XII International Conference on New Frontiers in Physics

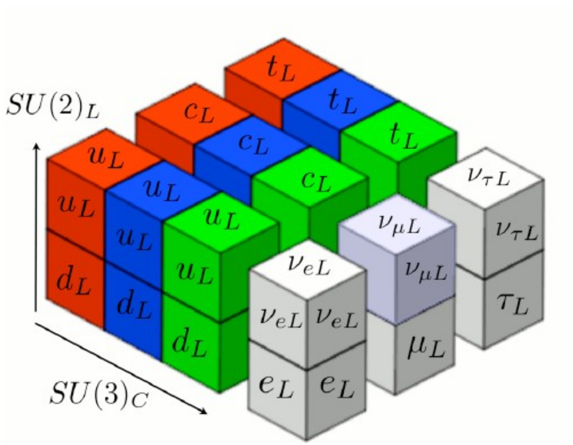
10-23 July 2023, Kolymbari

A new perspective on the flavor problem

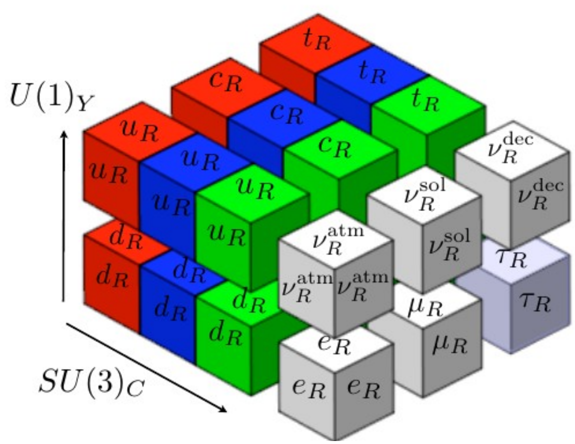
Davide Meloni
Dipartimento di Matematica e Fisica, Roma Tre

The Standard Model of Particle Physics

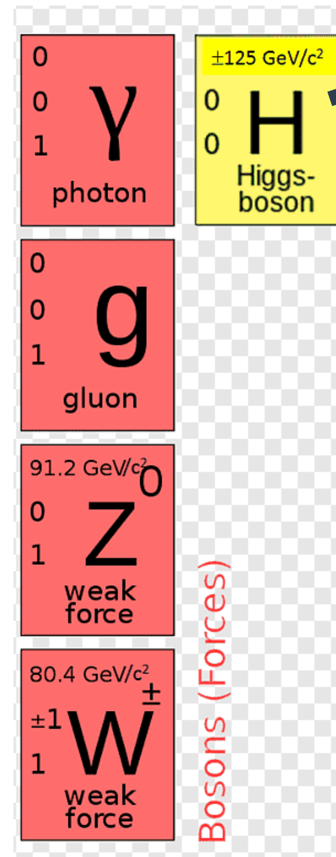
Left-handed



Right-handed



Gauge boson sector

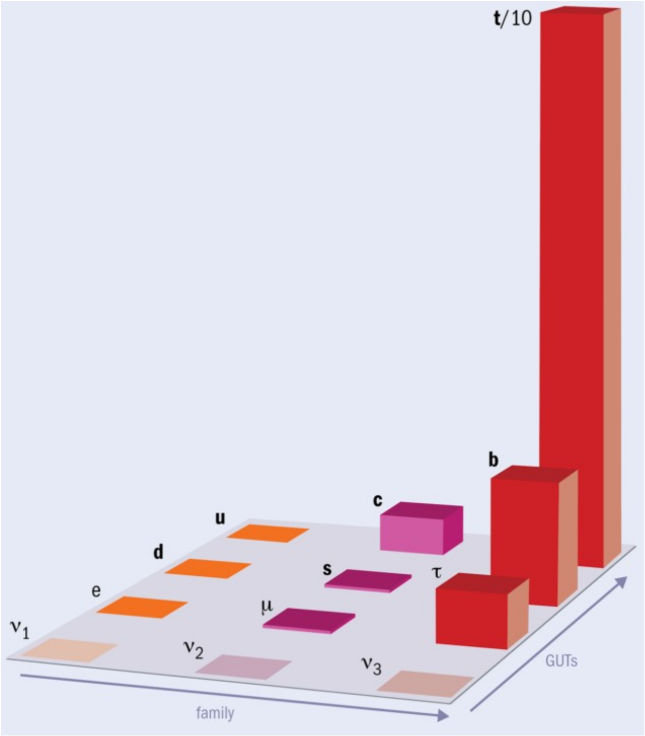


Scalar sector

S.King, talk at Bethe Forum on Modular Flavor Symmetries

The Flavor Problem

Mass hierarchies



$$m_d \ll m_s \ll m_b\,,\ \frac{m_d}{m_s} = 5.02 \times 10^{-2}\,,$$

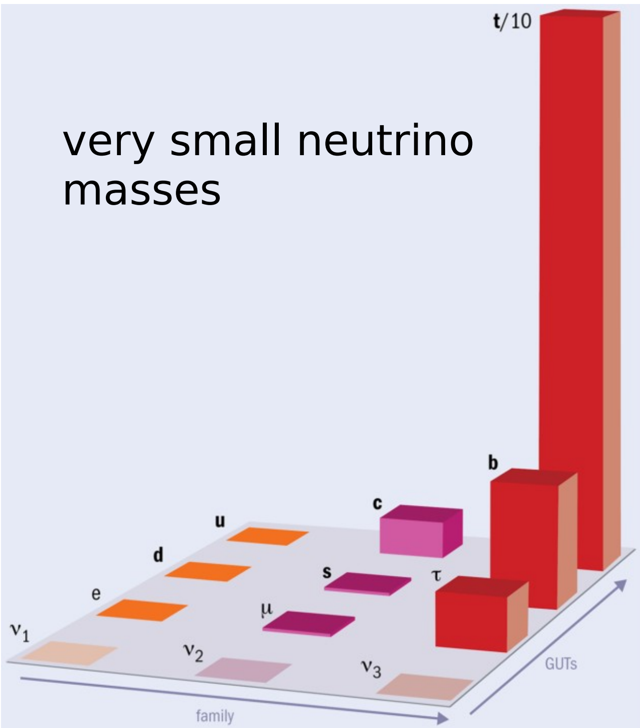
$$m_u \ll m_c \ll m_t\,,\ \frac{m_u}{m_c} = 1.7 \times 10^{-3}\,,$$

$$\frac{m_s}{m_b} = 2.22 \times 10^{-2}\,,\ m_b = 4.18\ \text{GeV};$$

$$\frac{m_c}{m_t} = 7.3 \times 10^{-3}\,,\ m_t = 172.9\ \text{GeV};$$

The Flavor Problem

Mass hierarchies



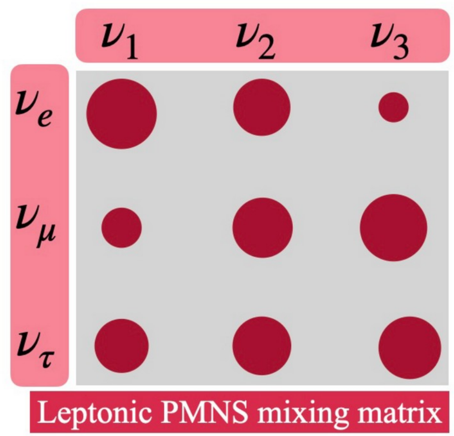
$$m_d \ll m_s \ll m_b \,, \quad \frac{m_d}{m_s} = 5.02 \times 10^{-2} \,,$$

$$m_u \ll m_c \ll m_t \,, \quad \frac{m_u}{m_c} = 1.7 \times 10^{-3} \,,$$

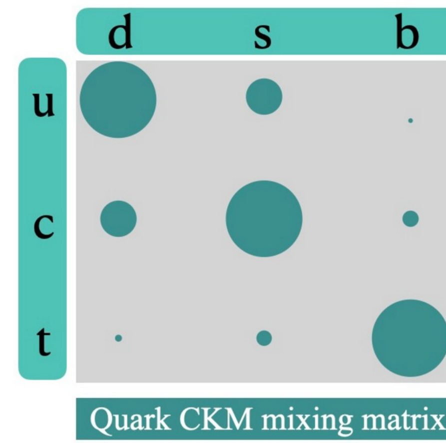
$$\frac{m_s}{m_b} = 2.22 \times 10^{-2} \,, \quad m_b = 4.18 \ \text{GeV};$$

$$\frac{m_c}{m_t} = 7.3 \times 10^{-3} \,, \quad m_t = 172.9 \ \text{GeV};$$

Fermion mixing



all mixing are large but the 13 element



almost a diagonal matrix

Suggested solutions

* Smallness of neutrino masses:

See-saw

$$\mathcal{M} = \begin{bmatrix} m_M^L & m_D \\ m_D & m_M^R \end{bmatrix}$$

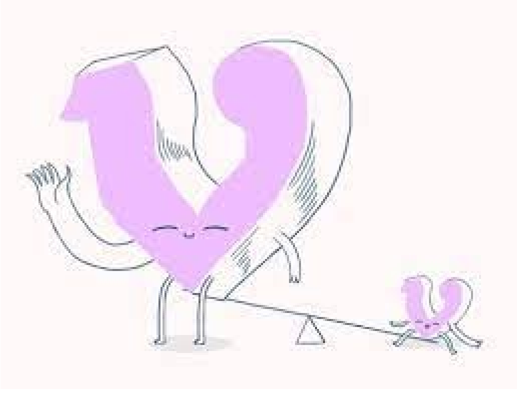
$$m_{light} \sim \frac{m_D^2}{M_M^R}$$

No clue on mixing !

Suggested solutions

* Smallness of neutrino masses:

See-saw



$$\mathcal{M} = \begin{bmatrix} m_M^L & m_D \\ m_D & m_M^R \end{bmatrix}$$

$$m_{light} \sim \frac{m_D^2}{M_M^R}$$

No clue on mixing !

* Hierarchical Pattern

Froggatt-Nielsen mechanism

$$L \sim \overline{\Psi}_L H \Psi_R \left(\frac{\theta}{\Lambda}\right)^n \rightarrow e^{(-q_L + q_H + q_R + n * q_\theta)}$$

Suggested solutions

* Smallness of neutrino masses:

See-saw

$$\mathcal{M} = \begin{bmatrix} m_M^L & m_D \\ m_D & m_M^R \end{bmatrix}$$

$$m_{light} \sim \frac{m_D^2}{M_M^R}$$

No clue on mixing !

* Hierarchical Pattern

Froggatt-Nielsen mechanism

$$L \sim \overline{\Psi}_L H \Psi_R \left(\frac{\theta}{\Lambda} \right)^n$$

Too many O(1) coefficients

Works better for small mixing

Suggested solutions

* Smallness of neutrino masses:

See-saw

$$\mathcal{M} = \begin{bmatrix} m_M^L & m_D \\ m_D & m_M^R \end{bmatrix}$$

$$m_{light} \sim \frac{m_D^2}{M_M^R}$$

No clue on mixing !

* Hierarchical Pattern

Froggatt-Nielsen mechanism

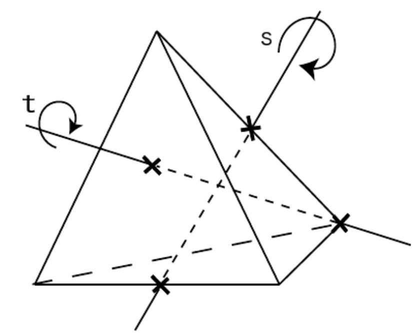
$$L \sim \overline{\Psi}_L H \Psi_R \left(\frac{\theta}{\Lambda}\right)^n$$

Too many O(1) coefficients

Works better for small mixing

* mixing angles

elegant explanation:
non-Abelian
discrete flavour symmetries



Complicated scalar sector

Modular Symmetry

We start from

Feruglio, 1706.08749

$$\Gamma(N)=\left\{\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL(2,Z), \begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} (Mod\, N)\right\}$$

the group of 2x2 matrices with integer entries
modulo N and determinant equals to one modulo N

Modular Symmetry

We start from

Feruglio, 1706.08749

$$\Gamma(N)=\left\{\begin{pmatrix} a & b \\ c & d \end{pmatrix}\in SL(2,Z),\begin{pmatrix} a & b \\ c & d \end{pmatrix}=\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}(Mod\,N)\right\}$$

the group of 2x2 matrices with integer entries modulo N and determinant equals to one modulo N

Γ(1)=SL(2, Z) = *special linear group* = the group of 2x2 matrices with integer entries and determinant equals to one, called ***homogeneous modular group*** Γ

Modular Symmetry

We start from

Feruglio, 1706.08749

$$\Gamma(N)=\left\{\begin{pmatrix}a & b\\ c & d\end{pmatrix}\in SL(2,Z),\begin{pmatrix}a & b\\ c & d\end{pmatrix}=\begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix}(Mod\,N)\right\}$$

the group of 2x2 matrices with integer entries
modulo N and determinant equals to one modulo N

Γ(1)=SL(2, Z) = *special linear group* = the group of 2x2 matrices with integer entries and determinant equals to one, called ***homogeneous modular group Γ***

Γ(N), N>=2 are infinite normal subgroups of Γ

Modular Symmetry

We start from

Feruglio, 1706.08749

$$\Gamma(N) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL(2,Z), \begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} (Mod\, N) \right\}$$

the group of 2x2 matrices with integer entries
modulo N and determinant equals to one modulo N

Γ(1)=SL(2, Z) = *special linear group* = the group of 2x2 matrices with integer entries and determinant equals to one, called ***homogeneous modular group Γ***

Γ(N), N>=2 are infinite normal subgroups of Γ

the group Γ(N) acts on the complex variable τ (Im τ >0)

$$\gamma\tau = \frac{a\tau + b}{c\tau + d}$$

Modular Symmetry

Important observation for N=1: a transformation characterized by parameters {a, b, c, d} is identical to the one defined by {-a, -b, -c, -d}

Γ(1) is isomorphic to **PSL(2, Z) = SL(2, Z)/{±1} = Γ**

inhomogeneous modular group **(or simply Modular Group)**

Modular Symmetry

Important observation for N=1: a transformation characterized by parameters {a, b, c, d} is identical to the one defined by {-a, -b, -c, -d}

Γ(1) is isomorphic to **PSL(2, Z) = SL(2, Z)/{±1} = Γ**

inhomogeneous modular group **(or simply Modular Group)**

<u>In addition:</u>

$$\overline{\Gamma}(2)=\Gamma(2)/\{1,-1\}$$

since 1 and -1 **cannot** be distinguished

$$\overline{\Gamma}(N)=\Gamma(N) \quad N>2$$

since 1 and -1 **can** be distinguished

Modular Symmetry

Important observation for N=1: a transformation characterized by parameters {a, b, c, d} is identical to the one defined by {-a, -b, -c, -d}

Γ**(1)** is isomorphic to **PSL(2, Z) = SL(2, Z)/{±1} =** $\bar{\Gamma}$

inhomogeneous modular group **(or simply Modular Group)**

In addition:

$$\bar{\Gamma}(2)=\Gamma(2)/\{1,-1\}$$

since 1 and -1 **cannot** be distinguished

$$\bar{\Gamma}(N)=\Gamma(N) \qquad N>2$$

since 1 and -1 **can** be distinguished

Finite Modular Group:

$$\Gamma_N=\frac{\bar{\Gamma}}{\bar{\Gamma}(N)}$$

Modular Symmetry

Generators of Γ_N : elements S and T satisfying

$$S^2 = 1, \quad (ST)^3 = 1, \quad T^N = 1$$

$$S = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \quad T = \begin{pmatrix} 1 & 1 \\ 0 & -1 \end{pmatrix}$$

corresponding to:

$$\tau \xrightarrow{S} -\frac{1}{\tau} \qquad \tau \xrightarrow{T} \tau + 1$$

Modular Symmetry

Generators of Γ_N : elements S and T satisfying

$$S^2=1, \quad (ST)^3=1, \quad T^N=1$$

$$S=\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \quad T=\begin{pmatrix} 1 & 1 \\ 0 & -1 \end{pmatrix}$$

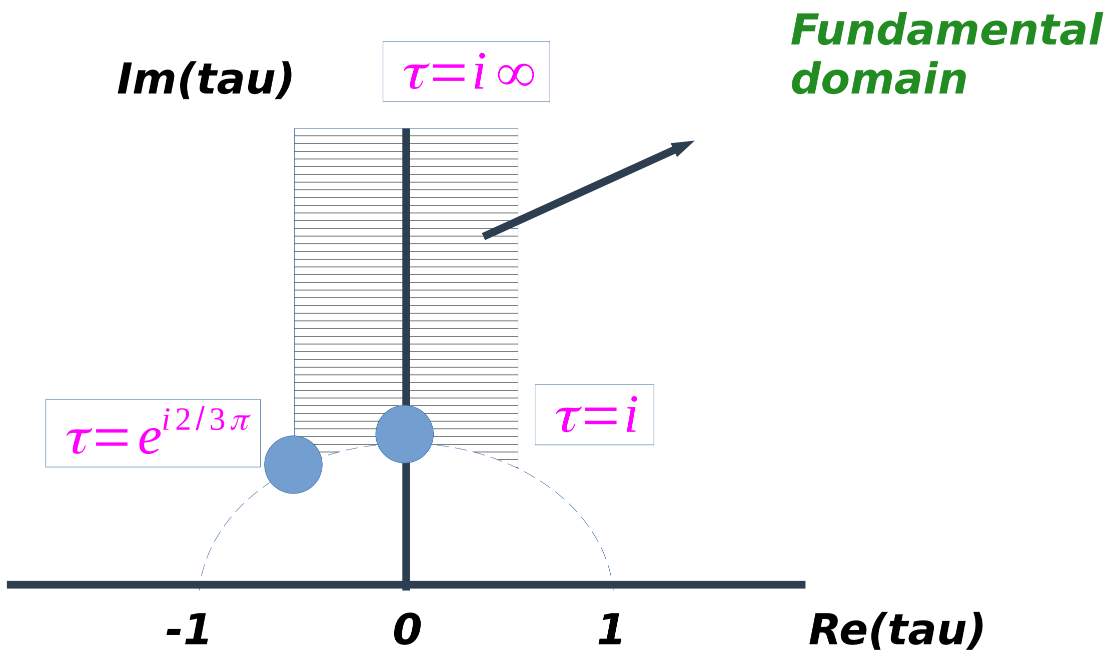
corresponding to:

$$\tau \xrightarrow{S} -\frac{1}{\tau} \qquad \tau \xrightarrow{T} \tau+1$$

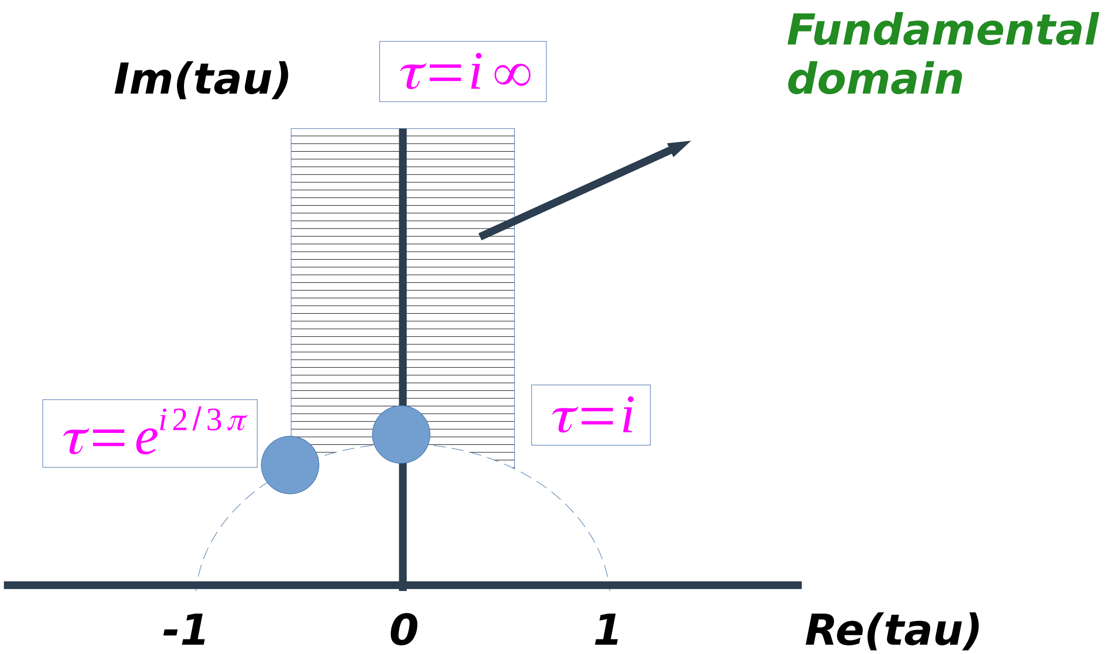
residual symmetry

modular invariance completely broken everywhere but at ***three*** *fixed points*	$\tau=i$	$\tau \xrightarrow{S} -\frac{1}{\tau}$	Z_4^S
	$\tau=e^{i2/3\pi}$	$\tau \xrightarrow{ST} -\frac{1}{\tau+1}$	$Z_2^{ST} \, x \, Z_2^{S^2}$
	$\tau=i\infty$	$\tau \xrightarrow{T} \tau+1$	$Z^T \, x \, Z_2^{S^2}$

Modular Symmetry



Modular Symmetry



relevant for model building:

for N ≤ 5, the finite modular groups Γ_N are isomorphic to non-Abelian discrete groups

$\Gamma_2 \simeq S_3$ $\Gamma_3 \simeq A_4$ $\Gamma_4 \simeq S_4$ $\Gamma_5 \simeq A_5$

Then the question is: why Modular Symmetry ?

Modular Forms

Modular Forms:

holomorphic functions of the complex variable τ with well-defined transformation properties under the group **Γ(N)**

$$f(\gamma\tau)=(c\tau+d)^k f(\tau), \qquad \gamma=\begin{pmatrix} a & b \\ c & d \end{pmatrix}\in\Gamma(N)$$

k = weigth, N = level

Modular Forms:

holomorphic functions of the complex variable τ with well-defined transformation properties under the group **Γ(N)**

$$f(\gamma\tau)=(c\tau+d)^k f(\tau), \qquad \gamma=\begin{pmatrix} a & b \\ c & d \end{pmatrix}\in\Gamma(N)$$

k = weigth, N = level

K < 0:
no modular forms

K = 0:
constant functions

K > 0:
linear space of finite dimension

N	$d_k(\Gamma(N))$	Γ_N
2	$k/2+1$	S_3
3	$k+1$	A_4
4	$2k+1$	S_4
5	$5k+1$	A_5

R. C. Gunning, Lectures on Modular Forms, Princeton, New Jersey USA, Princeton University Press 1962

Model Building

Key points:

1. Modular forms of weight 2k and level N ≥ 2 are invariant, up to the factor $(c\tau + d)^k$ under **Γ(N)** but they transform under Γ_N !

$$f_i(\gamma\tau) = (c\tau + d)^k \rho(\gamma)_{ij} f_j(\tau)$$

representative element of Γ_N

unitary representation of Γ_N

Model Building

<u>Key points:</u>

1. Modular forms of weight 2k and level N ≥ 2 are invariant, up to the factor $(c\tau + d)^k$ under **Γ(N)** but they transform under Γ_N !

$$f_i(\gamma\tau) = (c\,\tau + d)^k \rho(\gamma)_{ij} f_j(\tau)$$

representative element of Γ_N

unitary representation of Γ_N

2. in addition, one assumes that the fields of the theory χ_i transforms non-trivially under Γ_N

$$\chi(x)_i \rightarrow (c\,\tau + d)^{-k_i} \rho(\gamma)_{ij} \chi(x)_j$$

not modular forms !
No restrictions on ki

Model Building

Building blocks:

1. Modular forms and fields: $L_{eff} \in Y(\tau) \times \chi^{(1)} \dots \chi^{(n)}$

Yukawas are modular forms

Model Building

Building blocks:

1. Modular forms and fields: $L_{eff} \in Y(\tau) \times \chi^{(1)} \ldots \chi^{(n)}$

Yukawas are modular forms

2. Invariance under modular transformation requires:

$$k = \Sigma_i k_i$$

$$\rho_f \otimes \rho_{\chi_1} \otimes \ldots \otimes \rho_{\chi_n} \supset I$$

only few terms allowed in the potential

Model Building

Building blocks:

1. Modular forms and fields: $L_{eff} \in Y(\tau) \times \chi^{(1)} \ldots \chi^{(n)}$

Yukawas are modular forms

2. Invariance under modular transformation requires:

$$k = \Sigma_i k_i$$

$$\rho_f \otimes \rho_{\chi_1} \otimes \ldots \otimes \rho_{\chi_n} \supset I$$

only few terms allowed in the potential

To start playing the game:

Can someone give me the Modular Forms?

Simplest Case: $\Gamma_2 \sim S_3$

Let us find the functions f(τ) !

The group S_3 contains 1 + 1’ + 2

N	$d_k(\Gamma(N))$	Γ_N
2	$k/2+1$	S_3
3	$k+1$	A_4
4	$2k+1$	S_4
5	$5k+1$	A_5

two independent modular forms can fit into a doublet of S_3

Simplest Case: $\Gamma_2 \sim S_3$

Let us find the functions f(τ) !

The group S_3 contains 1 + 1' + 2

N	$d_k(\Gamma(N))$	Γ_N
2	$k/2+1$	S_3
3	$k+1$	A_4
4	$2k+1$	S_4
5	$5k+1$	A_5

— two independent modular forms can fit into a doublet of S_3

Dedekind eta functions $\eta(\tau) = q^{1/24} \prod_{n=1}^{\infty} (1-q^n)$ $q \equiv e^{i2\pi\tau}$

S: $\eta(-1/\tau) = \sqrt{-i\tau}\, \eta(\tau)$, T: $\eta(\tau+1) = e^{i\pi/12}\, \eta(\tau)$

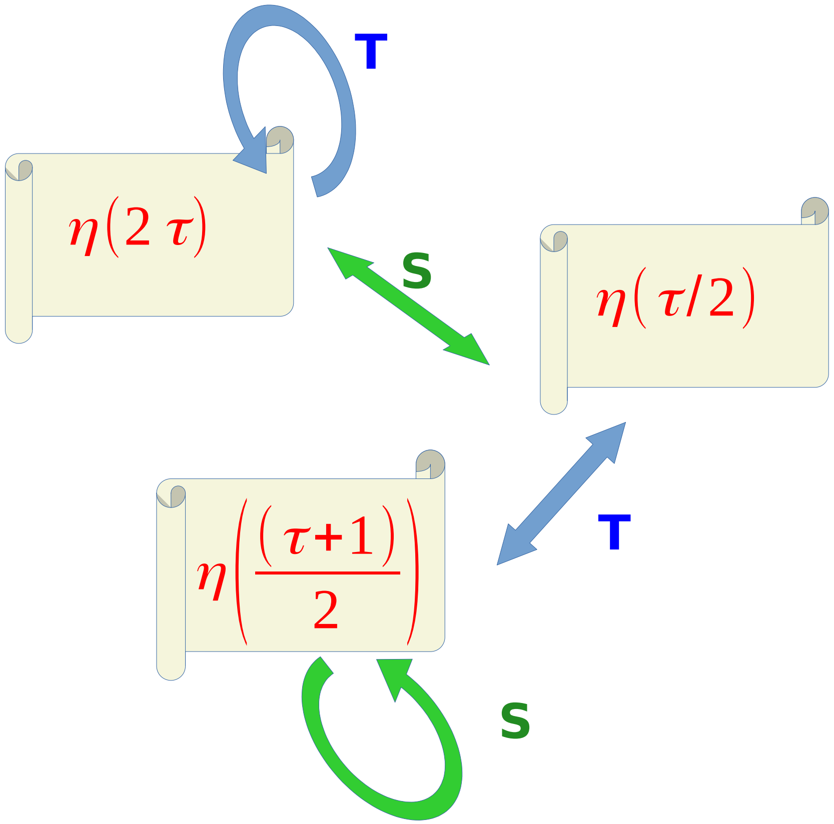
↓

η^{24} is a modular form of weight 12

Simplest Case: $\Gamma_2 \sim S_3$

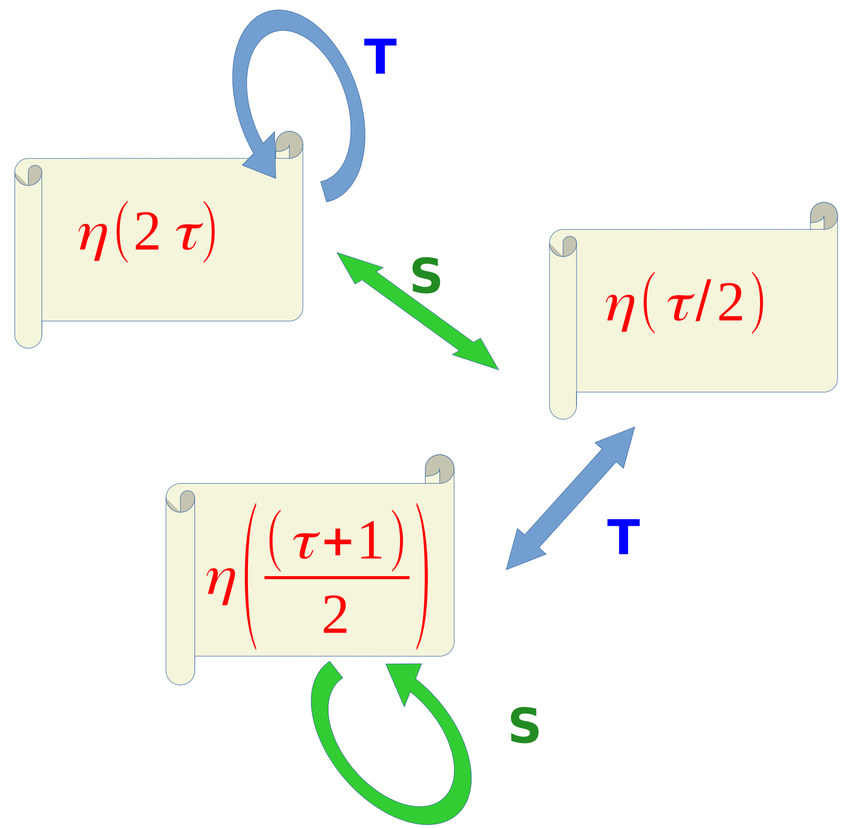
Constructing the Modular Forms

the system is closed under modular transformation



Simplest Case: $\Gamma_2 \sim S_3$

Constructing the Modular Forms



the system is closed under modular transformation

$$\begin{cases} Y_1(\tau) = \frac{c}{2}\left[\frac{\eta'(\tau/2)}{\eta(\tau/2)} + \frac{\eta'\left(\frac{\tau+1}{2}\right)}{\eta\left(\frac{\tau+1}{2}\right)} - \frac{8\eta'(2\tau)}{\eta(2\tau)}\right] \\ Y_2(\tau) = \frac{c}{2}\sqrt{3}\left[\frac{\eta'(\tau/2)}{\eta(\tau/2)} - \frac{\eta'\left(\frac{\tau+1}{2}\right)}{\eta\left(\frac{\tau+1}{2}\right)}\right] \end{cases}$$

doublet of S_3: Y $\begin{pmatrix} Y_1(\tau) \\ Y_2(\tau) \end{pmatrix}_{\mathbf{2}} = \begin{pmatrix} \frac{7}{100} + \frac{42}{25}q + \frac{42}{25}q^2 + \frac{168}{25}q^3 + \ldots \\ \frac{14\sqrt{3}}{25}q^{1/2}(1 + 4q + 6q^2 + \ldots) \end{pmatrix}$

$q = e^{2\pi i \, \mathrm{Re}(\tau)} \, e^{-2\pi \, \mathrm{Im}(\tau)}$

Simplest Case: $\Gamma_2 \sim S_3$

DM & Matteo Parriciatu, 2306.09028 [hep-ph]

Guiding principles for model buildings:

small number of operators (few free parameters) → ***predictability***

no new matter fields → ***minimality***

no new scalar fields beside Higgs(es) → ***symmetry breaking dictated by the vev of*** τ

charged lepton hierarchy by symmetry arguments → "***appealing***"

$$|Y_2(\tau)/Y_1(\tau)| \ll 1$$

for τ in D

fit to all low energy neutrino data → ***useful***

A case study: $\Gamma_2 \sim S_3$

Texture zeros – minimal # of free parameters

Charged leptons

$$M_l = \begin{pmatrix} X & X & 0 \\ X & X & 0 \\ 0 & 0 & X \end{pmatrix}$$

unspecified weight

LH fields

$$\begin{pmatrix} l_1 \\ l_2 \end{pmatrix} \rightarrow (c\,\tau + d)^{-kl} \rho(\gamma) \begin{pmatrix} l_1 \\ l_2 \end{pmatrix}$$

$$l_3 \sim 1', k_l$$

2 of S_3

RH fields

$$E_1^c \sim (1, 4 - k_l) \qquad E_2^c \sim (1', 2 - k_l) \qquad E_3^c \sim (1', -k_l)$$

A case study: $\Gamma_2 \sim S_3$

Charged leptons

modular forms of weight 4

$$M_\ell = \begin{pmatrix} \alpha(Y_{\mathbf{2}}^{(2)})_1 & \alpha(Y_{\mathbf{2}}^{(2)})_2 & 0 \\ \beta Y_2 & -\beta Y_1 & 0 \\ 0 & 0 & \gamma \end{pmatrix} v_d$$

$$|Y_2(\tau)/Y_1(\tau)| = \varepsilon$$

$$m_e = v_d \alpha \left(|Y_1^2| - \frac{7}{2}|Y_1^2||\epsilon|^2 + \mathcal{O}(\epsilon^3) \right)$$

$$m_\mu = v_d \alpha \left(\frac{\beta}{\alpha}|Y_1| + \frac{1}{2}\frac{\beta}{\alpha}|Y_1||\epsilon|^2 + \mathcal{O}(\epsilon^3) \right)$$

$$m_\tau = v_d \alpha \left(\frac{\gamma}{\alpha} \right) .$$

A case study: $\Gamma_2 \sim S_3$

Charged leptons

modular forms of weigth 4

$|Y_2(\tau)/Y_1(\tau)| = \varepsilon$

$$M_\ell = \begin{pmatrix} \alpha(Y_{\mathbf{2}}^{(2)})_1 & \alpha(Y_{\mathbf{2}}^{(2)})_2 & 0 \\ \beta Y_2 & -\beta Y_1 & 0 \\ 0 & 0 & \gamma \end{pmatrix} v_d$$

$$m_e = v_d \alpha \left(|Y_1^2| - \frac{7}{2}|Y_1^2||\epsilon|^2 + \mathcal{O}(\epsilon^3) \right)$$

$$m_\mu = v_d \alpha \left(\frac{\beta}{\alpha}|Y_1| + \frac{1}{2}\frac{\beta}{\alpha}|Y_1||\epsilon|^2 + \mathcal{O}(\epsilon^3) \right)$$

$$m_\tau = v_d \alpha \left(\frac{\gamma}{\alpha} \right) .$$

for $|Y_1| \sim 7/100$

$$(m_\tau, m_\mu, m_e) \sim m_\tau(1, |Y_1|, |Y_1|^2).$$

$$U_\ell \sim \begin{pmatrix} 1 - \frac{|\epsilon|^2}{2} & -\overline{\epsilon} & 0 \\ \epsilon & 1 - \frac{|\epsilon|^2}{2} & 0 \\ 0 & 0 & 1 \end{pmatrix} + \mathcal{O}(\epsilon^3)$$

Mass hierarchy scaling naturally reproduced !

(no fit so far...)

A case study: $\Gamma_2 \sim S_3$

Ready for Neutrinos: key ingredient is to fix k_l

several possible choices. The best one gives (k_l=2):

$$m_\nu^{k_\ell=2} = \frac{2gv_u^2}{\Lambda}\left[\begin{pmatrix} -(Y_2^2-Y_1^2) & 2Y_1Y_2 & \frac{g'}{2g}2Y_1Y_2 \\ 2Y_1Y_2 & (Y_2^2-Y_1^2) & -\frac{g'}{2g}(Y_2^2-Y_1^2) \\ \frac{g'}{2g}2Y_1Y_2 & -\frac{g'}{2g}(Y_2^2-Y_1^2) & 0 \end{pmatrix} + \right.$$
$$\left. + \begin{pmatrix} \frac{g''}{g}(Y_1^2+Y_2^2) & 0 & 0 \\ 0 & \frac{g''}{g}(Y_1^2+Y_2^2) & 0 \\ 0 & 0 & \frac{g_p}{g}(Y_1^2+Y_2^2) \end{pmatrix}\right]$$

Independent parameters: Re(τ), Im(τ), β/α, γ/α, g'/g, g''/g, g_p/g

A case study: $\Gamma_2 \sim S_3$

<u>Numerical fit</u>

Mass matrices against the experimental data

data

$r \equiv \Delta m^2_{\rm sol}/\vert\Delta m^2_{\rm atm}\vert$	0.0296 ± 0.0008
$\sin^2\theta_{12}$	$0.303^{+0.013}_{-0.013}$
$\sin^2\theta_{13}$	$0.0223^{+0.0007}_{-0.0006}$
$\sin^2\theta_{23}$	$0.455^{+0.018}_{-0.015}$
m_e/m_μ	0.0048 ± 0.0002
m_μ/m_τ	0.0565 ± 0.0045

$\chi^2 \sim O(0.1)$

fit results

${\rm Re}\,\tau$	$\pm 0.0895^{+0.0032}_{-0.0055}$
${\rm Im}\,\tau$	$1.697^{+0.023}_{-0.016}$
β/α	$14.33^{+0.58}_{-0.38}$
γ/α	$17.39^{+1.38}_{-0.87}$
g'/g	$31.57^{+27.59}_{-10.29}$
g''/g	$7.17^{+6.36}_{-2.36}$
g_p/g	$8.51^{+7.99}_{-3.03}$

A case study: $\Gamma_2 \sim S_3$

Numerical fit

Mass matrices against the experimental data

data

$r \equiv \Delta m^2_{\rm sol}/\lvert\Delta m^2_{\rm atm}\rvert$	0.0296 ± 0.0008
$\sin^2\theta_{12}$	$0.303^{+0.013}_{-0.013}$
$\sin^2\theta_{13}$	$0.0223^{+0.0007}_{-0.0006}$
$\sin^2\theta_{23}$	$0.455^{+0.018}_{-0.015}$
m_e/m_μ	0.0048 ± 0.0002
m_μ/m_τ	0.0565 ± 0.0045

$\chi^2 \sim$ O(0.1)

fit results

$\mathrm{Re}\,\tau$	$\pm 0.0895^{+0.0032}_{-0.0055}$
$\mathrm{Im}\,\tau$	$1.697^{+0.023}_{-0.016}$
β/α	$14.33^{+0.58}_{-0.38}$
γ/α	$17.39^{+1.38}_{-0.87}$
g'/g	$31.57^{+27.59}_{-10.29}$
g''/g	$7.17^{+6.36}_{-2.36}$
g_p/g	$8.51^{+7.99}_{-3.03}$

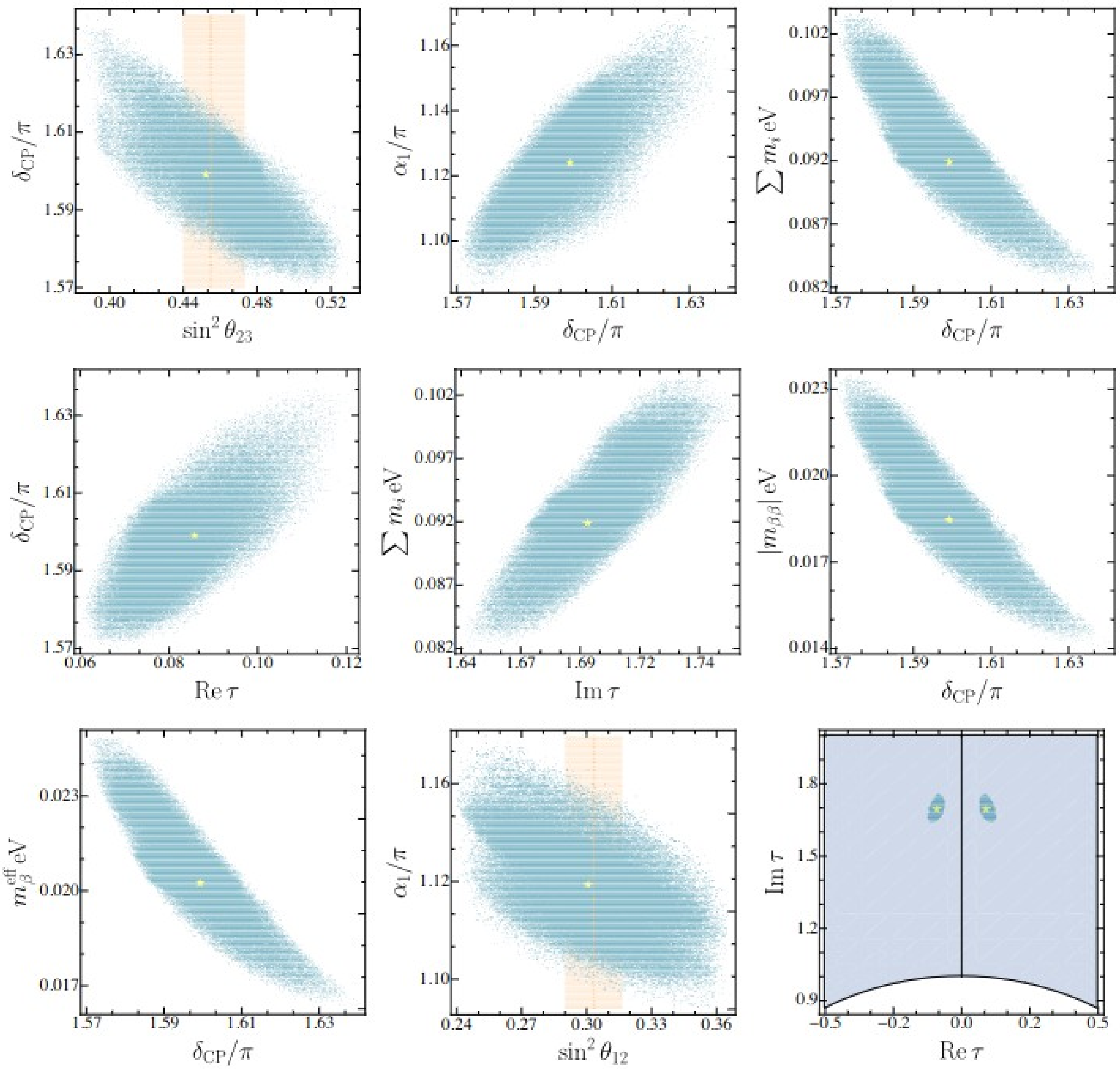
predictions

Ordering	NO
δ/π	$\pm 1.594^{+0.007}_{-0.010}$
m_1 [eV]	$0.0182^{+0.0018}_{-0.0014}$
m_2 [eV]	$0.0201^{+0.0017}_{-0.0013}$
m_3 [eV]	$0.0537^{+0.0006}_{-0.0005}$
$\sum_i m_i$ [eV]	$0.092^{+0.004}_{-0.003}$
$\lvert m_{\beta\beta}\rvert$ [meV]	$18.89^{+1.90}_{-1.47}$
$m_\beta^{\rm eff}$ [meV]	$20.26^{+1.69}_{-1.30}$
α_1/π	$\pm 1.124^{+0.014}_{-0.017}$
α_2/π	$\pm 0.949^{+0.005}_{-0.005}$

!!!

A case study: $\Gamma_2 \sim S_3$

correlations



- the CP-violating phase is in agreement (within the 2σ range) with the global analysis of oscillation data
- values of the sum of neutrino masses is around 0.090 eV, which is compatible with the present upper bound of 0.115 eV (95 % C.L.)
- the Majorana effective mass lies around ~20 meV, not too far from the recent KamLAND-Zen upper bound | mββ | < (36 − 156) meV
- Majorana phases α1, α2 live in narrow regions around ±1.13π, ±0.95π

Conclusions

Modular symmetries offer an alternative way for model building

Yukawa couplins dictated by modular forms

symmetry breaking by the vev of tau only

A lot to do:

mass hierarchy

unified description of quarks and leptons

more than one modulus

more pheno: leptogenesis, LFV...

Backup slides

Model Building

Constructing the Modular Forms

Crucial observation:

if $$g(\tau) \rightarrow e^{i\alpha}(c\tau+d)^k g(\tau)$$

then $$\frac{d}{d\tau}\log[g(\tau)] \rightarrow (c\tau+d)^2\frac{d}{d\tau}\log[g(\tau)]+kc(c\tau+d)$$

this term prevents of having a modular form of weight **2 k = 2**

The inhomogeneous term can be removed if we combine several $f_i(\tau)$ with weights k_i

$$\frac{d}{d\tau}\Sigma_i\log[g_i(\tau)] \rightarrow (c\tau+d)^2\frac{d}{d\tau}\Sigma_i\log[g_i(\tau)]+(\Sigma_i k_i)c(c\tau+d)$$

with $\Sigma_i k_i=0$

A case study: $\Gamma_2 \sim S_3$

Constructing the Modular Forms

Equations to be satisfied:

$$\begin{pmatrix} Y_1(-1/\tau) \\ Y_2(-1/\tau) \end{pmatrix} = \tau^2 \rho(S) \begin{pmatrix} Y_1(\tau) \\ Y_2(\tau) \end{pmatrix}, \qquad \begin{pmatrix} Y_1(\tau+1) \\ Y_2(\tau+1) \end{pmatrix} = \rho(T) \begin{pmatrix} Y_1(\tau) \\ Y_2(\tau) \end{pmatrix}$$

representation of generators

$$\rho(S) = \frac{1}{2} \begin{pmatrix} -1 & -\sqrt{3} \\ -\sqrt{3} & 1 \end{pmatrix}, \qquad \rho(T) = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

$$(\rho(S))^2 = \mathbb{I}, \qquad (\rho(S)\rho(T))^3 = \mathbb{I}, \qquad (\rho(T))^2 = \mathbb{I}$$

$$Y_1(\alpha, \beta, \gamma) \sim Y(1,1,-2) \qquad Y_2(\alpha, \beta, \gamma) \sim Y(1,-1,0)$$

$$\begin{aligned} Y_1(\tau) &= \frac{i}{4\pi} \left(\frac{\eta'(\tau/2)}{\eta(\tau/2)} + \frac{\eta'((\tau+1)/2)}{\eta((\tau+1)/2)} - \frac{8\eta'(2\tau)}{\eta(2\tau)} \right) \\ Y_2(\tau) &= \frac{\sqrt{3}i}{4\pi} \left(\frac{\eta'(\tau/2)}{\eta(\tau/2)} - \frac{\eta'((\tau+1)/2)}{\eta((\tau+1)/2)} \right), \end{aligned}$$

doublet of S3: Y

Kahler potential

Under Γ:
$$\begin{cases} \tau \to \dfrac{a\tau+b}{c\tau+d} \\ \varphi^{(I)} \to (c\tau+d)^{-k_I}\rho^{(I)}(\gamma)\varphi^{(I)} \end{cases}$$

The invariance of the action requires the invariance of the superpotential w(Φ) and the invariance of the Kahler potential up to a Kahler transformation:

$$\begin{cases} w(\Phi) \to w(\Phi) \\ K(\Phi,\bar{\Phi}) \to K(\Phi,\bar{\Phi}) + f(\Phi) + f(\overline{\Phi}) \end{cases}$$

Kahler potential:

$$\sum_I (-i\tau + i\bar{\tau})^{-k_I} |\varphi^{(I)}|^2$$

→

modular invariant kinetic terms

$$\frac{h}{\langle -i\tau + i\bar{\tau}\rangle^2}\partial_\mu\bar{\tau}\partial^\mu\tau + \sum_I \frac{\partial_\mu\overline{\varphi}^{(I)}\partial^\mu\varphi^{(I)}}{\langle -i\tau + i\bar{\tau}\rangle^{k_I}}$$

Some definitions

a <u>normal subgroup</u> (also known as an invariant subgroup or self-conjugate subgroup) is a *subgroup* which is invariant under conjugation by members of the group of which it is a part:

a subgroup N of the group G is normal in G if and only if $(g\, n\, g^{-1}) \in N$ for all $g \in G$ and $n \in N$

Γ(N), N>=2 are infinite normal subgroups of Γ, called *principal congruence subgroups*

the group Γ(N) acts on the complex variable τ (Im τ >0)

$$\gamma\,\tau = \frac{a\,\tau + b}{c\,\tau + d}$$

And it can be shown that the upper half-plane is mapped to itself under this action.
The complex variable is henceforth restricted to have positive imaginary part

Some definitions

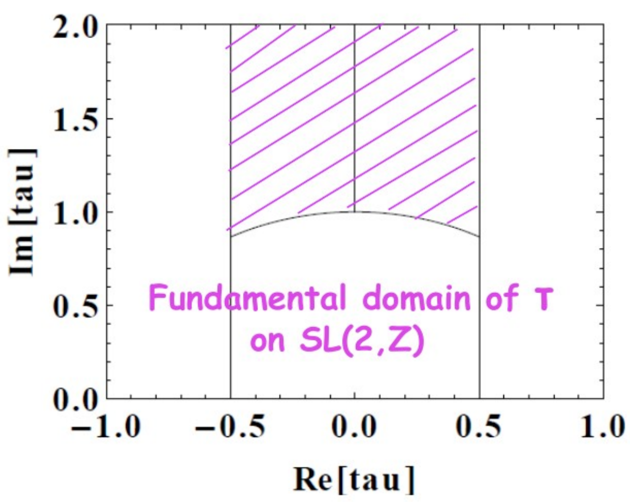
Modular Functions and Modular Forms
J. S. Milne

DEFINITION 0.2. A holomorphic function $f(z)$ on $\mathbb{H}$ is a *modular form of level N and weight* 2k if

(a) $f(\alpha z) = (cz+d)^{2k} \cdot f(z)$, all $\alpha = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma(N)$;

(b) $f(z)$ is "holomorphic at the cusps".

Fundamental domain of τ on SL(2,Z): connected open subset such that no two points of D are equivalent under SL(2,Z)



THEOREM 2.12. *Let* $D = \{z \in \mathbb{H} \mid |z| > 1, |\Re(z)| < 1/2\}$.

(a) *D is a fundamental domain for* $\Gamma(1) = \mathrm{SL}_2(\mathbb{Z})$*; moreover, two elements z and* z' *of* $\bar{D}$ *are equivalent under* $\Gamma(1)$ *if and only if*
  (i) $\Re(z) = \pm 1/2$ *and* $z' = z \pm 1$, *(then* $z' = Tz$ *or* $z = Tz'$*), or*
  (ii) $|z| = 1$ *and* $z' = -1/z = Sz$.

A case study: $\Gamma_2 \sim S_3$

Constructing the Modular Forms

Under **T**: $Y(\alpha,\beta,\gamma) \rightarrow Y(\gamma,\beta,\alpha)$

Under **S**: $Y(\alpha,\beta,\gamma) \rightarrow \tau^2\; Y(\gamma,\alpha,\beta)$

representation of generators

$$\rho(S) = \frac{1}{2}\begin{pmatrix} -1 & -\sqrt{3} \\ -\sqrt{3} & 1 \end{pmatrix}, \qquad \rho(T) = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

$$(\rho(S))^2 = \mathbb{I}, \qquad (\rho(S)\rho(T))^3 = \mathbb{I}, \qquad (\rho(T))^2 = \mathbb{I}$$

A case study: $\Gamma_2 \sim S_3$

Dedekind eta functions

$$\eta(\tau) = q^{1/24} \prod_{n=1}^{\infty} (1 - q^n) \qquad q \equiv e^{i2\pi\tau}$$

Under **T**:

$$\eta(2\tau) \rightarrow e^{i\pi/6}\,\eta(2\tau)$$

$$\eta(\tau/2) \rightarrow \eta((\tau+1)/2)$$

$$\eta((\tau+1)/2) \rightarrow e^{i\pi/12}\,\eta(\tau/2)$$

Under **S**:

$$\eta(2\tau) \rightarrow \sqrt{-i\,\tau/2}\;\eta(\tau/2)$$

$$\eta(\tau/2) \rightarrow \sqrt{-2i\,\tau}\;\eta(2\tau)$$

$$\eta\left(\frac{(\tau+1)}{2}\right) \rightarrow e^{-i\pi/12}\sqrt{-i\,\tau(\sqrt{3}-i)}\;\eta\left(\frac{(\tau+1)}{2}\right)$$

Mod

```
Id[a_, b_] := {{Mod[a, b], 0}, {0, Mod[a, b]}}
```

Id[-1, 2]

$$\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

Id[-1, 3]

$$\begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix}$$

Origin of modular symmetry

Two periods in complex functions f : C → C

elliptic function: $f(z+\omega_1) = f(z+\omega_2) = f(z)$

periods $\in$ C such that $\omega_2/\omega_1 \notin \Re$

a lattice Λ can be generated in the complex plane, spanned by the two directions ω_1, ω_2

$$\Lambda = \mathbb{Z}\omega_1 \oplus \mathbb{Z}\omega_2 = \{n_1\omega_1 + n_2\omega_2 \,|\, n_1, n_2 \in \mathbb{Z}\}$$

elliptic functions are translation-invariant in this lattice: f (z + λ) = f (z) for λ ∈ Λ

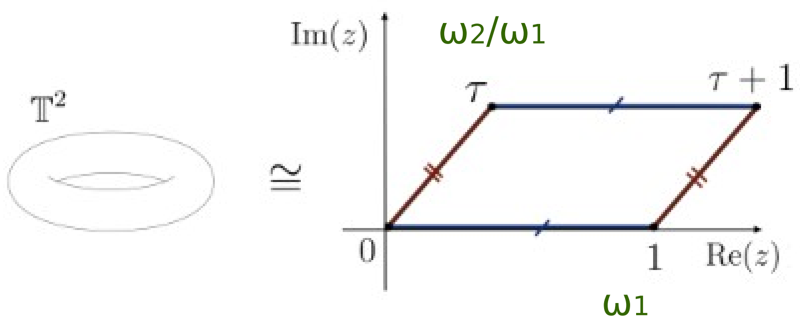
Thus, an elliptic function is single-valued on the quotient C/Λ, which is topologically known as a torus (T_2).

Origin of modular symmetry

Rescaling of the periods:

$\omega_1 = 1$ and $\omega_2/\omega_1 = \tau$, where τ is called the *modulus*

the torus is represented by a parallelogram with vertices $z = 0$, $z = 1$, $z = \tau$ and $z = \tau + 1$ where the opposite sides are pairwise identified



courtesy by Matteo Parriciatu, Master Thesis

The lattice Λ can be equivalently described by a different basis (ω'_1, ω'_2) related to the old by a linear map with integer parameters:

$$\begin{pmatrix} \omega'_2 \\ \omega'_1 \end{pmatrix} = \gamma \begin{pmatrix} \omega_2 \\ \omega_1 \end{pmatrix} \equiv \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} \omega_2 \\ \omega_1 \end{pmatrix} \quad , \qquad a, b, c, d \in \mathbb{Z}$$

Model Building

Long (not updated) list from **S.T. Petcov, Bethe Forum, University of Bonn, 04/05/2022**

For ($\Gamma_3 \simeq A_4$), the generating (basis) modular forms of weight 2 were shown to form a 3 of A_4 (expressed in terms of log derivatives of Dedekind η-function η'/η of 4 different arguments).

F. Feruglio, arXiv:1706.08749

For ($\Gamma_2 \simeq S_3$), the two basis modular forms of weight 2 were shown to form a 2 of S_3 (expressed in terms of η'/η of 3 different arguments).

T. Kobayashi, K. Tanaka, T.H. Tatsuishi, arXiv:1803.10391

For ($\Gamma_4 \simeq S_4$), the 5 basis modular forms of weight 2 were shown to form a 2 and a $3'$ of S_4 (expressed in terms of η'/η of 6 different arguments).

J. Penedo, STP, arXiv:1806.11040

For ($\Gamma_5 \simeq A_5$), the 11 basis modular forms of weight 2 were shown to form a 3, a $3'$ and a 5 of A_5 (expressed in terms of Jacobi theta function $\theta_3(z(\tau), t(\tau))$ for 12 different sets of $z(\tau), t(\tau)$).

P.P. Novichkov et al., arXiv:1812.02158; G.-J. Ding et al., arXiv:1903.12588

Multiplets of higher weight modular forms have been also constructed from tensor products of the lowest weight 2 multiplets:
i) for $N = 4$ (i.e., S_4), multiplets of weight 4 (weight $k \leq 10$) were derived in arXiv:1806.11040 (arXiv:1811.04933);
ii) for $N = 3$ (i.e., A_4) multiplets of weight $k \leq 6$ were found in arXiv:1706.08749;
iii) for $N = 5$ (i.e., A_5), multiplets of weight $k \leq 10$ were derived in arXiv:1812.02158.

Model Building

Highest level modular form:

$$Y_{\mathbf{2}}^{(1)}(\tau) \otimes Y_{\mathbf{2}}^{(1)}(\tau) = (Y_1^2(\tau) + Y_2^2(\tau))_{\mathbf{1}} \oplus \begin{pmatrix} Y_2^2(\tau) - Y_1^2(\tau) \\ 2Y_1(\tau)Y_2(\tau) \end{pmatrix}_{\mathbf{2}}$$