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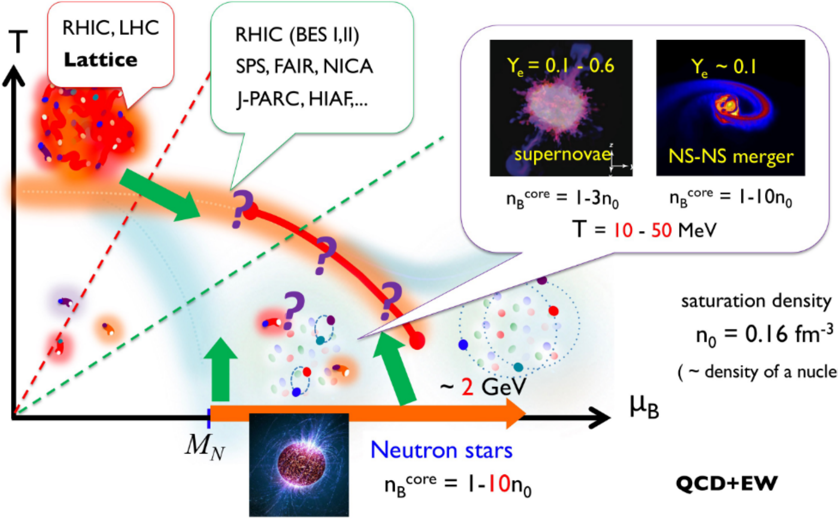
Color-superconductivity of asymptotically conformal quark matter as a portal between astrophysics and heavy ions collisions

Oleksii Ivanytskyi

OI & David Blaschke, PRD 2022
OI & David Blaschke, EPJ A 2022
OI & David Blaschke, Particles 2022

19th XQCD, Coimbra, 27 July 2023

Phase diagram of strongly interacting matter



RHIC, LHC
Lattice

RHIC (BES I,II)
SPS, FAIR, NICA
J-PARC, HIAF,...

$Y_e = 0.1 - 0.6$
supernovae

$Y_e \sim 0.1$
NS-NS merger

$n_B^{core} = 1\text{-}3n_0$

$n_B^{core} = 1\text{-}10n_0$

T = 10 - 50 MeV

T

~ 2 GeV

μ_B

M_N

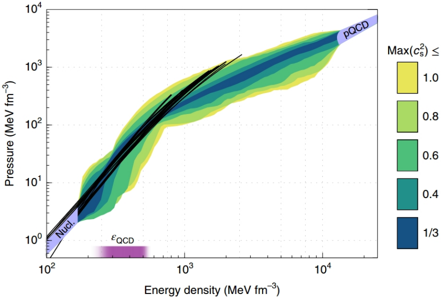
Neutron stars
$n_B^{core} = 1\text{-}10n_0$

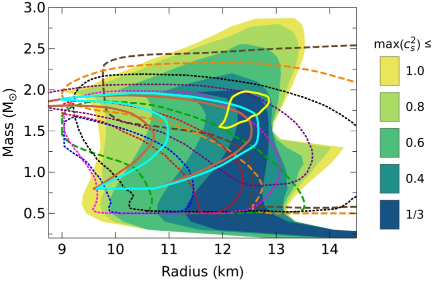
saturation density
$n_0 = 0.16\ \mathrm{fm}^{-3}$
(~ density of a nucleus)

QCD+EW

Figure from T. Kojo arXiv:1912.05326 [nucl-th]

pQCD vs $2M_\odot$ compact stars





E. Annala, T. Gorda, A. Kurkela, J. Nättilä, A. Vuorinen, Nature Physics 16, 907 (2020)

Existence of parameterization consistent with pQCD and $2M_\odot$

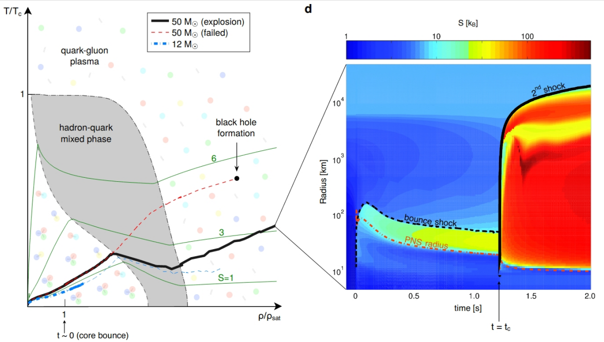
$\Downarrow$

Argument in favor of quark cores?

Quark matter in supernova explosions

- $2M_\odot$ stars formation? (accretion is too slow)
- Supernovae with progenitor mass $\sim 50\ M_\odot$
- Quark-hadron transition stabilizes collapse

T. Fischer et al., Nature Astronomy 2, 980–986 (2018)



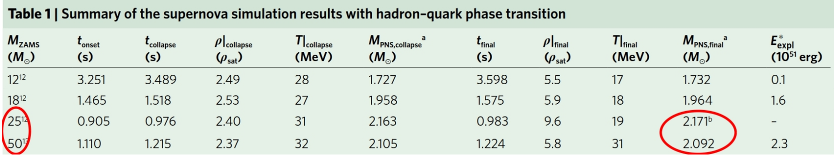


Table 1 | Summary of the supernova simulation results with hadron-quark phase transition

M_{ZAMS} $(M_\odot)$	t_{onset} (s)	$t_{collapse}$ (s)	$\rho\vert_{collapse}$ (ρ_{sat})	$T\vert_{collapse}$ (MeV)	$M_{PNS,collapse}$[a] $(M_\odot)$	t_{final} (s)	$\rho\vert_{final}$ (ρ_{sat})	$T\vert_{final}$ (MeV)	$M_{PNS,final}$[a] $(M_\odot)$	E^*_{expl} (10^{51} erg)
12[12]	3.251	3.489	2.49	28	1.727	3.598	5.5	17	1.732	0.1
18[12]	1.465	1.518	2.53	27	1.958	1.575	5.9	18	1.964	1.6
25[12]	0.905	0.976	2.40	31	2.163	0.983	9.6	19	2.171[b]	-
50[13]	1.110	1.215	2.37	32	2.105	1.224	5.8	31	2.092	2.3

Deconfinement is a supernova engine for massive blue giants

Hyperon puzzle

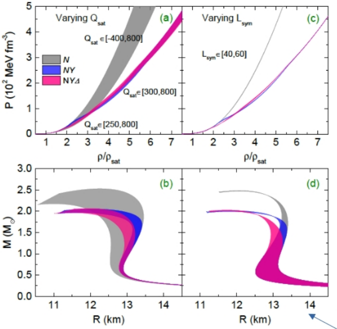


FIG. 4. EoS models and MR relations for N, NY, and $NY\Delta$ compositions of stellar matter. The bands are generated by varying the parameters Q_{sat} [MeV] (a, b) and L_{sym} [MeV] (c, d). The ranges of Q_{sat} and L_{sym} allowed by χEFT and maximum mass constraints are indicated in the figures.

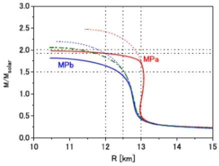


FIG. 7. Neutron-star masses as a function of the radius R. Solid (dashed) curves are with (without) hyperon (Λ and Σ^-) mixing for ESC+MPa and ESC+MPb. The dot-dashed curve for MPb is with Λ mixing only. Also see the caption of Fig. 3.

Yamamoto et al., Phys.Rev.C 96 (2017) 06580; arXiv:1708.06163 [nucl-th]
Yamamoto et al., Eur. Phys. J. A 52 (2016) 19; arXiv:1510.06099 [nucl-th]
Ji & Sedrakian, Phys. Rev. C 100 (2019) 015809; arXiv:1903.06057 [astro-ph.HE]

Examples for realistic hadronic EoS which suggest a Berlin Wall is inferior to the line M = 2.0 M_sun

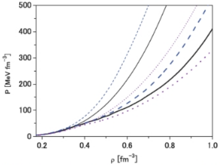


Fig. 8. Pressure P as a function of baryon density ρ. Thick (thin) curves are with (without) hyperon mixing. Solid, dashed and dotted curves are for MPa, MPa$^+$ and MPb.

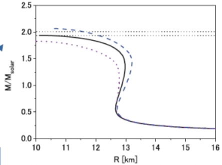


Fig. 9. Neutron-star masses as a function of the radius R. Solid, dashed and dotted curves are for MPa, MPa$^+$ and MPb. Two dotted lines show the observed mass $(1.97 \pm 0.04)M_\odot$ of J1614-2230.

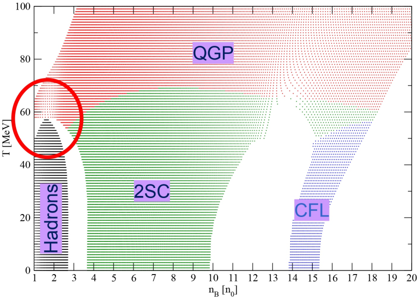
Hyperons soften EoS, prevent neutron stars from reaching $2M_\odot$

False quark dominance in hybrid quark-hadron EoS

- **Hadronic EoS consistent with astro (DDf4) + NJL model**

$$\Downarrow$$

False quark onset already @ T $\simeq$ 60 MeV



- **Hadron decays are energetically favorable**

$$\begin{array}{lcc} M_q \simeq 330 MeV & & M_{meson} > 2M_q \\ M_\omega = 783 MeV & \Rightarrow & quarks\ are\ too\ light \\ M_\rho = 775 MeV & & to\ be\ confined \end{array}$$

Effective quark "confinement" is needed

Model

$$\mathcal{L} = \overline{q}(i\partial\!\!\!/ - \hat{m})q - \mathcal{U} + \mathcal{L}_V + \mathcal{L}_D$$

- **Scalar & pseudoscalar interaction channels**

  $\mathcal{U}$ – χ-symmetric density functional (details below)

- **Vector-isoscalar interaction channel**

  $$\mathcal{L}_V = -G_V(\overline{q}\gamma_\mu q)^2$$

  (motivated by gluon exchange, stiff EoS needed to reach $2M_\odot$)

- **Diquark interaction channel**

  $$\mathcal{L}_D = G_D \sum_{A=2,5,7} (\overline{q} i\gamma_5\tau_2\lambda_A q^c)(\overline{q}^c i\gamma_5\tau_2\lambda_A q)$$

  (motivated by Cooper theorem, color superconductivity)

Relativistic density functional

$$\mathcal{U} = D_0 \left[(1+\alpha)\langle\bar{q}q\rangle_0^2 - (\bar{q}q)^2 - (\bar{q}i\vec{\tau}\gamma_5 q)^2\right]^{\varkappa}$$

- **Parameters**

  D_0 - dimensionfull coupling, controls interaction strength

  α - dimensionless constant, controls vacuum quark mass

  $\langle\bar{q}q\rangle_0$ - χ-condensate in vacuum (introduced for the sake of convenience)

$$\varkappa = 1/3 \qquad\qquad \varkappa = 1$$

$$\Downarrow \qquad\qquad \Downarrow$$

motivated by String Flip model — Nambu–Jona-Lasinio model

$$\mathcal{U}_{SFM} \propto \langle q^+q\rangle^{2/3}$$

$$\Sigma_{SFM} = \frac{\partial\mathcal{U}_{SFM}}{\partial\langle q^+q\rangle} \propto \langle q^+q\rangle^{-1/3} \propto separation$$

- **Dimensionality**

$$\begin{array}{l}[\mathcal{U}] = energy^4 \\ [\bar{q}q] = energy^3\end{array} \quad\Rightarrow\quad [D_0]_{\varkappa=1/3} = energy^2 = [string\ \ tension]$$

self energy $=$ **string tension** $\times$ **separation** $\Rightarrow$ **confinement**

Expansion around $\langle\bar{q}q\rangle$ and $\langle\bar{q}i\vec{\tau}\gamma_5 q\rangle=0$

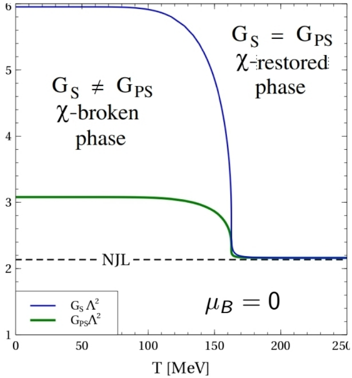
$$\mathcal{U} = \underbrace{\mathcal{U}_{MF}}_{0^{\text{th}}\ \text{order}} + \underbrace{\left(\bar{q}q - \langle\bar{q}q\rangle\right)\Sigma_S}_{1^{\text{st}}\ \text{order}} - \underbrace{G_S\left(\bar{q}q - \langle\bar{q}q\rangle\right)^2 - G_{PS}\left(\bar{q}i\vec{\tau}\gamma_5 q\right)^2}_{2^{\text{nd}}\ \text{order}} + \dots$$

- **Mean-field scalar self-energy**

$$\Sigma_S = \frac{\partial\mathcal{U}_{MF}}{\partial\langle\bar{q}q\rangle}$$

- **Effective medium dependent couplings**

$$G_S = -\frac{1}{2}\frac{\partial^2\mathcal{U}_{MF}}{\partial\langle\bar{q}q\rangle^2}, \quad G_{PS} = -\frac{1}{6}\frac{\partial^2\mathcal{U}_{MF}}{\partial\langle\bar{q}i\vec{\tau}\gamma_5 q\rangle^2}$$



Expansion around $\langle\overline{q}q\rangle$ and $\langle\overline{q}i\vec{\tau}\gamma_5 q\rangle=0$

$$\mathcal{U} = \underbrace{\mathcal{U}_{MF}}_{0^{\text{th}}\ \text{order}} + \underbrace{(\overline{q}q - \langle\overline{q}q\rangle)\,\Sigma_{MF}}_{1^{\text{st}}\ \text{order}} - \underbrace{G_S\,(\overline{q}q - \langle\overline{q}q\rangle)^2 - G_{PS}\,(\overline{q}i\vec{\tau}\gamma_5 q)^2}_{2^{\text{nd}}\ \text{order}} + \ldots$$

- **Mean-field scalar self-energy**

$$\Sigma_S = \frac{\partial\mathcal{U}_{MF}}{\partial\langle\overline{q}q\rangle}$$

- **Effective medium dependent couplings**

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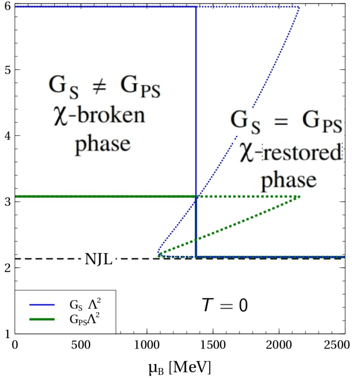


Comparison to NJL model

$$\mathcal{L} = \bar{q}(i\partial\!\!\!/ - \underbrace{(m+\Sigma_S)}_{\textbf{effective mass m}^*})q + G_S(\bar{q}q)^2 + G_{PS}\,(\bar{q}i\vec{\tau}\gamma_5 q)^2 + \dots + \mathcal{L}_V + \mathcal{L}_D$$

- **Similarities:**

  - current-current interaction
  - (pseudo)scalar, vector, diquark, ... channels

- **Differences:**

  - high m^* at low T, $\mu \Rightarrow$ **"confinement"**

$$\langle \bar{q}q \rangle = \langle \bar{q}q \rangle_0 \;\Rightarrow\; m^* = m - \frac{2G_0}{3\alpha^{2/3}\langle \bar{q}q \rangle_0^{1/3}}$$

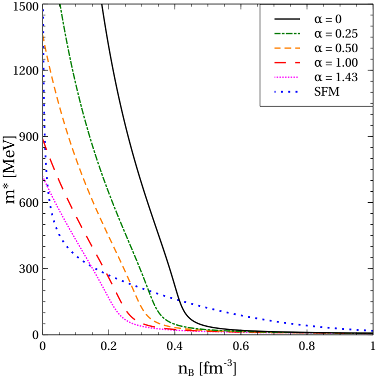
$$\Downarrow$$

$$m^* \to \infty \;at\; \alpha \to 0$$

  - medium dependent couplings:

$$\begin{array}{lllll} \text{low} & T,\ \mu, \Rightarrow & G_S \neq G_{PS} & \Rightarrow & \chi\text{-broken} \\ \text{high} & T,\ \mu, \Rightarrow & G_S = G_{PS} & \Rightarrow & \chi\text{-symmetric} \end{array}$$

$T = 0$



Model setup

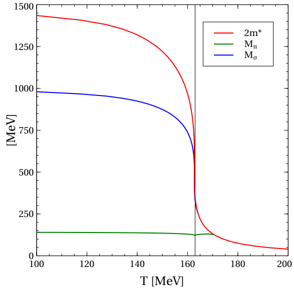
- **(Pseudo)scalar interaction channels (chiral condensate & π, σ mesons)**

m [MeV]	Λ [MeV]	α	$D_0\Lambda^{-2}$
4.2	573	1.43	1.39

M_π [MeV]	F_π [MeV]	M_σ [MeV]	$\langle\bar{l}l\rangle_0^{1/3}$ [MeV]
140	92	980	-267

Pseudocritical temperature

$$\boldsymbol{T_c = 163 \text{ MeV}}$$



- low T: $2m_{quark} > M_\pi, M_\sigma$
  (stable mesons, confined quarks)
- high T: $2m_{quark} < M_\pi, M_\sigma$
  (unstable mesons, deconfined quarks)

- **Vector repulsion channel (ω-meson)**

$$M_\omega = 783 \text{ MeV} \Rightarrow \ \eta_V \equiv \frac{G_V}{G_S} = 0.452$$

- **Diquark pairing channel (Fierz transformation)** $\eta_D \equiv \frac{G_D}{G_S} = 1.5\eta_V = 0.678$

High density asymptotic at constant G_V and G_D

- $\mathbf{G_V \neq 0}$

$$p \to G_V \langle q^+ q \rangle^2 \propto \mu_B^2, \quad c_S^2 \to 1$$

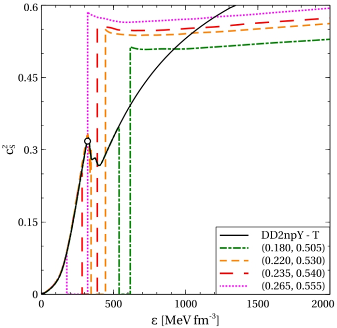
- $\mathbf{G_V = 0}$, $\mathbf{G_D \neq 0}$

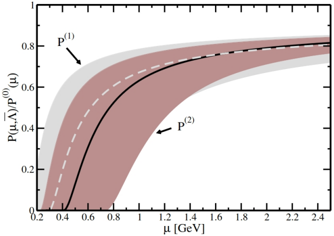
$$p \to G_D |\langle \overline{q}^c i\gamma_5 \tau_2 \lambda_2 q \rangle^2| \propto \mu_B^6, \quad c_S^2 \to \frac{1}{5}$$

- **Perturbative QCD**

$$p \to 0.8 p_{SB} \propto \mu_B^4, \quad c_S^2 \to \frac{1}{3}$$

A. Kurkela, P. Romatschke, A. Vuorinen, Phys. Rev. D 81, (2010)





Medium dependent couplings?

High density asymptotic at constant G_V and G_D

- $\mathbf{G_V \neq 0}$

$$p \to G_V \langle q^+ q \rangle^2 \propto \mu_B^2, \quad c_S^2 \to 1$$

- $\mathbf{G_V = G_I = 0,\ G_D \neq 0}$

$$p \to G_D |\langle \overline{q}^c i\gamma_5 \tau_2 \lambda_2 q \rangle^2| \propto \mu_B^6, \quad c_S^2 \to \frac{1}{5}$$

- **Perturbative QCD**

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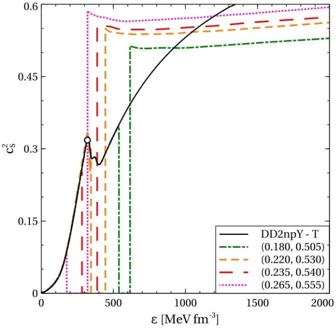
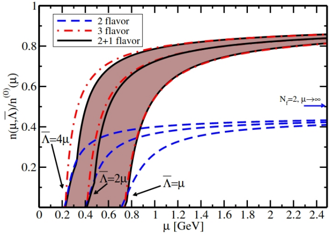




Medium dependent couplings?

Effective repulsion from non-perturbative gluon exchange

- **Fock energy with $D_{gluon} \propto \frac{1}{k^2 - M^2_{gluon}}$**

$$\varepsilon_{repulsion} = G_V \langle q^+ q\rangle^2 \quad with \quad G_V = \frac{4\pi\alpha_s/3}{9M^2_{gluon} + 8k_F^2}, \quad \alpha_s \; - \; frozen$$

Y. Song, G. Baym, T. Hatsuda, and T. Kojo Phys. Rev. D 100, 034018 (2019)

- **Density dependent coupling**

$$k_F = \left(\frac{6\pi^2\langle q^+ q\rangle}{2\cdot 2\cdot 3}\right)^{1/3} \quad \Rightarrow \quad G_V = \frac{G_V^{vacuum}}{1 + \frac{8}{9M^2_{gluon}}\left(\frac{\pi^2\langle q^+ q\rangle}{2}\right)^{2/3}}$$

- **High density asymptotic**

$$\varepsilon_{repulsion} \propto \langle q^+ q\rangle^{4/3} \quad \Rightarrow \quad c_S^2 \rightarrow \frac{1}{3}$$

Medium dependent couplings

$$G_V(n_V) = G_V^{vacuum} \cdot f(n_V), \qquad G_D(n_D) = G_D^{vacuum} \cdot f(n_D)$$

$$n_V = \langle q^+ q \rangle, \qquad n_D = |\langle \overline{q}^c i\tau_2 \gamma_5 \lambda_2 q \rangle|$$

- **Medium dependence**

$$f(n) = \left[1 + \frac{8}{9M_{gluon}^2} \left(\frac{\pi^2 n}{2} \right)^{2/3} \right]^{-1}$$

- **Rearrangement terms** (needed for thermodynamic consistency)

$$\langle f^+ f \rangle = -\frac{\partial \Omega}{\partial \mu_f}$$

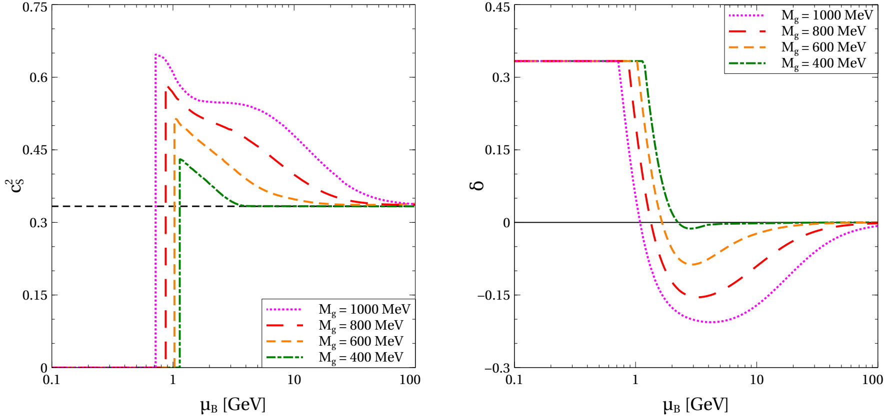
$$\Downarrow$$

$$\Omega \to \Omega - \Theta_V + \Theta_D \quad with \quad \Theta_i = \int_0^{n_i} dn \; n^2 \frac{\partial G_i(n)}{\partial n}$$

Asymptotically conformal EoS (symmetric matter @ T=0)

- **Conformal matter**

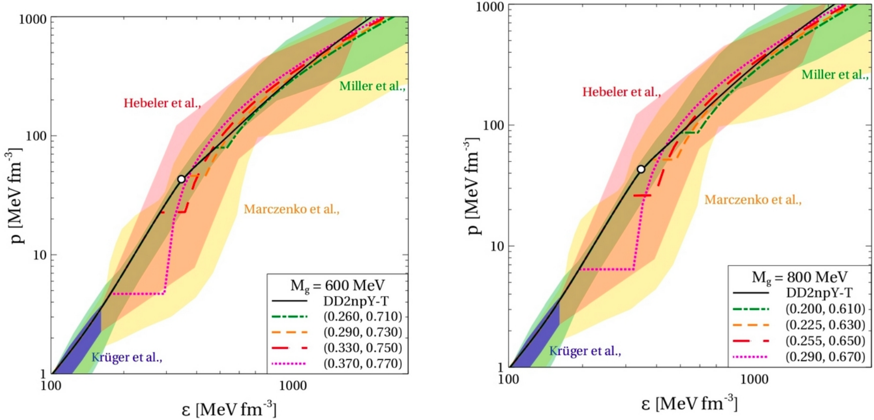
$$\varepsilon \propto \mu^{d+1} \quad \Rightarrow \quad c_S^2 \to \frac{1}{3}, \quad \delta \equiv \frac{T^\mu_\mu}{3\varepsilon} = \frac{1}{3} - \frac{p}{\varepsilon} \to 0$$



Conformality is reached at $\mu_B/3 \gg M_{gluon}$

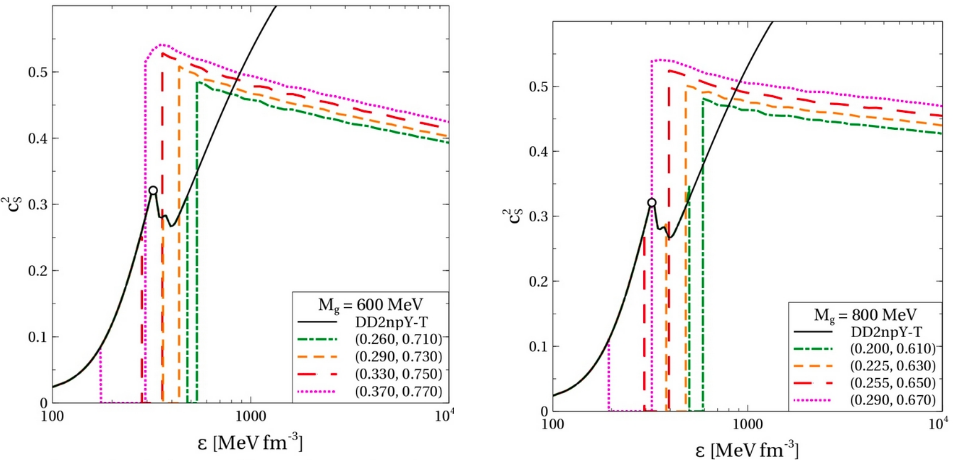
Asymptotically conformal EoS of neutron stars

- **Setup:** electric neutrality, β-equilibrium, Maxwell construction with DD2 EoS
- **Scanning over** $\eta_V = \frac{G_V}{G_S}\Big|_{vacuum}$ **and** $\eta_D = \frac{G_D}{G_S}\Big|_{vacuum}$

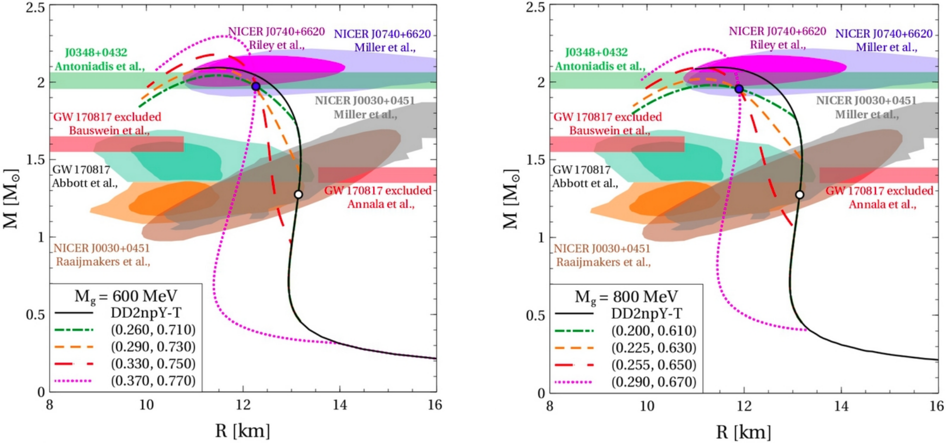


The ω-meson value of η_V and the Fierz value of η_D prefer early deconfinement?

Speed of sound

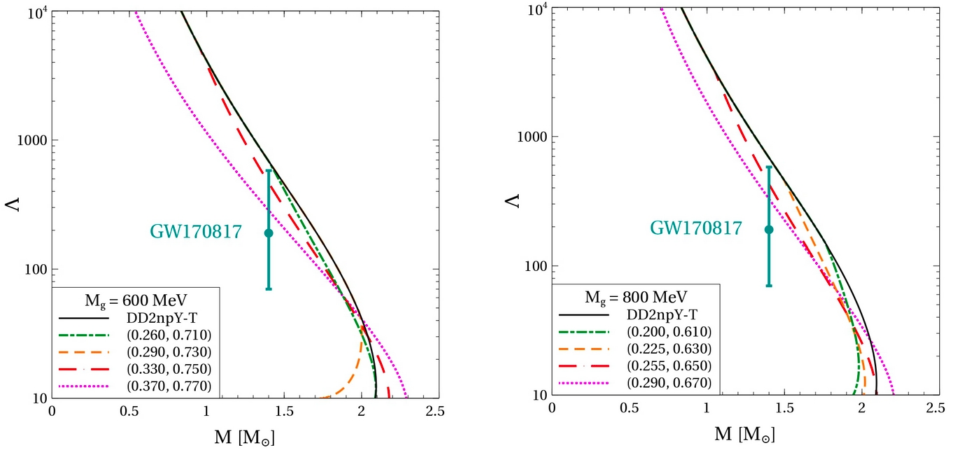


Mass-radius diagram



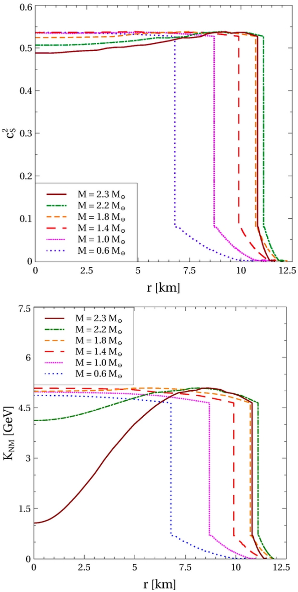
Observational data prefer early deconfinement?

Tidal deformability



Observational data prefer early deconfinement?

Conformality in neutrons stars?



- **Speed of sound**

  $c_S^2 = \frac{1}{3}$ in conformal matter

- **Compression modulus**

  $$K_{NM} = 9n_B^2 \frac{\partial^2}{\partial n_B^2} \frac{E}{A}$$

  $K_{NM} = -\frac{3\mu_B}{2} < 0$ in conformal matter

Both c_S^2 and K_{NM} contradict conformality in neutron stars

Phase diagram (Q-neutral, β-equilibrium, $M_{gluon} \to \infty$)

- **Normal quark matter**

  2 spin × 2 flavor × 3 color $=12$

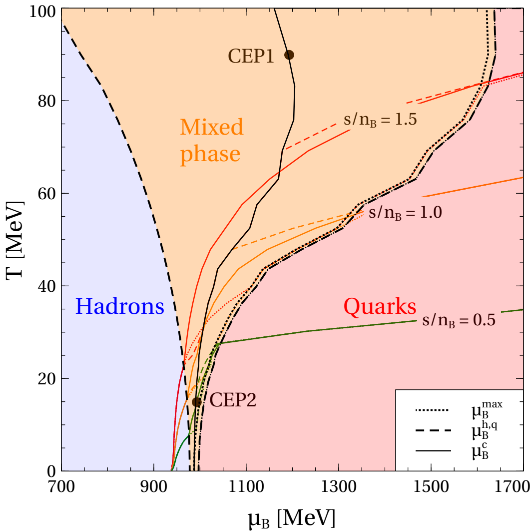
- **2SC quark matter**

  2 spin × 2 flavor × 1 color$+1=5$

Quark pairing reduces number of quark states

$\Downarrow$

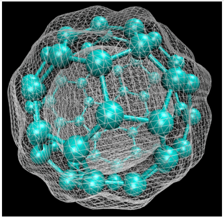
requires higher T along adiabat

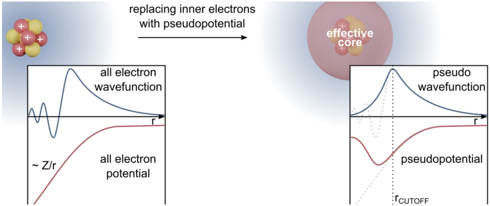


Conclusions

- Effective "confining" chiral model with color superconductivity is derived based on the χ-symmetric density functiobal
- Medium dependent quark-meson couplings provide conformal limit
- Agreement with the observational data on compact stars implies early onset of quark matter
- Neutron star matter is unlikely to be conformal
- Deconfinement to color superconducting quark matter leads to the temperature growth along the adiabats
- Perspectives: supernovae
  neutron star mergers
  unified quark-hadron EoS
  ...

Context: Density functional theory





(Dirac)Brueckner-Hartree-Fock T-, G-matrix based theories

$$\Updownarrow$$

Density functional theory

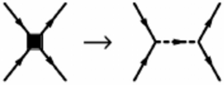
- **Many body problems**
- **Quantum chemistry**
- **Skyrme-type models for nuclear physics**
- **String Flip model for quark matter**
- **...**

Bosonization

- **Hubbard-Stratonovich transformation**

$$\exp\left[\int dx\ G(\bar{q}\hat{\Gamma}q)^2\right] = \int [D\phi] \exp\left[-\int dx \left(\frac{\phi^2}{4G} + \phi\bar{q}\hat{\Gamma}q\right)\right]$$

- **Vertexes:** $\hat{\Gamma}_S = 1 \ \Rightarrow$ scalar-isoscalar σ-field
  $\hat{\Gamma}_{PS} = i\gamma^5\vec{\tau} \ \Rightarrow$ pseuscalar-isoscalar $\vec{\pi}$-field
  $\hat{\Gamma}^{\mu}_{V} = \gamma^{\mu} \ \Rightarrow$ vector-isoscalar ω^{μ}-field
  $\hat{\Gamma}^{\mu}_{I} = \gamma^{\mu}\vec{\tau} \ \Rightarrow$ vector-isovector $\vec{\rho}^{\mu}$-field
  $\hat{\Gamma}^{A}_{D} = i\gamma^5\lambda_A\tau_2 \ \Rightarrow$ scalar diquark Δ_A-field



- **Bosonized Lagrangian ($m^* = m + \Sigma_S$ - effective mass, $\mathbf{Q}^{\mathbf{T}} = (\mathbf{q}\ \mathbf{q}^{\mathbf{c}})/\sqrt{2}$)**

$$\mathcal{L} + q^{+}\hat{\mu}q = \overline{Q}\hat{S}^{-1}Q - \frac{\sigma^2}{4G_S} - \frac{\vec{\pi}^2}{4G_{PS}} + \frac{\omega^2}{4G_V} + \frac{\vec{\rho}^2}{4G_I} - \frac{\Delta_A\Delta_A^*}{4G_D} - \mathcal{U}_{MF} + \langle\bar{q}q\rangle(\Sigma_S + \sigma)$$

$$\hat{S}^{-1} = \begin{pmatrix} \hat{\mathcal{S}}^{-1}_{+} & i\Delta_A\gamma_5\tau_2\lambda_A \\ i\Delta^*_A\gamma_5\tau_2\lambda_A & \hat{\mathcal{S}}^{-1}_{-} \end{pmatrix}, \quad \hat{\mathcal{S}}^{-1}_{\pm} = i\not{\partial} - m^* - \sigma - i\gamma^5\vec{\pi}\cdot\vec{\tau} \pm (\gamma_0\hat{\mu} + \not{\omega} + \vec{\not{\rho}}\cdot\vec{\tau})$$

Mean field

- **Field equations for σ and $\vec{\pi}$**

$$\begin{cases} \sigma = 2G_S(\langle\overline{q}q\rangle - \overline{q}q) \\ \vec{\pi} = -2G_{PS}\overline{q}i\vec{\tau}\gamma_5 q \end{cases} \Rightarrow \langle\sigma\rangle = \langle\vec{\pi}\rangle = 0 \Rightarrow \sigma, \vec{\pi} - \textbf{beyond MF}$$

comment: $\langle\sigma\rangle = 0$ does not assume χ-symmetry since $\langle\overline{q}q\rangle \neq 0$

- **Thermodynamic potential**

$$\langle\omega_\mu\rangle = \delta_{\mu 0}\omega, \quad \langle\rho^a_\mu\rangle = \delta_{\mu 0}\delta_{a3}\rho, \quad |\langle\Delta_A\rangle| = \delta_{A2}\Delta$$

$$\Downarrow$$

$$\Omega = -\frac{1}{2\beta V} Tr\ln(\beta\hat{S}^{-1}) - \frac{\omega^2}{4G_V} - \frac{\rho^2}{4G_I} + \frac{\Delta^2}{4G_D} + \mathcal{U}_{MF} - \langle\overline{q}q\rangle\Sigma_S$$

- **Vector fields, diquark gap, χ-condensate**

$$\frac{\partial\Omega}{\partial\omega} = 0, \quad \frac{\partial\Omega}{\partial\rho} = 0, \quad \frac{\partial\Omega}{\partial\Delta} = 0, \quad \langle\overline{q}q\rangle = \sum_f \frac{\partial\Omega}{\partial m_f}$$

Superconductivity onset

- **Single quark energy and distribution**

$$E_f^{\pm} = sgn(E_f \mp \mu_f)\sqrt{(E_f \mp \mu_f)^2 + \Delta^2}$$

$$f_f^{\pm} = [\exp(E_f^{\pm}/T) + 1]^{-1}$$

- **Gap equation**

$$\frac{\partial\Omega}{\partial\Delta} = \frac{\Delta}{2G_D} - 2\Delta \sum_{f,a=\pm} \int \frac{d\mathbf{k}}{(2\pi)^3} \frac{1-2f_f^a}{E_f^a} = 0$$

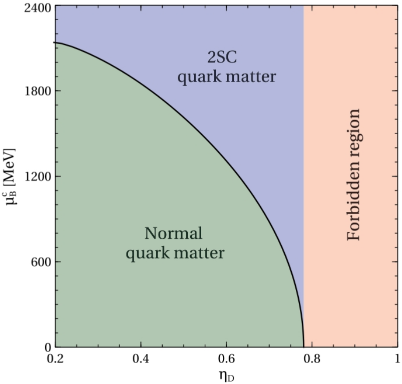
$$\Downarrow$$

two solutions : $\Delta = 0$ **or** $\Delta \neq 0$

- **Two solutions coincide** $\Rightarrow$ **SC onset**

$$\left.\frac{\partial^2\Omega}{\partial\Delta^2}\right|_{\Delta=0} = 0 \quad \Rightarrow \quad \mu_B = \mu_B(G_D)$$

$T = 0$



No vacuum superconductivity

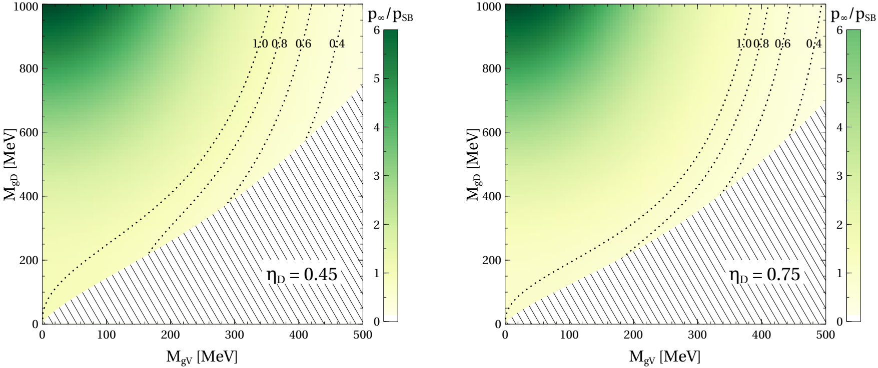
$$\Downarrow$$

$$\eta_D \lesssim 0.78$$

(agrees with the Fierz value)

Asymptotic of pressure

- **Perturbative QCD:** $p/p_{SB} \simeq 0.8$, $\delta \to +0$ in symmetric matter
- **Scanning over M_{gD} vs M_{gV} at different η_D**



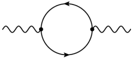
hatchet region: $\delta \to -0$ - contradiction with pQCD

$M_{gV} \simeq 400$ MeV and $M_{gV} \simeq 800$ MeV?

How to define G_V, G_I and G_D?

- **Mesonic correlations**

$$\mathcal{L} = \ldots + \bar{q}(\sigma + i\gamma_5\vec{\pi}\cdot\vec{\tau} + \not{\omega} + \vec{\not{\rho}}\cdot\tau)q - \frac{\sigma^2}{4G_S} - \frac{\vec{\pi}^2}{4G_{PS}} + \frac{\omega^2}{4G_V} + \frac{\vec{\rho}^2}{4G_I}$$

$D_i^{-1}(p^2) = \frac{1}{2G_i} -$  - one-loop mesonic propagator

$$D_i^{-1}(M_i^2) = 0 \;\Rightarrow\; \text{mesonic masses}$$

- **Fierz transformation** - rearrangement of Dirac, color and flavor indexes

$$\begin{aligned}(\gamma^\mu)_{mn}(\gamma_\mu)_{m'n'} = \mathbf{1}_{mn'}\mathbf{1}_{m'n} &+ (i\gamma_5)_{mn'}(i\gamma_5)_{mn'}\\ &-\frac{1}{2}(\gamma^\mu)_{mn'}(\gamma_\mu)_{m'n}\\ &-\frac{1}{2}(\gamma^\mu\gamma_5)_{mn'}(\gamma_\mu\gamma_5)_{m'n}\end{aligned}$$

$$\mathbf{1}_{ij}\mathbf{1}_{kl} = \frac{1}{3}\mathbf{1}_{il}\mathbf{1}_{kj} + \frac{1}{2}(\tau_a)_{il}(\tau_a)_{kj}$$

$$\lambda_\alpha^{ab}\lambda_\alpha^{a'b'} = \frac{16}{9}\mathbf{1}_{ab'}\mathbf{1}_{a'b} - \frac{1}{3}\lambda_\alpha^{ab'}\lambda_\alpha^{a'b}$$

coefficients - proportional to couplings

$$\mathbf{G_S : G_V : G_I : G_D = 1 : 0.5 : 0.5 : 0.75}$$