First order stable and causal anisotropic hydrodynamics

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Causality vs. the gradient expansion

Hydrodynamics

Hydrodynamics is an effective theory for understanding the collective behavior of fluids not far from equilibrium.

- Equations of hydrodynamics are conservation laws

$$\nabla_\mu T^{\mu\nu} = 0 \qquad \nabla_\mu J_I^\mu = 0$$

- We assume an uncharged fluid $J_I^\mu = 0$
- The fluid that we have in mind is the QGP produced in HICs

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* ∇_μ is the covariant derivative

* $T^{\mu\nu}$ is the energy-momentum tensor, and J_I^μ are the charge currents.

Perfect fluid

A perfect fluid is spatially isotropic for a comoving observer.

- In local rest frame (LRF):

$$T^{\mu\nu} = \begin{bmatrix} \varepsilon & & & \\ & P & & \\ & & P & \\ & & & P \end{bmatrix}$$

- Conformal fluid $g_{\mu\nu}T^{\mu\nu} = 0 \implies \varepsilon = 3P$
- Arbitrary observer sees
  $T_0^{\mu\nu} = \varepsilon u^\mu u^\nu + P\Delta^{\mu\nu}$ with $\Delta^{\mu\nu} = g^{\mu\nu} + u^\mu u^\nu$

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* ε is the energy density, and P is the pressure

* The metric $g_{\mu\nu}$ has mostly plus sign $\mathrm{diag}\,(-+++)$, therefore $u^\mu u_\mu = -1$

A real-life fluid is dissipative

$$T^{\mu\nu} = T_0^{\mu\nu} + T_1^{\mu\nu} + \mathcal{O}\left(\partial^2\right)$$

- Identify basic quantities $(\varepsilon, u^\mu) \sim \mathcal{O}(1)$
- ... from (Weyl-covariant) derivatives of these quantities ($\mathcal{D}_\mu \varepsilon$ and $\mathcal{D}_\mu u_\nu$)
- ... build scalar $(u^\mu \mathcal{D}_\mu \varepsilon, \mathcal{D}_\mu u^\mu)$, vector $(\mathcal{D}_\mu \varepsilon, u^\nu \mathcal{D}_\nu u_\mu)$, and tensor $(\sigma_{\mu\nu} = \frac{1}{2}\left(\mathcal{D}_\mu u_\nu + \mathcal{D}_\nu u_\mu\right))$ combinations
- ... coupled with transport coefficients (like shear viscosity η)

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* Weyl covariant-derivative **[Loganayagam (2008)]**:
$\mathcal{D}_\mu u_\nu = \Delta_\mu^\alpha \nabla_\nu u_\nu - \Delta_{\mu\nu} \nabla_\alpha u^\alpha / 3$ and
$\mathcal{D}_\mu \varepsilon = \nabla_\mu \varepsilon + 4\varepsilon(u^\alpha \nabla_\alpha u_\mu - u_\mu \nabla_\alpha u^\alpha / 3)$

A demonstrative example: the diffusion equation

Let's assume something simpler instead of hydrodynamics

- Continuity equation (EOM) $\frac{\partial n}{\partial t} + \nabla \cdot \mathbf{J} = 0$
- The number density $n = \mathcal{O}(1)$ is the only basic object
- ...we should build $\mathbf{J} = \mathcal{O}(\partial)$ out of it
- The EOM $\implies \frac{\partial n}{\partial t} = \mathcal{O}\left(\partial^2\right)$ **on-shell**
- Hence $\mathbf{J} = -D\vec{\nabla} n + \mathcal{O}\left(\partial^2\right)$ and

$$\frac{\partial n}{\partial t} = D\nabla^2 n$$

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* The example adopted from [Kostädt and Liu (2000)] and [Romatschke and Romatschke (2019)]

Is the diffusion equation stable?

The same prescription gives rise to **Navier-Stokes** hydrodynamics. But there is a problem that was observed by **[Hiscock and Lindblom (1985)]**.

- Consider a 1+1 world $n(t,x) = n(k)\exp(-i\omega t + ikx)$
- Diffusion equation $\implies \omega = -iDk^2$: looks stable
- ... but in a boosted frame $\omega(k=0) \propto i/D \implies$ instability
- The asymptotic ($k \to \infty$) group velocity exceeds the speed of light **[Pu et al. (2010)]**

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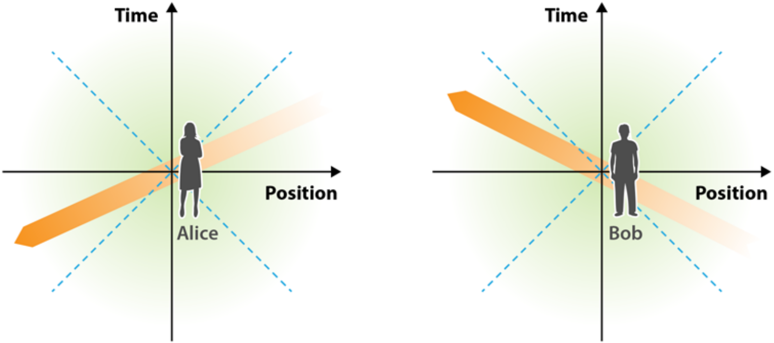
* HL showed that if $\mathrm{Im}\,\omega(k) > 0$ for some domain of k, then the spatial norm of perturbations grows with time, without a bound
* Here, NS refers to the first-order on-shell theories (Landau-Lifshitz and Eckart)
* Nonhydro (gapped) mode: $\omega(k=0) \neq 0$

The diffusion equation and relativity

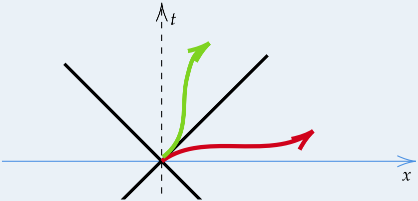
One observer sees stable modes, and the other sees unstable ones!



G. Denicol (https://physics.aps.org/articles/v15/149)

The mathematician knows

- (One) family of characteristics found from highest-order derivatives in (parabolic) $\frac{\partial n}{\partial t} = D\nabla^2 n$
- Eq. of characteristics: $(\partial_x \phi)^2 = 0 \implies \phi(t,x) = t = \text{const}$
- Domain of influence ...
- $n(t_0, x_0)$ affects anywhere in $t > t_0$ (inside and outside)



How can we rescue it?

- A hyperbolic equation is required $\rightarrow$ two families
- MIS approach (promote $\mathbf{J}$):

$$\partial_t n + \nabla \cdot \mathbf{J} = 0 \qquad \tau_J \partial_t \mathbf{J} + \mathbf{J} + D\vec{\nabla} n = 0$$

- BDNK approach (offshell regulators):

$$n \rightarrow \mathcal{N} = n + \tau_n \partial_t n + \mathcal{O}\left(\partial^2\right) \quad \mathbf{J} = -D\nabla n + \mathcal{O}\left(\partial^2\right)$$

in

$$\partial_t \mathcal{N} + \nabla \cdot \mathbf{J} = 0$$

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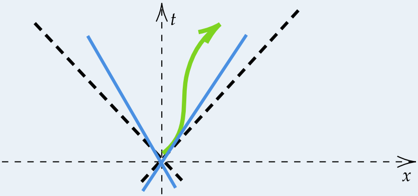
* For example [Israel and Stewart (1979)]
* DNMR [Denicol et al. (2012)] is used in state of the art simulations
* BDN [Bemfica et al. (2018)] [Bemfica et al. (2022)] K [Kovtun (2019)]

BDNK prescription

- Telegraph equation $\partial_t n + \tau_n \partial_t^2 n - D\partial_x^2 n = 0$
- Characteristics $\phi(t,x) = x \pm t\sqrt{D/\tau_n}$
- Causality demands off-shell term $(\tau_n \neq 0)$ and $\tau_n > D > 0$
- [Gavassino (2022)] Causality + Stability in LRF $\implies$ stability in any other frame

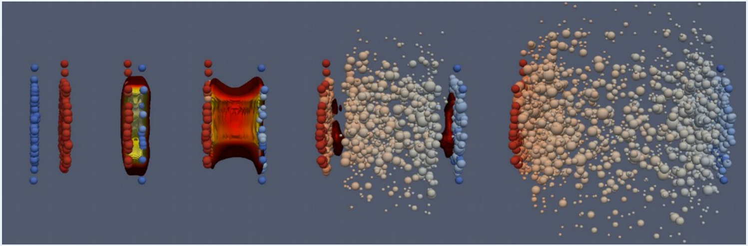


Causality of Anisotropic BDNK

Anisotropy

[Martinez and Strickland (2010)] - [Florkowski and Ryblewski (2011)]:

In a heavy collision, there is a preferred spatial direction.



MADAI collaboration, Hannah Petersen and Jonah Bernhard

- Anisotropic hydrodynamics

$$T^{\mu\nu} = \begin{bmatrix} \varepsilon & & & \\ & P_\perp & & \\ & & P_\perp & \\ & & & P_l \end{bmatrix}$$

- More generally

$$T^{\mu\nu} = \varepsilon\, u^\mu u^\nu + (P_l - P_\perp)\, l^\mu l^\nu + P_\perp\, \Delta^{\mu\nu}$$

- Spacelike anisotropy vector $l_\mu l^\mu = 1$ and $l_\mu u^\mu = 0$
- Dissipative fluxes from kinetic theory [Molnár et al. (2016)]

General form

- Can we use BDNK expansion without kinetic theory?
- A general form adopted from [Molnár et al. (2016)]

$$T^{\mu\nu} = \mathcal{E}\, u^\mu u^\nu + (\mathcal{P}_l - \mathcal{P}_\perp)\, l^\mu l^\nu + \mathcal{P}_\perp\, \Delta^{\mu\nu} + 2\, M\, u^{(\mu}\, l^{\nu)}$$
$$+ 2\, W_{\perp u}^{(\mu} u^{\nu)} + 2\, W_{\perp l}^{(\mu}\, l^{\nu)} + \pi_\perp^{\mu\nu}$$

- For example $\mathcal{E} = \varepsilon + \mathcal{O}(\partial) \quad \mathcal{P}_{l,\perp} = P_{l,\perp} + \mathcal{O}(\partial)$
- Second law of thermodynamics $\rightarrow$ zeroth-order, and first-order on-shell terms
- Zeroth-order EOM $\rightarrow$ off-shell terms

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* $A^{(\mu\nu)} = \frac{1}{2}\,(A^{\mu\nu} + A^{\nu\mu})$

* The vectors $W^\mu_{\perp\, l}$, $W^\mu_{\perp\, u}$, and the symmetric traceless tensor $\pi_\perp^{\mu\nu}$ are orthogonal to u and l

Entropy current

- The generating functional is $W_0 = \int \mathrm{d}^4x \sqrt{-g} P_\perp$
- Therefore $TS^\mu = P_\perp u^\mu - u_\nu T^{\mu\nu}$
- $\nabla_\mu S^\mu \geq 0$
- Domain of applicability $\mathcal{O}\left(\partial^2\right)$
- Difference between NS and BDNK should not be measurable
  $\to$ On-shell terms only
- No external agent $\to P_\perp = P_l = \varepsilon/3$

$$\nabla_\mu S^\mu = -\frac{\pi_\perp^{\mu\nu}\sigma_{\perp\mu\nu}}{T} - \frac{2W_{\perp l}^{\mu} l^\nu \sigma_{\mu\nu}}{T} - \frac{\varepsilon - 3P_\perp}{4T}\frac{u^\alpha \mathcal{D}_\alpha \varepsilon}{\varepsilon} - \frac{(\mathcal{P}_l - \mathcal{P}_\perp) l^\mu l^\nu \sigma_{\mu\nu}}{T} - \frac{\mathcal{E}^{(1)} u^\nu + W_{\perp u}^{\nu} + M\, l^\nu}{4T}\frac{\mathcal{D}_\nu \varepsilon}{\varepsilon} + \mathcal{O}\left(\partial^3\right)$$

The final form

- Add off-shell terms from zeroth order EOM

$$\mathcal{E}^{(1)} = \frac{3\chi}{4}\frac{u^\mu \mathcal{D}_\mu \varepsilon}{\varepsilon} \qquad \mathcal{P}_l^{(1)} = \frac{\chi_l}{4}\frac{u^\mu \mathcal{D}_\mu \varepsilon}{\varepsilon} - 2\eta_{ll} l^\alpha l^\beta \sigma_{\alpha\beta}$$

- 7 independent transport coefficients
  $\{\chi, \chi_l, \eta_l > 0, \eta_{ll} > 0, \eta_\perp > 0, \lambda_l, \lambda_\perp\} \propto \varepsilon^{3/4}$

Entropy production and the other terms

$$\nabla_\mu S^\mu = \frac{\pi_\perp^{\mu\nu}\pi_{\perp\mu\nu}}{2\eta_\perp T} + \frac{W_{\perp l}^\mu W_{\perp l\,\mu}}{\eta_l T} + \frac{\left(\mathcal{P}_\perp^{(1)} - \mathcal{P}_l^{(1)}\right)^2}{3\eta_{ll} T} + \mathcal{O}(\partial^3) \geq 0$$

$$\mathcal{P}_\perp^{(1)} = \frac{\chi_\perp}{4}\frac{u^\mu \mathcal{D}_\mu \varepsilon}{\varepsilon} + \eta_{ll} l^\alpha l^\beta \sigma_{\alpha\beta} \qquad \pi_\perp^{\mu\nu} = -2\eta_\perp \sigma_\perp^{\mu\nu} \qquad W_{\perp l}^\mu = -2\eta_l \Xi_\lambda^\mu l_\nu \sigma^{\lambda\nu}$$

$$W_{\perp u}^\mu = \frac{\lambda_\perp}{4}\Xi^{\mu\nu}\frac{\mathcal{D}_\nu \varepsilon}{\varepsilon} \qquad M = \frac{\lambda_l}{4}\frac{l^\nu \mathcal{D}_\nu \varepsilon}{\varepsilon} \qquad \Xi^{\mu\nu} = \Delta^{\mu\nu} - l^\mu l^\nu$$

$$\sigma_{\perp\mu\nu} = \Xi_{\mu\nu}^{\alpha\beta}\mathcal{D}_\alpha u_\beta \qquad \Xi_{\mu\nu}^{\alpha\beta} = (\Xi_\mu^\alpha \Xi_\nu^\beta + \Xi_\mu^\beta \Xi_\nu^\alpha - \Xi^{\alpha\beta}\Xi_{\mu\nu})/2$$

- Method of characteristics

$$\nabla_\mu T^{\mu\nu} = 0 \qquad R^{\mu\nu} - \frac{1}{2} g^{\mu\nu} R = (8\pi G/c^4) T^{\mu\nu}$$

- Pure gravity sector decouples $\rightarrow$ lightlike surfaces

$$\text{characteristics equation} = \text{gravity sector} \times \text{matter sector}$$

- Furthermore $\text{matter sector} = 4 \text{ roots} \times 6 \text{ roots}$
- 4 roots $\rightarrow$ off-shell terms are required for causality

$$\lambda_\perp > \max(\eta_l, \eta_\perp) \geq 0 \qquad \chi > 0$$

- For the 6 roots part we provide an *algorithm*

Stable and Causal example

- For example

$$\eta_\perp = \eta > 0 \quad \eta_l = \frac{2\eta}{3} \quad \eta_{ll} = \frac{5\eta}{6} \quad \lambda_\perp = \frac{13\eta}{2}\lambda_l = 6\eta$$
$$\chi = 5\eta \quad \chi_\perp = \frac{11\eta}{2} \quad \chi_l = \frac{16\eta}{3}$$

  is a causal set
- Also linearly stable in LRF
- . . . and in a moving frame

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* Plane wave $\delta X(t, x^i) \to e^{-iT_0 x^\mu k_\mu}\delta X(k^\mu)$ where $k^\mu = (\omega, k^i)$

* EOM $\to M^{AB}\delta X^B = 0 \to \det(M) = 0 \to \omega = \omega(k)$

Concluding remarks

Summary

- First-order stable and causal anisotropic hydrodynamics for a conformal uncharged fluid.
- Constraints for the nonlinear causality of the theory
- Off-shell terms are required if the theory is to be causal
- Linear stability analysis using plane wave perturbations around a homogeneous background
- Bjorken : $\mathcal{A} = \left(\mathcal{P}_\perp - \mathcal{P}_l\right)/P$ has off-shell second-order contributions, in contrast to isotropic BDNK
- Bjorken: The behavior of attractors is closely related to causality conditions: If the causality conditions are violated a reheating occurs in very early times

Outlook

- Anisotropy at the zeroth order
- General case without conformal invariance
- Charged fluids
- BDNK theory of resistive dissipative magnetohydrodynamics

Backup

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Bjorken flow I

- In Milne coordinates

$$u^\mu = (1,0,0,0) \qquad l^\mu = \frac{1}{\tau}\,(0,0,0,1)$$

- Bjorken symmetries

$$\pi_\perp^{\mu\nu} = 0 \qquad W_{\perp l}^\mu = 0 \qquad W_{\perp u}^\mu = 0 \qquad M = 0$$

- EOM with scaled transport coefficients

$$9\tilde{\chi}\frac{\tau^2\ddot{T}}{T} + 18\tilde{\chi}\frac{\tau^2\dot{T}^2}{T^2} + \left(\frac{3\tau(9\tilde{\chi}-\tilde{\chi}_\perp)}{T} + 12\tau^2\right)\dot{T} + 4\tau T + 3\tilde{\chi} - 2\tilde{\chi}_\perp - 4\tilde{\eta}_{ll} = 0$$

- Reduction of order $w = T\tau$ and $f(w) = \frac{\tau}{w}\frac{\mathrm{d}w}{\mathrm{d}\tau}$ **[Heller and Spalinski (2015)]**

$$\frac{9\tilde{\chi}}{4}f(w)^2 + wf(w)\left(1 + \frac{3}{4}\tilde{\chi}f'(w)\right) - \frac{6\tilde{\chi}+\tilde{\chi}_\perp}{2}f(w) + \frac{3\tilde{\chi}+\tilde{\chi}_\perp-\tilde{\eta}_{ll}}{3} - \frac{2w}{3} = 0$$

Bjorken flow II

- Late times solution

$$f(w) = \frac{2}{3} + \frac{\tilde{\eta}_{ll}}{3w} + \frac{\tilde{\eta}_{ll(\tilde{\chi}_l + 5\tilde{\chi}_\perp)}}{18w^2} + \mathcal{O}\left(\frac{1}{w^3}\right)$$

- Linear corrections to the series

$$\delta f(w) \sim \exp\left(-\frac{2w}{\tilde{\chi}}\right) w^{\frac{\tilde{\eta}_{ll} + \tilde{\chi}_\perp}{\tilde{\chi}}}$$

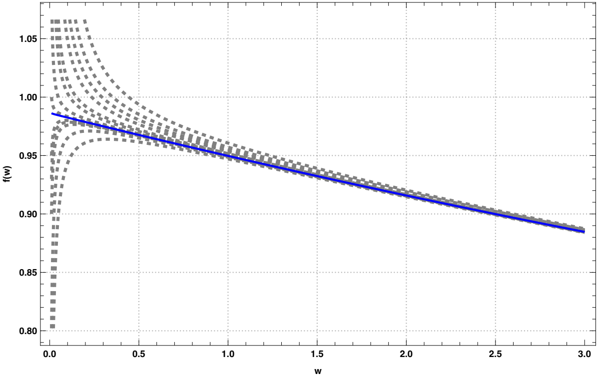
- Slow-roll attractor

$$f(w)_{\text{slowroll}} = \frac{7}{9} - \frac{\tilde{\chi} - \tilde{\chi}_\perp}{9\tilde{\chi}} - \frac{2w}{9\tilde{\chi}} + \frac{\sqrt{(2w - \tilde{\chi}_\perp)^2 + 12\tilde{\eta}_{ll}\tilde{\chi}}}{9\tilde{\chi}}$$

Bjorken flow III

- Initial condition for the numerical attractor

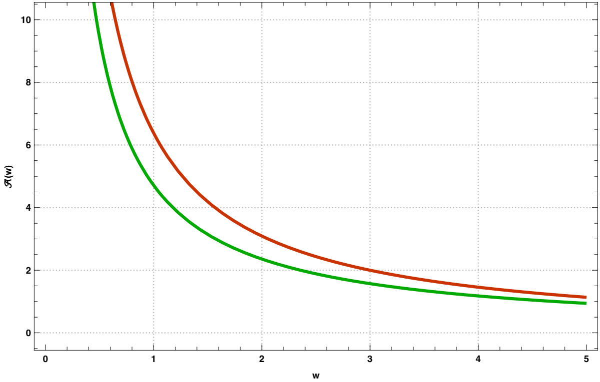
$$f(w \ll 1) = \frac{7}{9} - \frac{\tilde{\chi} - \tilde{\chi}_{\perp}}{9\tilde{\chi}} + \frac{\sqrt{12\tilde{\eta}_{ll}\tilde{\chi} + \tilde{\chi}_{\perp}^2}}{9\tilde{\chi}}$$



Bjorken flow IV

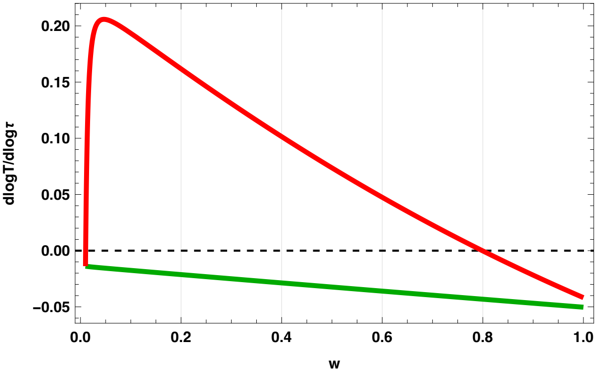
- Pressure anisotropy

$$\mathcal{A} = \frac{2}{3}\frac{\tilde{\chi}_{\perp} - \tilde{\chi}_{l}}{w}\left(f - \frac{2}{3}\right) + \frac{6\tilde{\eta}_{ll}}{w}$$



- Causality conditions prevent reheating

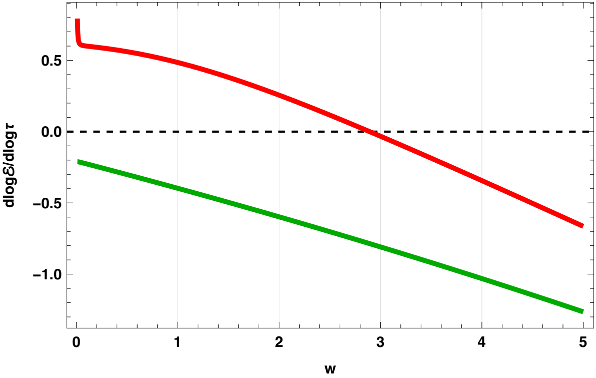
$$\frac{2}{3} < f(0) < 1 \implies \chi_l > 4\eta_{ll} > 0$$



Bjorken flow VI

- The same condition is required for

$$\frac{\mathrm{d}\mathcal{E}}{\mathrm{d}w} < 0$$



Bjorken flow VII

- Only for the causal set, the offshell (nonphysical) entropy decreases initially

$$\frac{\nabla \cdot S_{\text{off}}}{T^4}\Big|_{w=0} = 6\chi\left(f(0)-\frac{2}{3}\right)\left(f(0)-1\right)$$