Helicity conservation in relativistic fluids

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Outline

- Classical helicity conservation for perfect fluids
- Quantum chiral anomaly and chiral transport
- Dynamical helicity evolution for chiral fluids

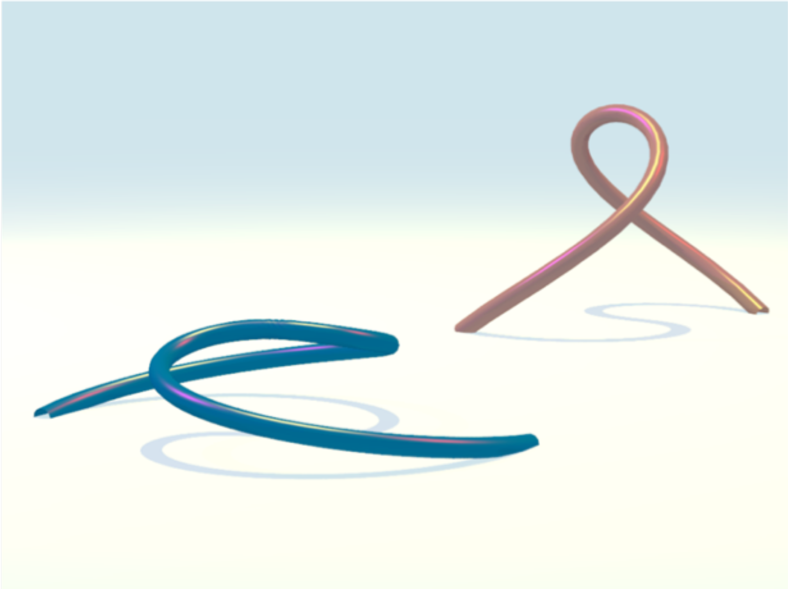
Based on work with Juan Torres-Rincon, arXiv:2211.13697
PRD107, 16003

Magnetic Helicity Density

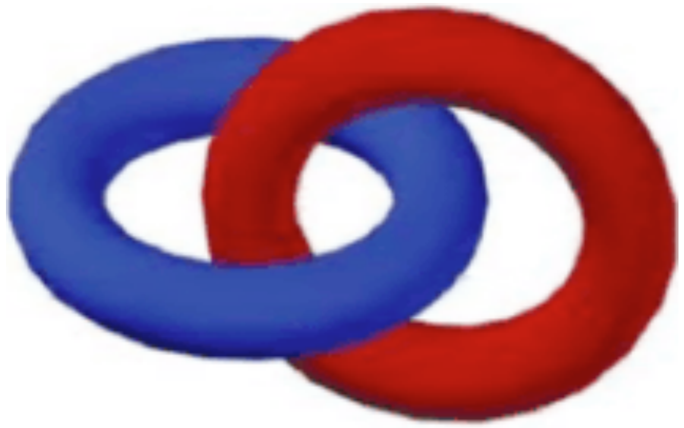
(or Chern-Simons number)

$$\mathcal{H}(t) = \frac{1}{V}\int_V d^3x\,\mathbf{A}\cdot\mathbf{B} \qquad \mathbf{B} = \nabla\times\mathbf{A}$$

gives a measure of a non-trivial topology of the B lines



gauge invariant if B=0 on ∂V (or B n=0)



$H_M = \int \mathbf{A}\cdot\mathbf{B}dV = 2\phi\cdot\phi$

$$\Phi = \int \mathbf{B}\cdot\mathbf{dS}$$

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$$\int \mathbf{A}\cdot\mathbf{B}\,dV \;= \int A_1 dl_1 \int B_1 dS_1 + \int A_2 dl_2 \int B_2 dS_2$$

$$\int A dl = \int (\nabla \times \mathbf{A})\cdot dS = \Phi$$

Magnetic helicity
gives a measure of magnetic linkage

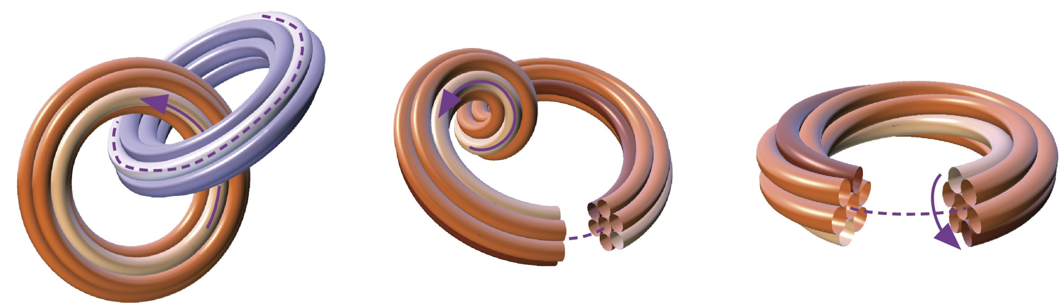
Helicities in hydrodynamics

Vorticity in a fluid (~ magnetic field)

$$\omega = \nabla \times \mathbf{v} \qquad\qquad \mathbf{B} = \nabla \times \mathbf{A}$$

Fluid helicity ~ similar to magnetic helicity

$$\mathcal{H}_f = \frac{1}{\Gamma}\int d^3x\, \mathbf{v}\cdot\omega$$



measures the linking and winding of fluid lines

While the mixed helicity

$$\mathcal{H}_{mf} = \frac{1}{\Gamma}\int d^3x\, \mathbf{v}\cdot\mathbf{B}$$

measures linking and winding of fluid and magnetic field lines

In ideal flow, the fluid helicity is conserved

Kelvin

Magnetic helicity and mixed helicities are also conserved in perfect MHD

Woltjer, Moffat 60`s

Chiral anomaly in a classical barotropic Euler fluid??

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Axial-Current Anomaly in Euler Fluids

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We argue that a close analog of the axial-current anomaly of quantum field theories with fermions occurs in the classical Euler fluid. The conservation of the axial current (closely related to the helicity of inviscid barotropic flow) is anomalously broken by the external electromagnetic field as $\partial_\mu j^\mu_A = 2\boldsymbol{E}\cdot\boldsymbol{B}$, similar to that of the axial current of a quantum field theory with Dirac fermions, such as QED.

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Introduction.—Axial-current anomaly of QED asserts that, while the electric (vector) current of Dirac fermions $j^\mu = \bar\psi\gamma^\mu\psi$ is conserved, the axial current $j^\mu_A = \bar\psi\gamma_5\gamma^\mu\psi$ is not

$$\partial_\mu j^\mu = 0, \qquad (1)$$

$$\partial_\mu j^\mu_A = \frac{k}{4}\,{}^\star FF, \qquad (2)$$

where ${}^\star F^{\mu\nu} = \frac{1}{2}\epsilon^{\mu\nu\lambda\rho}F_{\lambda\rho}$ is the dual electromagnetic tensor. The constant k is integer valued when electromagnetic tensor F is measured in units of the magnetic flux quantum $\Phi_0 = hc/e$. In QED it is $k = 2$, the number of Weyl fermions in the Dirac multiplet. In terms of electric and magnetic field, the anomaly (2) reads

$$\partial_\mu j^\mu_A = k\boldsymbol{E}\cdot\boldsymbol{B}. \qquad (3)$$

The term "anomaly" emphasizes that, while simultaneous transformation $\psi_{L,R} \to e^{i\alpha}\psi_{L,R}$ of left and right components of the Dirac multiplet by virtue of the Noether theorem yields the conservation of the electric charge $Q = \int j^0 d\boldsymbol{x}$, the axial transformation

$$\psi_{L,R} \to e^{\pm i\alpha_A}\psi_{L,R} \qquad (4)$$

does not warrant the conservation of the chirality $Q_A = \int j^0_A d\boldsymbol{x}$, even though it leaves the classical Dirac equation unchanged. It follows from (2) that

$$\frac{d}{dt}Q = 0, \qquad \frac{d}{dt}Q_A = 2\int \boldsymbol{E}\cdot\boldsymbol{B}d\boldsymbol{x}. \qquad (5)$$

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Obtained in 1969 by Adler [1] in QED and Bell and Jackiw [2] in the liner σ model, the axial-current anomaly (or partial conservation of axial current) is a fundamental nonperturbative result in gauge field theories that goes well beyond QED, proven experimentally at different scales of high energy. The most recent advances take place in heavy-ion collision [3], the field that initiated a search for anomalies in relativistic hydrodynamics [4] (see also [5]). Anomalies also have important applications in quantum fluids, the most notably in the superfluid He^3 [6].

Main results.—In this Letter, we show that axial-current anomaly is also a property of a classical Euler's hydrodynamics of the ordinary inviscid barotropic fluid. Such fluid is described by the Euler equations with Lorentz force

$$\dot\rho + \boldsymbol{\nabla}(\rho\boldsymbol{v}) = 0, \qquad (6)$$

$$(\partial_t + \boldsymbol{v}\cdot\boldsymbol{\nabla})m\boldsymbol{v} + \boldsymbol{\nabla}\mu = e\boldsymbol{E} + (e/c)\boldsymbol{v}\times\boldsymbol{B}, \qquad (7)$$

where μ, a function of the density ρ, is the chemical potential related to pressure as $dp = \rho d\mu$. The fluid is assumed to be electrically charged responding to electromagnetic field.

Like QED, the barotropic fluid possesses two locally conserved charges. One is electric charge (the mass in units of e) $Q = \int \rho d\boldsymbol{x}$. Its current, a four-vector $j^\mu = (\rho, \rho\boldsymbol{v})$ is manifestly divergence-free as it is stated by the continuity equation (6) and expressed by (1).

The axial charge is the fluid helicity defined in [7] as

$$\mathcal{H} = (1/\Gamma^2)\int \boldsymbol{v}\cdot(\nabla\times\boldsymbol{v})d\boldsymbol{x}. \qquad (8)$$

It is conserved in the absence of external fields. If we assume that the vorticity is concentrated in thin vortex (closed) filaments of an equal circulation Γ, the helicity is twice the linking number of the filaments [7]. In a superfluid, $\Gamma = h/m$ is Onsager circulation quantum (h is the

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Not really:
classical conservation law
of the total helicity

Total conservation of helicity for a perfect non-relativistic fluid

For isentropic fluids

$$\frac{d}{dt}\left(\mathcal{H}+\mathcal{H}_f+\mathcal{H}_{mf}\right)=0$$

Abanov-Wiegmann equations

$$\partial_t\rho_A+\nabla\mathbf{j}_A=\mathbf{E}\cdot\mathbf{B}$$

$$\rho_A=\left(m\mathbf{v}\cdot\omega+2m\mathbf{v}\cdot\mathbf{B}\right) \qquad \omega=m(\nabla\times\mathbf{v})$$

$$\mathbf{j}_A=\mathbf{v}\rho_A+(\mathbf{w}+2\mathbf{B})(\mu-\frac{1}{2}mv^2)-m\mathbf{v}\times(\mathbf{E}+\mathbf{v}\times\mathbf{B})$$

Relativistic Ideal Hydrodynamics

Conservation laws

$$\partial_\mu j^\mu(x) = 0 \ , \qquad \partial_\mu T^{\mu\nu}(x) + j_\rho(x) F^{\rho\nu}(x) = 0$$

$$j^\mu(x) = n(x)\, u^\mu(x) \qquad\qquad u_\mu u^\mu = 1$$

$$T^{\mu\nu}(x) = \left[\epsilon(x) + P(x)\right] u^\mu(x)\, u^\nu(x) - P(x)\, g^{\mu\nu} \ ,$$

$$d\epsilon = T ds + \mu dn \ , \qquad \epsilon + p = Ts + \mu n$$

Enthalpy per particle $\quad h \equiv \dfrac{\epsilon + p}{n} = \mu + T\bar{s}$

Chern-Simons current

$$J^{\mu}_{CS} = \frac{1}{2}\epsilon^{\mu\nu\alpha\beta}A_{\nu}\partial_{\alpha}A_{\beta}$$

$$\partial_{\mu}J^{\mu}_{CS} = \frac{1}{4}F^{\alpha\beta}\widetilde{F}_{\alpha\beta}$$

Vorticity

$$\omega^{\mu} = \frac{1}{2}\epsilon^{\mu\nu\alpha\beta}u_{\nu}\partial_{\alpha}u_{\beta}$$

$$B^{\mu} = \frac{1}{2}\epsilon^{\mu\nu\alpha\beta}u_{\nu}F_{\alpha\beta}$$

$$u^{\mu} = \gamma(1, v)\ , \qquad \gamma = (1 - v^2)^{-1/2}\ ,$$

$$\omega^{\mu} = \gamma^2\left(\frac{1}{2}\mathbf{v}\cdot(\nabla\times\mathbf{v}), \frac{1}{2}(\nabla\times\mathbf{v}) + \frac{1}{2}\mathbf{v}\times\partial_t\mathbf{v}\right)\ ,$$

$$B^{\mu} = \gamma\left(\mathbf{v}\cdot\mathbf{B}, \mathbf{B} - \mathbf{v}\times\mathbf{E}\right)$$

Hydrodynamical equations say that

$$\boxed{\partial_\mu(h^2\omega^\mu + hB^\mu + J^\mu_{CS}) = T(\partial_\alpha \bar{s})(2h\omega^\alpha + B^\alpha)}$$

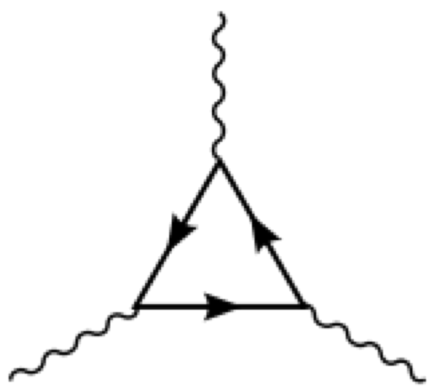
Isentropic fluids obey

$$\boxed{\partial_\mu(h^2\omega^\mu + hB^\mu) = -\frac{1}{4}F^{\alpha\beta}\widetilde{F}_{\alpha\beta}}$$

Quantum chiral anomaly

$(\hbar = c = k_B = 1)$

Adler, Bell, Jackiw, 1969



QFT requires a UV regulator

- it is impossible (with m=0 fermions) to keep both gauge invariance and preserve the chiral number

$$\boxed{\partial_\mu J_5^\mu = -\frac{e^2}{16\pi^2}\epsilon^{\alpha\beta\mu\nu}F_{\alpha\beta}F_{\mu\nu}}$$

$$\boxed{\partial_t \rho_5 + \nabla \mathbf{j}_5 = \frac{e^2}{2\pi^2}\mathbf{E}\cdot\mathbf{B}}$$

Main success: explaining anomalous decays

$$\pi^0 \to \gamma\gamma$$

CHIRAL HYDRODYNAMICS

Son and Surowka, 09

Hydrodynamical equations for systems containing (quasi) massless fermions have to be modified to take into account chiral anomalous effects

CHIRAL TRANSPORT THEORY

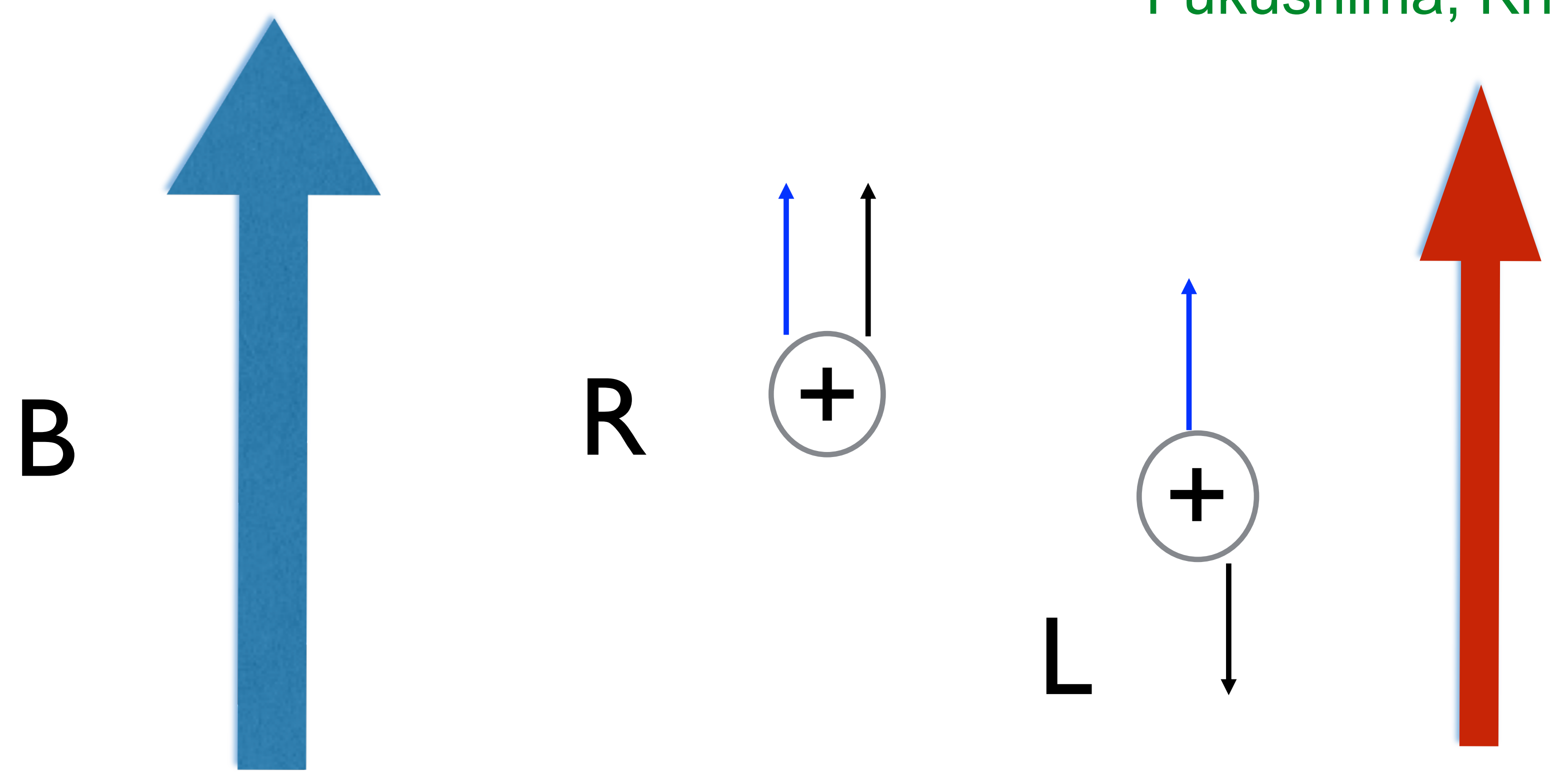
A microscopic semi-classical approach can be used, based on transport theory, describing all these effects

$$f(x,p) \Rightarrow T^{\mu\nu}(x) \ , J^{\mu}(x)$$

Son and Yamamoto, 2012
Stephanov and Yin, 2012
Chen, S. Pu, Q. Wang and X.N. Wang, 2013
Torres-Rincon and CM, 2014
Hidaka, S. Pu and D. L.Yang, 2017
Mueller and Venugopalan 2017
Carignano, CM, Torres-Rincon 2018

Chiral Magnetic/Vortical Effect

Fukushima; Kharzeev, Warringa, 08



$$\mathbf{J} = \frac{e^2\,\mu_5}{4\pi^2}\,\mathbf{B}$$

In a B a misbalance in the population of L/R handed fermions leads to an e.m. current || to B

Similarly, in the presence of fluid vorticity

$$\mathbf{J} = \frac{e}{4\pi}\mu\mu_5\omega$$

Also interesting to point out that these new chiral phenomena does not represent dissipation in the system!
(no increase of entropy)

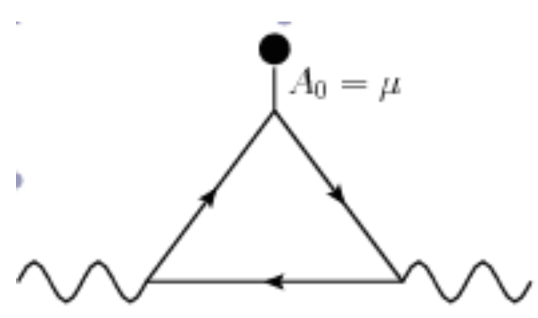
$$\mathbf{J} = \frac{e^2\,\mu_5}{4\pi^2}\,\mathbf{B}$$

$\mathcal{T}$ even

$$\mathbf{J} = \sigma\mathbf{E}$$

$\mathcal{T}$ odd

The coefficient of the chiral conductivity is completely fixed by the chiral quantum anomaly



in the presence of chemical potentials, at T=0

Sadofyev et al, ‘11

$$\delta H = \mu u_\mu J^\mu + \mu_5 u_\mu J_5^\mu$$

$$\partial_\mu j^\mu = -\frac{1}{2\pi^2}\epsilon_{\mu\nu\alpha\beta}\left(\partial^\mu(eA^\nu + \mu u^\nu)\partial^\alpha(\mu_5 u^\beta)\right)$$

$$\partial_\mu j_5^\mu = -\frac{1}{4\pi^2}\epsilon_{\mu\nu\alpha\beta}\left(\partial^\mu(eA^\nu + \mu u^\nu)\partial^\alpha(eA^\beta + \mu u^\beta) + \partial^\mu(\mu_5 u^\nu)\partial^\alpha(\mu_5 u^\beta)\right)$$

$$\partial_\mu\left(j^\mu + \frac{e\mu_5}{\pi^2}B^\mu + \frac{\mu\mu_5}{\pi^2}\omega^\mu\right) = 0\ ,$$

$$\partial_\mu\left(j_5^\mu + \frac{e\mu}{2\pi^2}B^\mu + \frac{\mu^2+\mu_5^2}{2\pi^2}\omega^\mu\right) = -\frac{e^2}{16\pi^2}\epsilon_{\mu\nu\alpha\beta}F^{\mu\nu}F^{\alpha\beta}\ ,$$

Helicity conservation in chiral fluids

$$j^\mu = n u^\mu + \xi \omega^\mu + \xi_B B^\mu \ ,$$
$$j_5^\mu = n_5 u^\mu + \xi_5 \omega^\mu + \xi_{B,5} B^\mu$$

$$h = \frac{\epsilon + P}{n} = \frac{sT + n\mu + n_5\mu_5}{n}$$

$$dP = n dh - T n d\bar{s} - \mu_5 n dx_5$$

$$\boxed{\partial_\mu (h^2 \omega^\mu + h B^\mu + J^\mu_{CS}) = (2h\omega^\mu + B^\mu)(T \partial_\mu \bar{s} + \mu_5 \partial_\mu x_5)}$$

$$\partial_\mu (n_5 u^\mu + \xi_5 \omega^\mu + \xi_{B,5} B^\mu) = -\frac{C}{4} F^{\mu\nu} \widetilde{F}_{\mu\nu}$$

Summary

- Total helicity conservation law for isentropic classical hydrodynamics - interesting to study back reaction and dissipation, as helicity might be transferred among sectors
- In chiral fluids, chiral anomaly effects are also to be be considered: emergence of chiral plasma instabilities can make these transfers more effective!

- Our considerations might be relevant for HIC where most of the conditions are met