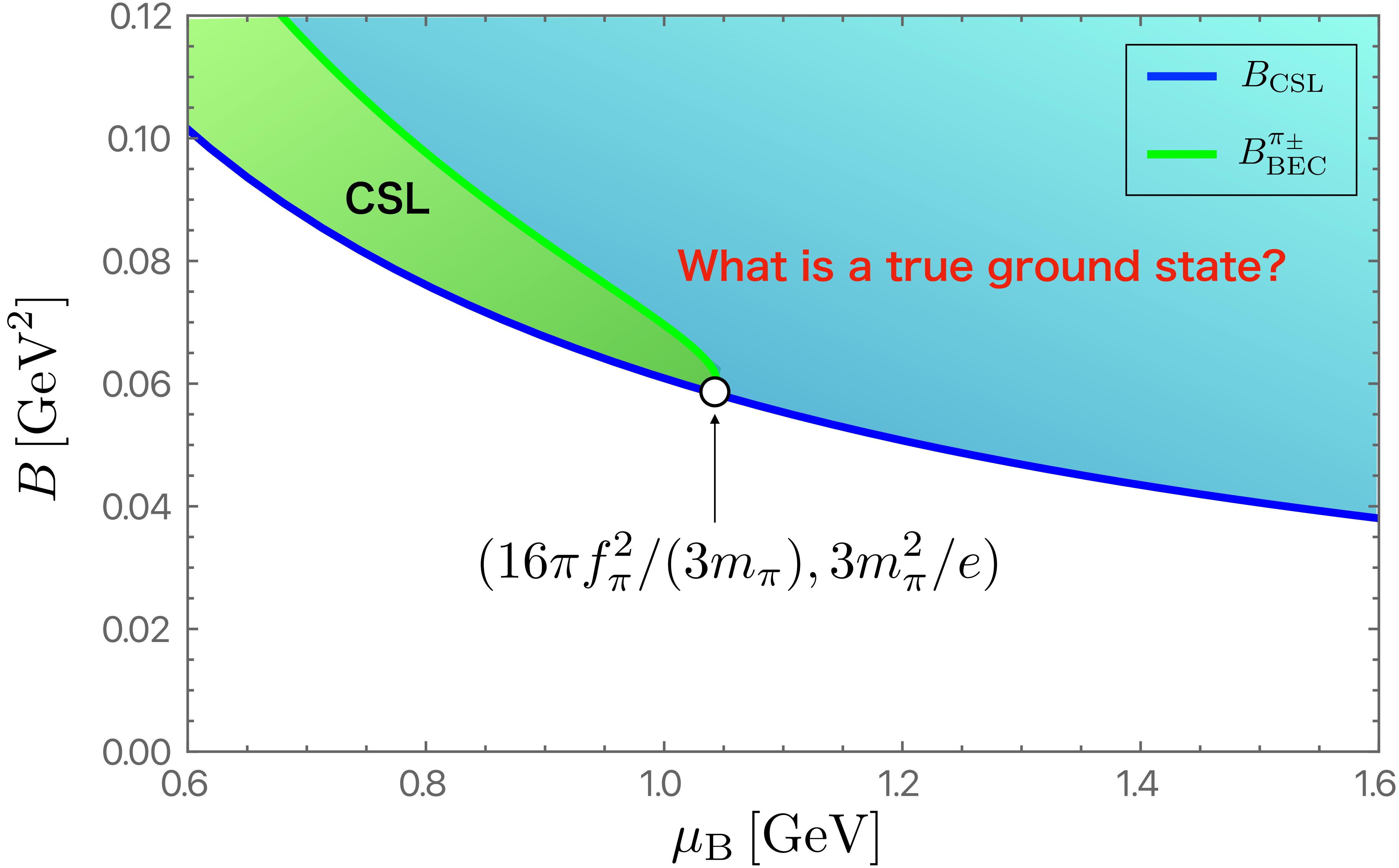
- B≠0 → Finite 1st derivative term → Favor ϕ inhomogenity
- What is a ground state at finite B? Brauner and Yamamoto (2017)

Chiral Soliton Lattice (CSL)

- **EOM :** $\partial_z^2 \phi_3 = m_\pi^2 \sin\phi_3$ $\quad \phi = 4\tan^{-1}\exp m_\pi (z - z_0)$
- **Energy :** $E = \int_{-\infty}^{\infty} \mathrm{d}z\, \mathcal{H} = 8 m_\pi^2 f_\pi - \frac{e\mu_\mathrm{B} B}{2\pi}$
- **Critical magnetic field :** $B_\mathrm{c} = \frac{16\pi f_\pi m_\pi^2}{e\mu_\mathrm{B}}$



μ_B-B phase diagram



Brauner and Yamamoto (2017)

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- Domain wall Skyrmion Eto, KN and Nitta (2023)

What we overlooked

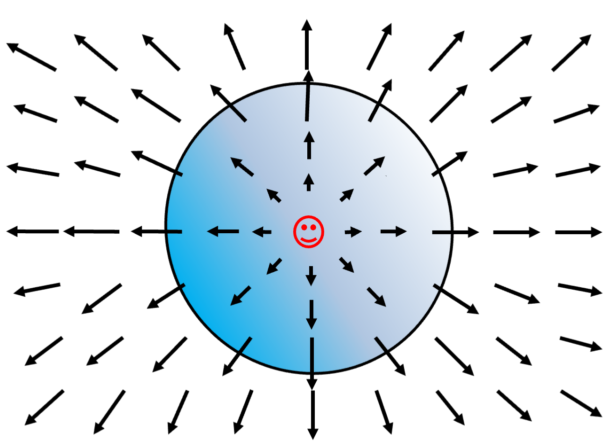
- The baryon current contains the Skyrmion charge, which is O(p²).

$$\mathcal{L}_{\rm B} = -\mu_{\rm B}\frac{\epsilon^{0ijk}}{24\pi^2}\mathrm{tr}\{\underline{L_iL_jL_k} - \underline{3\mathrm{i}e\partial_i\left[A_jQ(L_k+R_k)\right]}\}$$

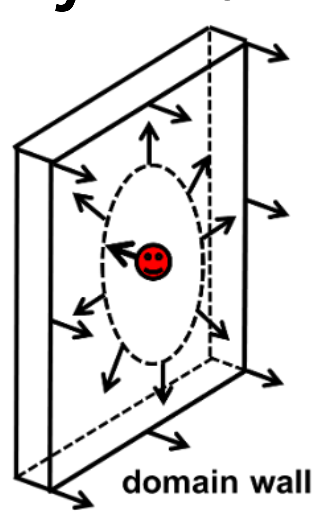
$$\mu_{\rm B}B\partial_z\phi_3 \subset \mathcal{L}_{\rm B}$$

  - In the unstable region, $\pi_\pm$ is important element.
  - The Skyrmion charge term becomes finite only when $\pi_\pm$ is considered.

Skyrmion : $\pi_3(S^3)$



Baby Skyrmion : $\pi_2(S^2)$



become stable at finite $\mu_{\rm B}$!

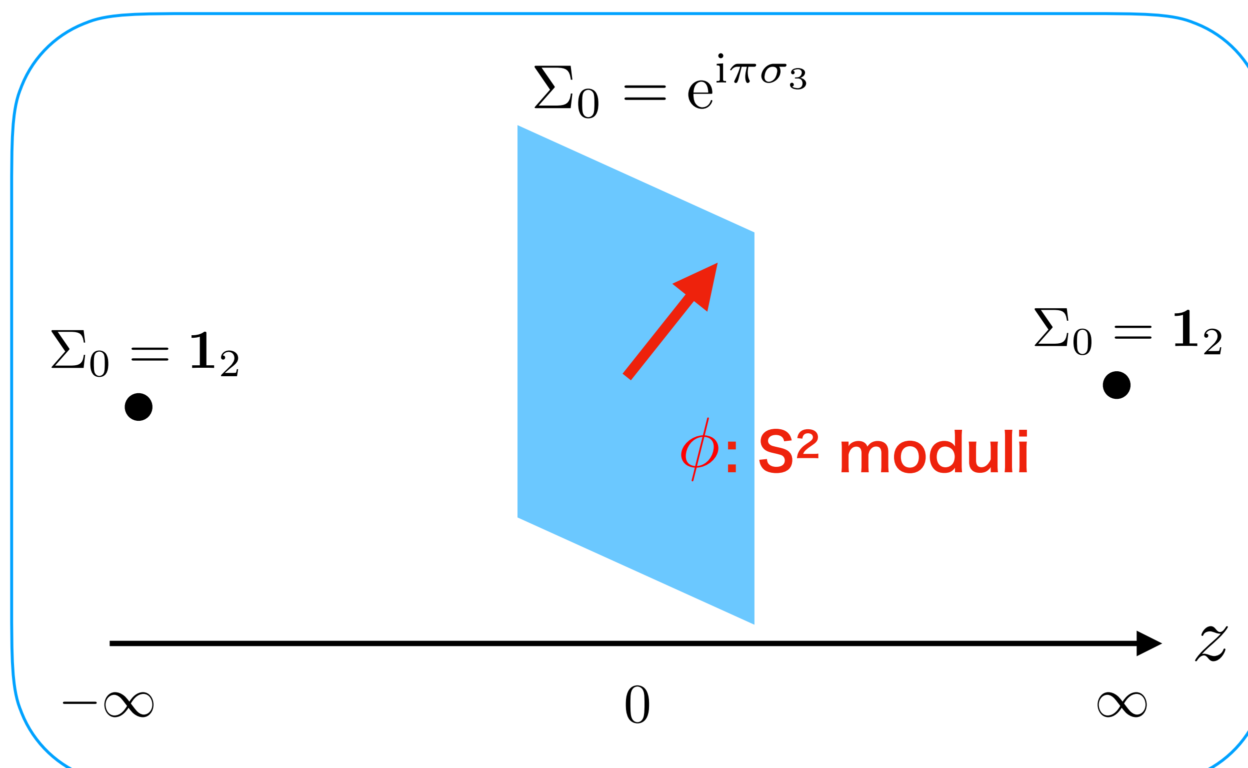
Non-abelian soliton

- The single soliton : $\Sigma_0 = \mathrm{e}^{\mathrm{i}\sigma_3\theta}\,,\ \ \theta = 4\tan^{-1}\mathrm{e}^{m_\pi z}$
- More general solution : $\Sigma = g\Sigma_0 g^\dagger = \exp(\mathrm{i}\theta g\tau_3 g^\dagger)\,,\quad g \in \mathrm{SU}(2)_\mathrm{V}$
- Σ_0 is invariant under U(1) transformation : $g = \mathrm{e}^{\mathrm{i}\tau_3\theta}$
- SSB of $\mathrm{SU(2)_V} \to \mathrm{U(1)}$ —Moduli (NG mode)→ $\mathrm{SU}(2)/\mathrm{U}(1) \cong S^2$

- The collective coordinate :

$$\phi \in \mathbb{C}^2\,, \qquad \phi^\dagger\phi = 1$$

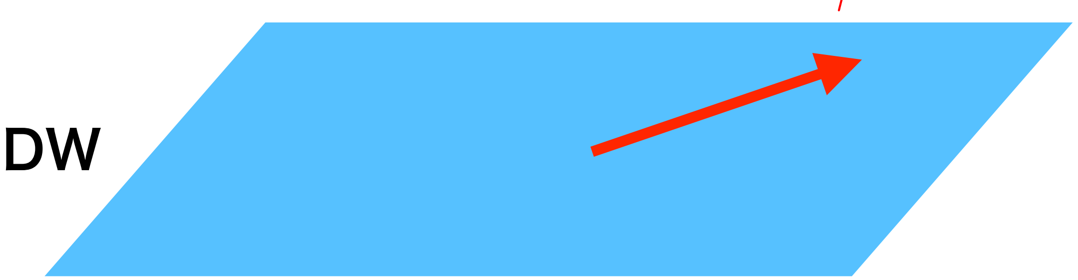
$$g\sigma_3 g^\dagger = 2\phi\phi^\dagger - 1$$



Nitta (2015); Eto and Nitta (2015)

EFT of the DW

- Construct DW world effective theory via the moduli approximation.



- Promote the moduli to a field on D=2+1 world volume

$$\phi \to \phi(x^\alpha)\,, \quad (\alpha = 0, 1, 2)$$

- The center of mass is also moduli, but it is not important in this talk.

- Integrating over the codimension z : $\mathcal{L}_{\mathrm{EFT}} = \int_{-\infty}^{\infty} \mathrm{d}z\, \underline{\mathcal{L}_{\mathrm{ChPT}} + \mathcal{L}_{\mathrm{B}}}$

Substitution

$$\Sigma = \exp(2\mathrm{i}\theta\phi\phi^\dagger)u^{-1}$$

- **Effective Lagrangian :** $\mathcal{L}_{\rm EFT} = \mathcal{L}_{\rm const} + \mathcal{L}_{\rm kin} + \mathcal{L}_{\rm topo}$ Eto, KN and Nitta (2023)

- **DW tension :** $\mathcal{L}_{\rm const} = -8m_\pi^2 f_\pi + \dfrac{e\mu_{\rm B} B}{2\pi}$

  - The condition that the DW tension is negative gives $B_{\rm CSL}$.

- **Kinetic term :** $\mathcal{L}_{\rm kin} = \dfrac{16 f_\pi^2}{3m_\pi}[(\phi^\dagger D_\alpha \phi)^2 + D^\alpha \phi^\dagger D_\alpha \phi]$

  - This is the same as the CP^1 Lagrangian.

- **Topological terms :** $-\mu_{\rm B}\dfrac{\epsilon^{0ijk}}{24\pi^2}\displaystyle\int_{-\infty}^{\infty} {\rm d}z\, {\rm tr}\{L_i L_j L_k - 3{\rm i}e\partial_i\left[A_j Q(L_k + R_k)\right]\}$

O(3) nonlinear sigma model

$$n_a \equiv \phi^\dagger \sigma_a \phi$$

$$\mathcal{L}_{\rm topo} = 2\mu_{\rm B} q + \frac{e\mu_{\rm B}}{2\pi}\epsilon^{03jk}\partial_j[A_k(1-n_3)]$$

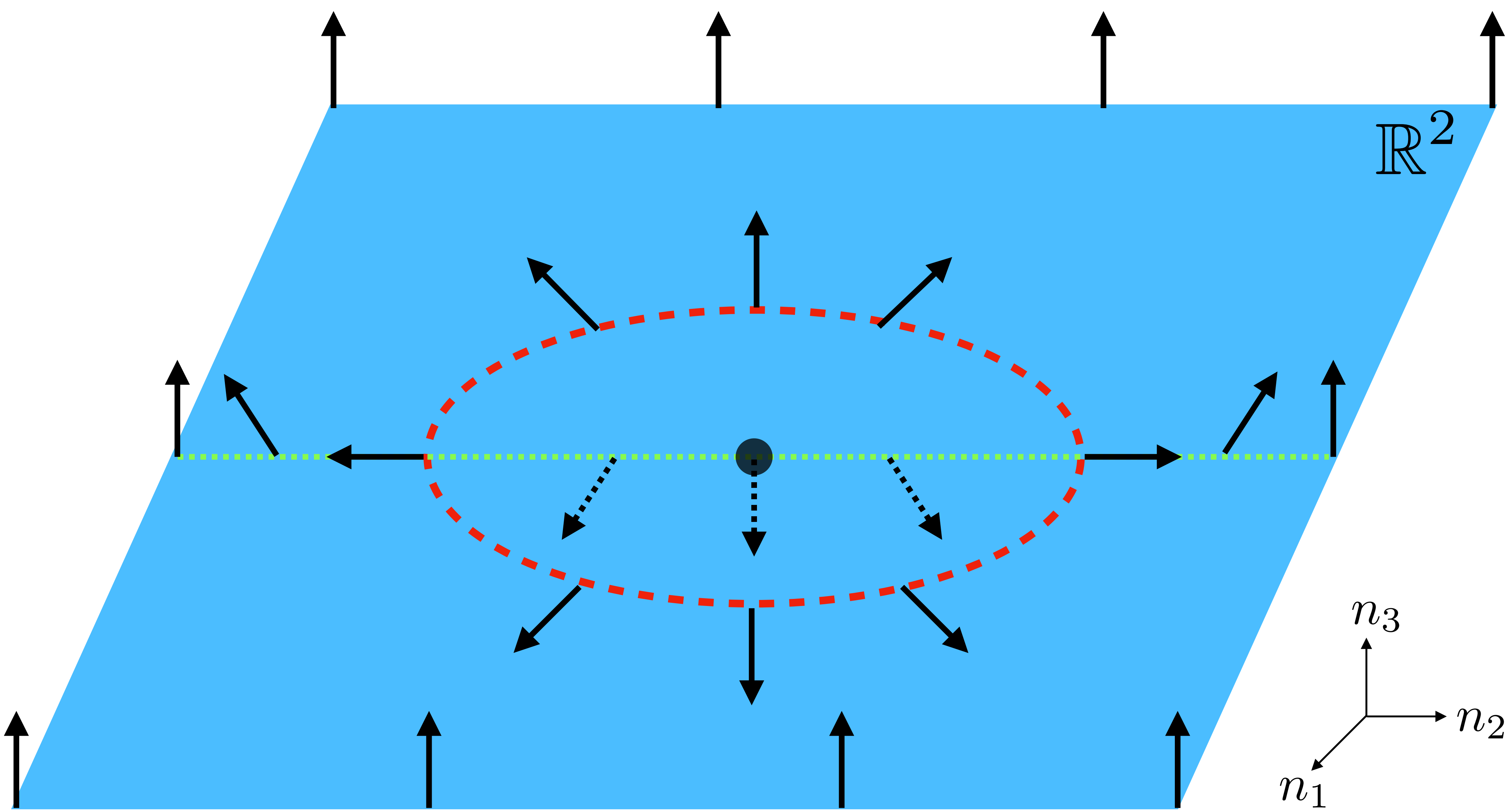
Stabilizing the baby Skyrmion at finite $\mu_{\rm B}$

$\pi_2(S^2)$ topological charge (density) : $q \equiv -\dfrac{\rm i}{2\pi}\epsilon^{ij}\partial_i\phi^\dagger\partial_j\phi\,, \quad k = \displaystyle\int {\rm d}^2x\, q \in \mathbb{Z}$

Baby Skyrmion

- **Configuration on DW surrounding S²:** ↑ = n_a $\boldsymbol{n}^2 = 1$

- **The figure below shows the baby Skyrmion with k = 1.**



Bogomol'nyi bound

- The effective Hamiltonian :

$$\mathcal{H}_{\rm EFT} = \frac{4f_\pi^2}{3m_\pi}(\partial_i \boldsymbol{n})^2 - 2\mu_{\rm B} q - \frac{e\mu_{\rm B}}{2\pi}\epsilon^{03jk}\partial_j[A_k(1-n_3)]$$

- BPS equation : $(\partial_i \boldsymbol{n})^2 = \frac{1}{2}\underline{(\partial_i \boldsymbol{n} \pm \epsilon_{ij}\boldsymbol{n}\times\partial_j \boldsymbol{n})^2} \pm 8\pi q \geq \pm 8\pi q$

$= 0$ → BPS equation → Baby Skyrmion!

- Bogomol'nyi bound :

$$E_{\rm EFT} \equiv \int \mathrm{d}^2x\, \mathcal{H}_{\rm EFT} \geq \underline{\frac{32 f_\pi^2 \pi |k|}{3m_\pi} - 2\mu_{\rm B} k} + \underline{\frac{e\mu_{\rm B} B}{4\pi}\oint \mathrm{d}S_i\, x^i (n_3 - 1)}$$

Can it be negative here? Nontrivial constraint

Eto, KN and Nitta (2023)

Constraint on baby Skyrmion

- **k-Baby Skyrmion solution :** $n_3 = \frac{1-|f|^2}{1+|f|^2}, \quad f = \frac{b_{k-1}w^{k-1} + \cdots + b_0}{w^k + a_{k-1}w^{k-1} + \cdots + a_0}$

- **Total energy :** $E_{\rm DWSk} = \frac{32\pi f_\pi^2}{3m_\pi}|k| - 2\mu_{\rm B}k + e\mu_{\rm B}B|b_{k-1}|^2$

- **k=2 :** $b_1 = 0 \ (|b_0| = 4/(eB))$
  - The baby Skyrmion in this case can appear in the ground state!

- **k=1 :** $b_0 = \sqrt{2/(eB)}$
  - A cancelation between the second and third terms happens!
  - $E_{\rm DWSk}$ is always positive, and k=1 case is not energetically stable.

Eto, KN and Nitta (2023)

Critical chemical potential

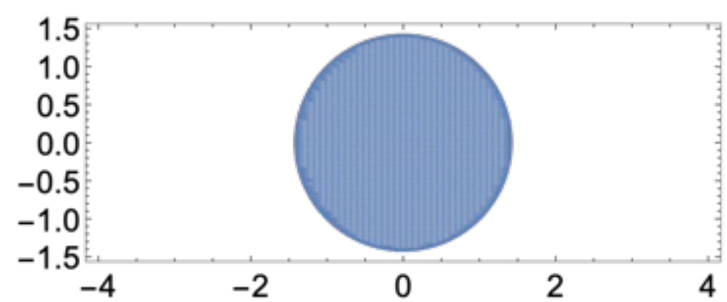
- The energy with k=>2 : $E_{\rm DWSk} = \frac{32\pi f_\pi^2}{3m_\pi}|k| - 2\mu_{\rm B} k$

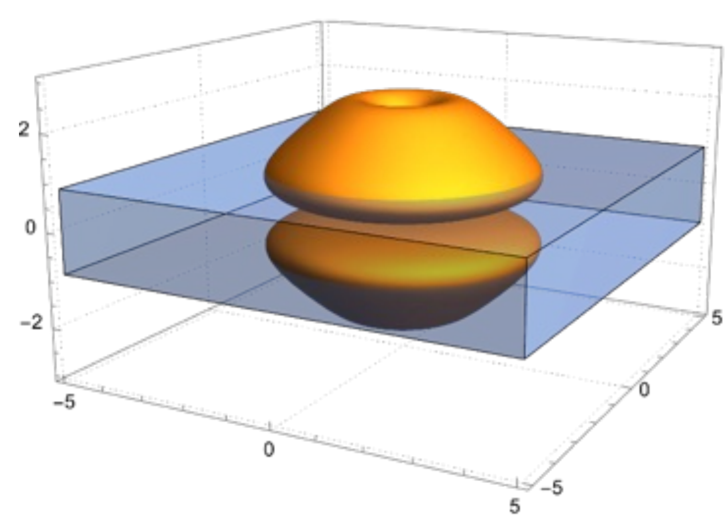
- The condition that baby Skyrmions appear on the DW in the GS :

$$\mu_{\rm B} \geq \mu_{\rm c} = \frac{16\pi f_\pi^2}{3m_\pi} \sim 1.03 \ \ {\rm GeV}$$

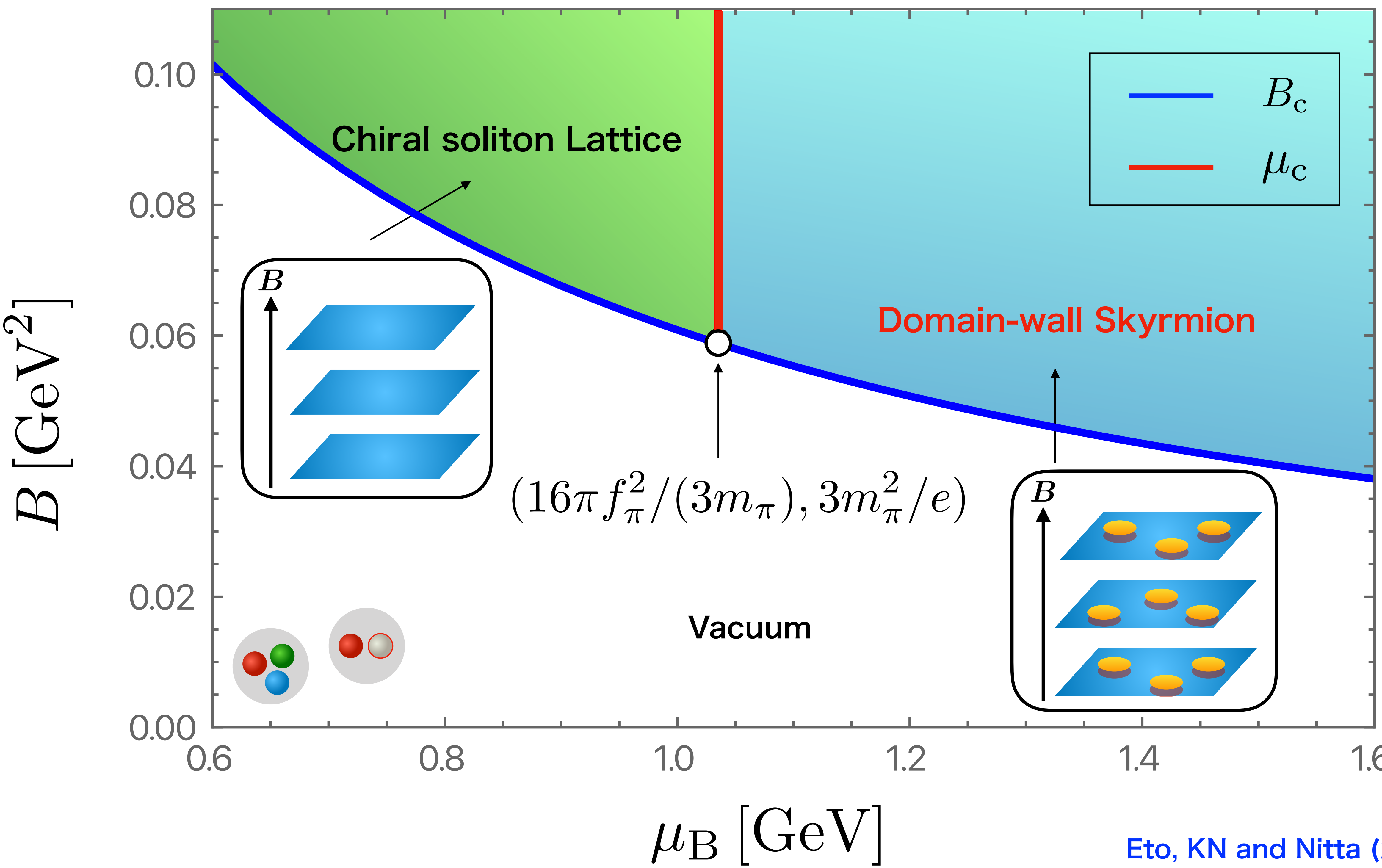
Eto, KN and Nitta (2023)

- Configuration of the baby Skyrmion with k=2 :





Summary



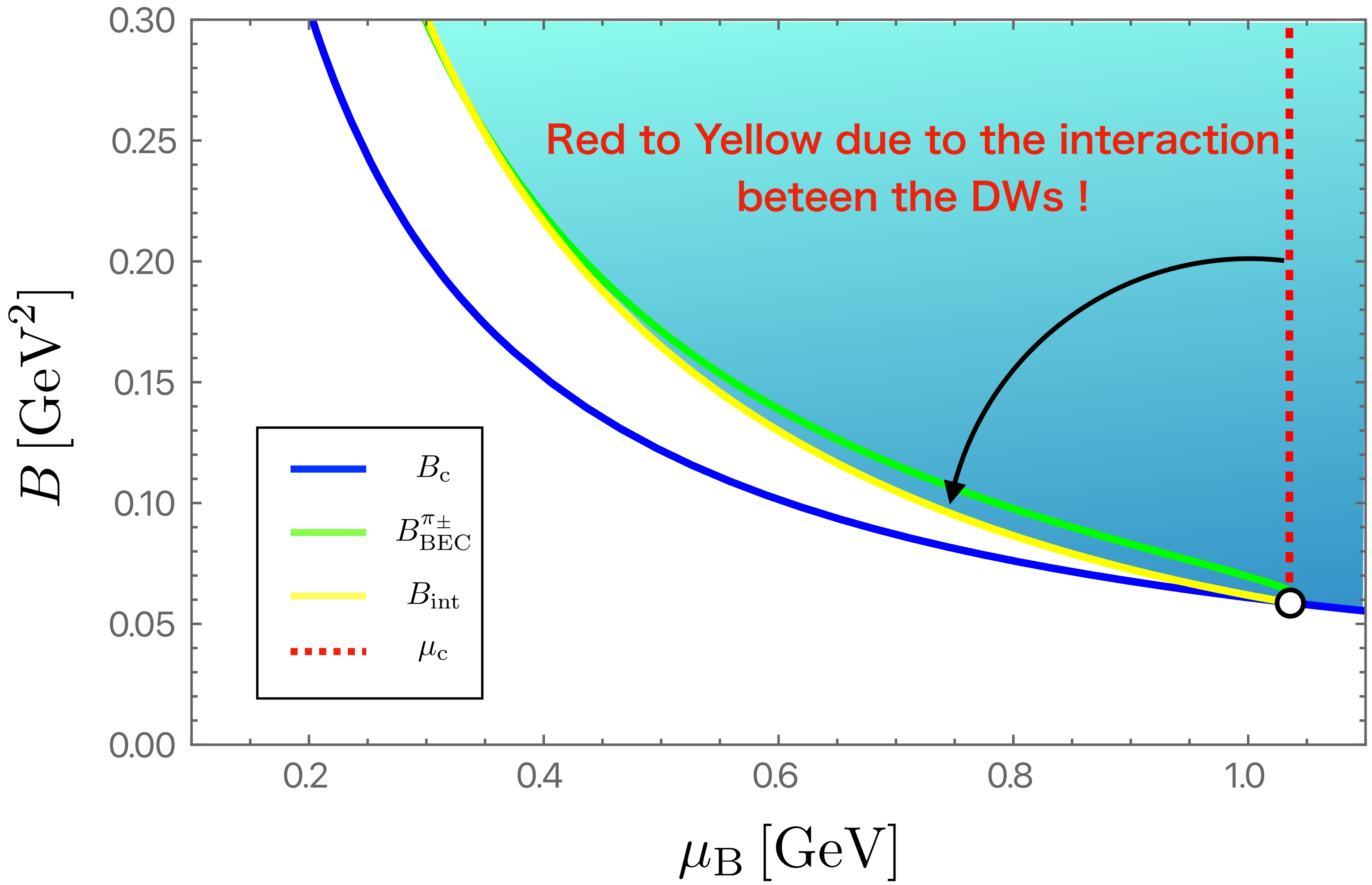
Eto, KN and Nitta (2023)

Future direction

- **Interaction effect :**

- **When considering single DW, there is no interaction.**

Eto, KN and Nitta, in preparation



- **O(p^4) terms, such as dynamical elemag fields and Skyrm term**

Evans and Schmitt (2022) Chen, Fukushima and Qiu (2021)

- **Quantization of the baby Skyrmion**

Cf) Quantization of the Skyrmion Witten (1983)

Thank you
for your attention!

Back up

sine-Gordon theory with the topological term

- Ignoring charged pions : $\Sigma = \mathrm{e}^{\mathrm{i}\phi_3 \tau_3}$

- Reduced Hamiltonian (B is oriented in z-direction) :

$$\mathcal{H} = \frac{f_\pi^2}{2}\left[(\partial_x \phi_3)^2 + (\partial_y \phi_3)^2 + (\partial_z \phi_3)^2\right] + f_\pi^2 m_\pi^2 (1 - \cos\phi_3) - \frac{e\mu_\mathrm{B}}{4\pi^2} B \partial_z \phi_3$$

- The last term stems from the 2nd term of the skyrmion term.

$$\mathcal{L}_\mathrm{B} = -\mu_\mathrm{B} \frac{\epsilon^{0ijk}}{24\pi^2} \mathrm{tr}\{L_i L_j L_k - 3\mathrm{i}e\partial_i \left[A_j Q(L_k + R_k)\right]\}$$

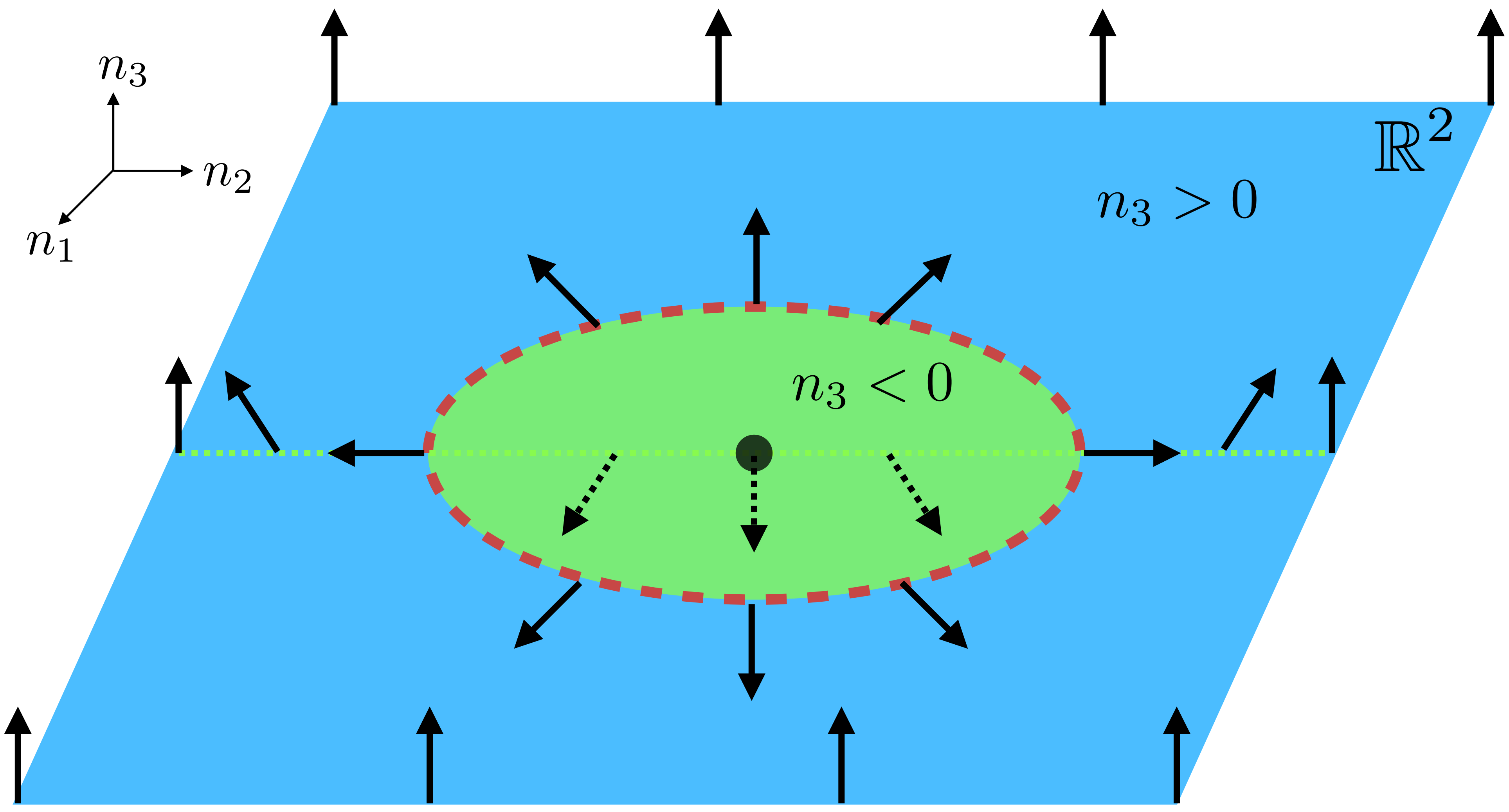
- B=0 → spatially homogeneous + potential minimum → ϕ=0

- B≠0 → Finite 1st derivative term → Favor ϕ inhomogenity

- What is a ground state at finite B? Brauner and Yamamoto (2017)

Superconducting ring

- n_3 is zero on the red line → $\pi_{\pm}$ is condensate → superconducting ring !



Area quantization

- n_1 and n_2 without n_3 are represented by the following equation :

$$n_1 + \mathrm{i}n_2 = \mathrm{e}^{\mathrm{i}\psi}$$

- Minimization condition on the red line :

$$|D_\alpha(n_1 + \mathrm{i}n_2)|^2 = 0 \to \partial_\alpha \psi = eA_\alpha$$

- Magnetic flux of integration form :

Quantization of the size of the baby Skyrmion!

$$BS_D = \int_D \mathrm{d}^2x\, B = \oint_C \mathrm{d}x^i A_i = \frac{1}{e}\oint_C \mathrm{d}x^i \partial_i \psi = \frac{2\pi k}{e}$$

z-axis symmetric solution

- **k=1 :** $f = b_0/w\,,\, n_3 = (1-|f|^2)/(1+|f|^2) = (|w|^2 - |b_0|^2)/(|w|^2 + |b_0|^2)$
- **SC ring :** $n_3 = 0 \to |w| = |b_0|$
- **Area of SC ring :** $B(\pi|b_0|^2) = 2\pi/e \to |b_0| = \sqrt{2/(eB)}$

z-axis symmetric solution

- **k=2 :**

$$f = b_0/w^2\,,\ n_3 = (1-|f|^2)/(1+|f|^2) = (|w|^4 - |b_0|^2)/(|w|^4 + |b_0|^2)$$

- **SC ring :** $n_3 = 0 \to |w|^2 = |b_0|$

- **Area of SC ring :** $B(\pi|b_0|) = 4\pi/e \to |b_0| = 4/(eB)$

Chiral Soliton Lattice

- **EOM = Pendulum**

$$\partial_z^2 \phi_3 = m_\pi^2 \sin \phi_3$$

- **Analytic solution :**

$$\bar{\phi} = 2\mathrm{am}\left((z - z_0)/k, k\right) + \pi$$

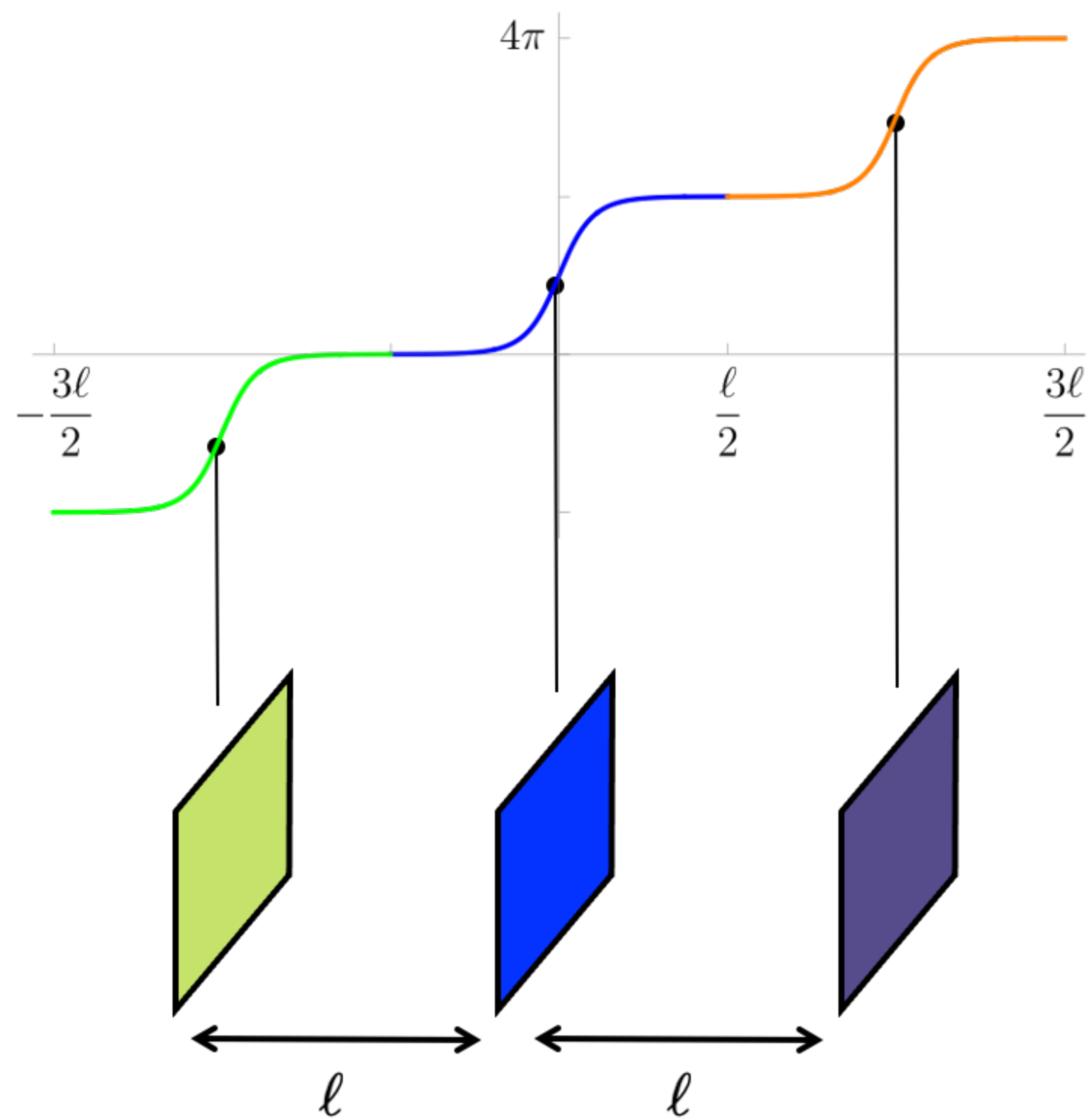
k : Elliptic modulus

- **Period :**

$$\phi(z + \ell) = \phi(z) + 2\pi$$

$$\ell = 2kK(k)$$

$K(k)$: The complete elliptic integral of the first kind



Brauner and Yamamoto (2017)

Minimization of the total energy

- **What is the most energetically stable period?**

$$\mathcal{E}_{\rm tot} = \int_0^\ell \mathrm{d}z \left[\frac{f_\pi^2}{2}(\partial_z\phi)^2 + f_\pi^2 m_\pi^2 (1-\cos\phi) - \frac{\mu_{\rm B}}{4\pi^2} B\partial_z\phi \right]$$

positive **negative!** $\phi(\ell) - \phi(0) = 2\pi$

- **Energy minimization condition :**

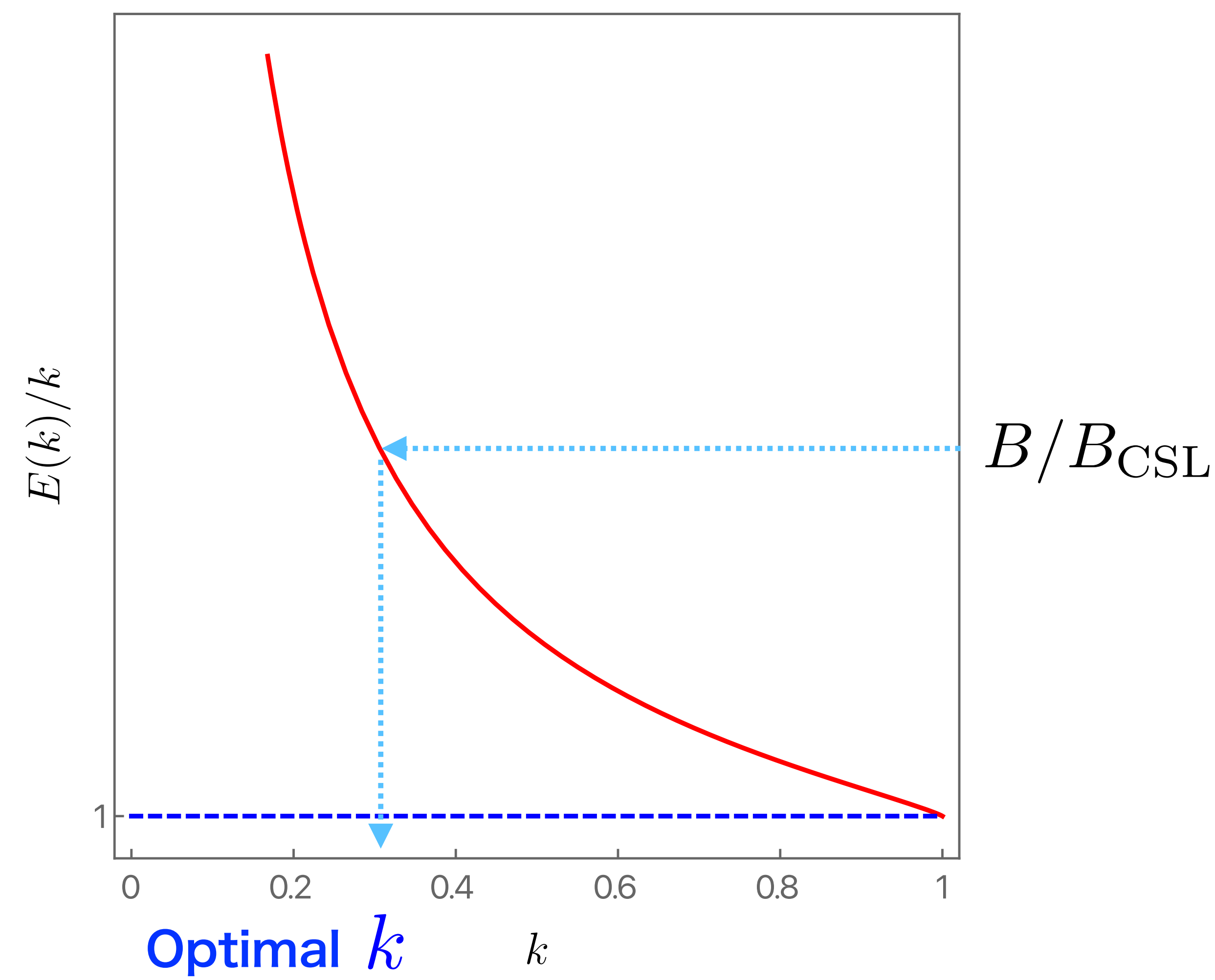
$$\frac{\mathrm{d}}{\mathrm{d}k}\left(\frac{\mathcal{E}_{\rm tot}}{\ell}\right) \rightarrow \frac{E(k)}{k} = \frac{B}{B_{\rm CSL}}$$

$E(k)$: The complete elliptic integral of the 2nd kind

- **Critical magnetic field :**

$$B_{\rm CSL} = 16\pi f_\pi^2 m_\pi / \mu_{\rm B}$$

Brauner and Yamamoto (2017)



Fluctuations of $\pi_{\pm}$

- **Fluctuation around the CSL background :**

- **CSL is unstable against fluctuations of $\pi_{\pm}$ above $B^{\pi\pm}{}_{\mathrm{BEC}}$**

$$\frac{E(k)}{k} = \frac{\mu_{\mathrm{B}} B_{\mathrm{BEC}}^{\pi\pm}}{16\pi m_\pi f_\pi^2}$$

$$B_{\mathrm{BEC}}^{\pi\pm} = \frac{m_\pi^2}{k^2}\left(2 - k^2 + 2\sqrt{1 - k^2 + k^4}\right)$$

$$k = k(B_{\mathrm{BEC}}^{\pi\pm}) \longrightarrow$$

- **Derive the effective action up to the 2nd of the fluctuations from the CSL**
- **Calculate the dispersion relation ω**
- **When $\omega^2<0$, the fluctuation is tachyonic and CSL becomes unstable.**

Brauner and Yamamoto (2017)

EOM of the fluctuations

- Fluctuation around the CSL background :

$$\omega^2\pi_+ = \underbrace{\left[-\partial_x^2 + B^2\left(x - \frac{p_y}{B}\right)^2\right]}_{\text{Giving the Landau quantization}}\pi_+ + (\partial_z^2 + 2\mathrm{i}\partial_z + m_\pi^2 \mathrm{e}^{\mathrm{i}\phi_3})\pi_+$$

- Chiral limit : $\omega^2 = p_z^2 - \underbrace{\frac{\mu_B B p_z}{2\pi^2 f_\pi^2}}_{\text{Deducing the energy!}} + (2n+1)B$

- $\omega^2<0$: $B_{\mathrm{BEC}} = \frac{16\pi^4 f_\pi^2}{\mu_B^2}$ Brauner and Yamamoto (2017)

Our Lagrangian

- The leading order Lagrangian up to O(p^2) : $\mathcal{L}_{\rm tot} = \mathcal{L}_{\rm ChPT} + \mathcal{L}_{\rm B}$
  Son and Stephanov (2008); Brauner and Yamamoto (2017);
  Eto, KN and Nitta (2023)

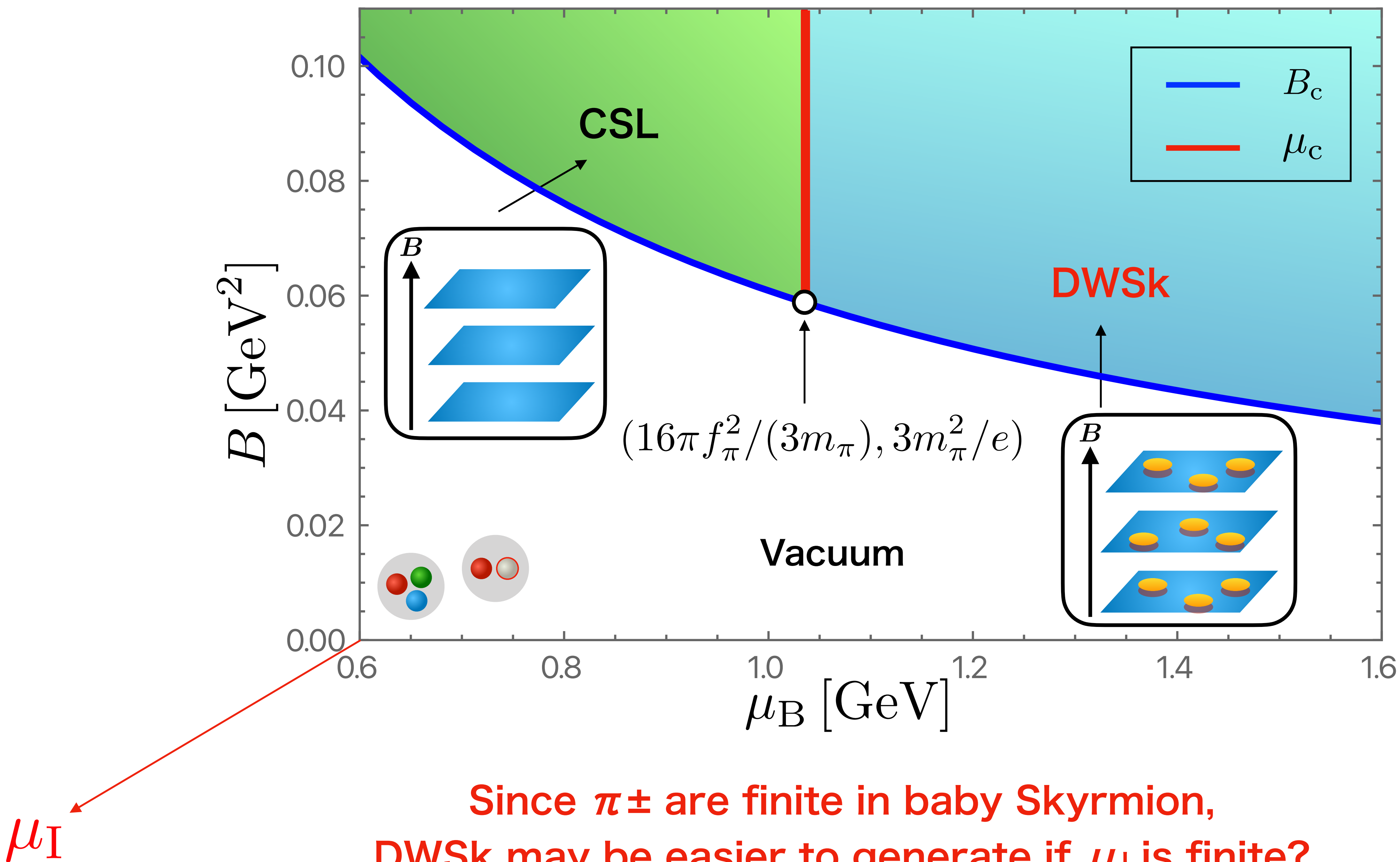
- The chiral anomaly is captured by Wess-Zumino-Witten term, but O(p^4)
  Wess and Zumino (1971); Witten (1983)

  - For example, decay of π^0 to two photon : $\pi_0 \boldsymbol{E} \cdot \boldsymbol{B}$

- The Maxwell term F^2 is also O(p^4). Then, the EM field is not dynamical.

Isospin chemical potential

- **Both μ_B and μ_I are finite in the interior of the neutron stars.**



Since $\pi\pm$ are finite in baby Skyrmion,
DWSk may be easier to generate if μ_I is finite?

Elliptic integrals and functions

- **The elliptic integral of the first kind :** $k' = \sqrt{1-k^2}$

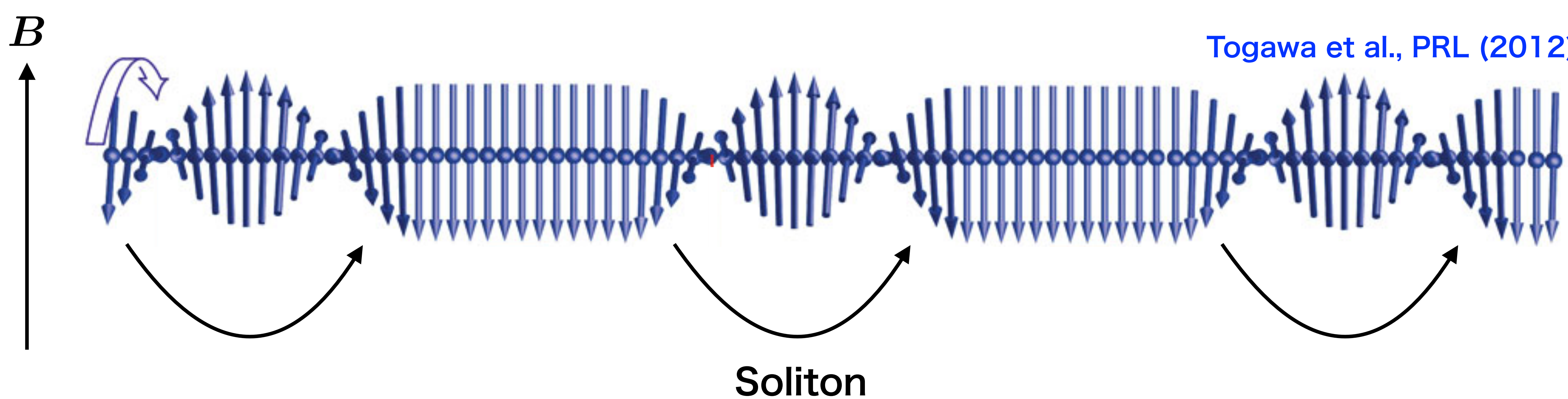
$$K(k) = \int_0^{\pi/2} \mathrm{d}\theta \, \frac{1}{\sqrt{1-k^2\sin^2\theta}} \simeq \ln\frac{4}{k'^2} + \frac{k'^2}{4}\left(\ln\frac{4}{k'^2} - 1\right)$$

- **The elliptic integral of the second kind :**

$$E(k) = \int_0^{\pi/2} \mathrm{d}\theta \, \sqrt{1-k^2\sin^2\theta} \simeq 1 + \frac{k'^2}{2}\left(\ln\frac{4}{k'^2} - \frac{1}{2}\right)$$

Chiral soliton lattice (CSL)

	NG mode	Explicit breaking (periodic potential)	surface term (stabilizing solitons)
Chiral magnets [1]	Magnon	magnetic field	Dzyaloshiinsky-Moriya interaction
Dense QCD under magnetic fields [2]	π_0 meson	Quark masses	Anomaly-related topological term



[1] Togawa et al., PRL (2012)
[2] Son and Stephanov, PRD (2008); Brauner and Yamamoto, JHEP (2017)

Counting scheme

- **Consistent counting scheme :** Brauner, Kolesova and Yamamoto (2021)

$$\partial_\mu \,, m_\pi \,, A_\mu = \mathcal{O}(p^1) \,, \;\; A_\mu^{\mathrm{B}} = \mathcal{O}(p^{-1})$$