Quantum Corrections to

Binding Energies of BPS Vortices

H. Weigel

(Stellenbosch University, Institute for Theoretical Physics)

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Presentation mainly based on:
N. Graham, HW, Phys. Rev. **D101** (2020) 076006; **D104** (2021) L011901; **D106** (2022) 076013

recent review: N. Graham, HW, Int. J. Mod. Phys. **A37** (2022) 2241004

see also: N. Graham, M. Quandt, HW, Springer Lect. Notes Phys. **777** (2009)

Introduction

⋆ starting point: field theory with classical localized solutions (solitons, solitary waves)

⋆ quantum corrections: typically small; decisive when solitons are classically degenerate

⋆ soliton polarizes its harmonic fluctuations

$\Longrightarrow$ shift in zero-point energies of harmonic fluctuations
cannot be normal-ordered away

⋆ vacuum polarization energy (VPE):

$$E_{\rm VPE} = \frac{\hbar}{2}\sum_k \left[\omega_k - \omega_k^{(0)}\right]_{\rm ren.}$$

⋆ sum contains bound and scattering states

⋆ requires renormalizable theory (obviously)

VPE and Scattering Data (aka spectral methods)

- ⋆ (static) soliton generates potential $V(r)$ for harmonic fluctuations $\psi_\ell(k,r)$
- ⋆ scattering described by momentum dependent phase $\delta_\ell(k)$ in channel ℓ
- ⋆ change in state density: $$\Delta\frac{\delta n_\ell(k)}{\delta k} = \frac{1}{\pi}\frac{d}{dk}\delta_\ell(k)$$ (Krein formula)
- ⋆ change in state density $\longleftrightarrow$ change in energy: $$E_{\rm VPE} \sim \int \frac{dk}{2\pi}\,\omega_k\,\frac{d}{dk}\delta_\ell(k)$$
  (scattering part)

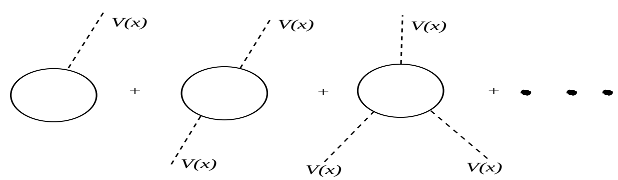
⋆ QFT derivation: matrix element of energy-momentum tensor & analytic properties of Green's functions

$$E_{\rm VPE}[V] = \sum_\ell D_\ell \Big[\sum_j \frac{\epsilon_{\ell j}}{2} + \int_0^\infty \frac{dk}{2\pi} \sqrt{k^2+m^2}\, \frac{d}{dk} [\delta_\ell(k)]_N \Big] + E_{\rm FD}^N[V] + E_{\rm CT}[V]$$

– D_ℓ degeneracy factor in ℓ-th partial wave *e.g.* $2\ell+1$

– $\epsilon_{\ell j}$ bound state energies

– $[\delta_\ell(k)]_N = \delta_\ell(k) - \delta_\ell^{(1)}(k) - \ldots - \delta_\ell^{(N)}(k)$
phase shifts with first N-Born terms subtracted (expansion in V)
$\Longrightarrow$ finite momentum integral (choose N large enough)

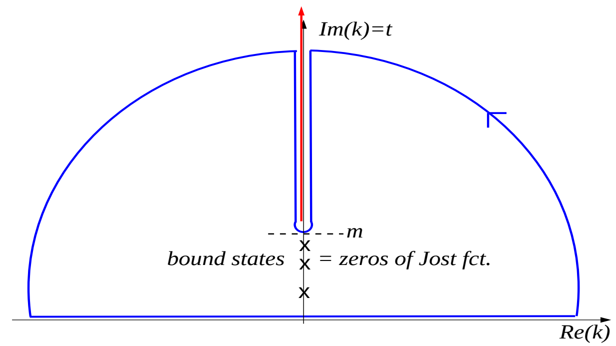
– $E_{\rm FD}^N[V] =$  (expansion of the effective action up to order N in $V(r)$)

– $E_{\rm CT}[V]$ counterterm contribution $\Rightarrow$ standard renormalization conditions
$E_{\rm FD}^N[V] + E_{\rm CT}[V]$ finite by renormalization

⋆ Jost solution to scattering problem: $\lim_{x\to\infty} f(k,x)\mathrm{e}^{-\mathrm{i}kx} = 1$; analytic for $\mathsf{Im}(k) \ge 0$

⋆ Jost function: $F_\ell(k) = |F_\ell(k)|\mathrm{e}^{\mathrm{i}\delta_\ell(k)}$

- -) from $f(k,0)$ and/or $f'(k,0)$
- -) for real k: real part even, imaginary part odd
- -) $F_\ell(\mathrm{i}\kappa_j) = 0$ bound state wave-numbers



⋆ integration in complex plane

- -) semi-circle does not contribute because of Born subtractions
- -) poles from $\frac{d}{dk}\ln(F(k))$ at $\mathrm{i}\kappa_j$ cancel (explicit) bound state contribution
- -) avoid cut from $\omega = \sqrt{m^2 - t^2} = 0$ (m is smallest of all masses)

$$E_{\mathrm{VPE}} = \int_m^\infty \frac{tdt}{4\pi} W(t)\,[\nu(t)]_N + E_{\mathrm{FD}}^{(N)} + E_{\mathrm{CT}} \quad \text{with} \quad [\nu(t)]_N = \lim_{L\to\infty}\sum_{\ell=-L}^{L}[\ln(F_\ell(\mathrm{i}t))]_N$$

$$W(t) = \frac{2}{\sqrt{t^2 - m^2}} \quad \text{for} \quad D = 2+1 \qquad \text{and} \qquad W(t) = 1 \quad \text{for} \quad D = 3+1$$

Classical ANO Vortex

⋆ appearance of vortices

-) cosmic strings in $SU(2)$ electro-weak theory: Higgs and massive gauge bosons

(frustration at interfaces of regions with different Higgs VEVs)

-) magnetic flux in a superconductor propto topological charge n

type I: $E_n < nE_1$ vortices coalesce

type II: $E_n > nE_1$ isolated, single vortices

scalar field is order parameter for condensate of Cooper pairs

⋆ model Lagrangian (scalar ED)

$$\mathcal{L} = -\frac{1}{4}F_{\mu\nu}F^{\mu\nu} + |D_\mu\Phi|^2 - \frac{\lambda}{4}\left(|\Phi|^2 - v^2\right)^2$$

$$F_{\mu\nu} = \partial_\mu A_\mu - \partial_\nu A_\mu \qquad \text{and} \qquad D_\mu\Phi = \left(\partial_\mu - \mathrm{i}eA_\mu\right)\Phi$$

⋆ mass parameters

$$M_H^2 = \lambda v^2 \qquad \text{and} \qquad M_A^2 = 2v^2e^2$$

⋆ profile functions for winding number n

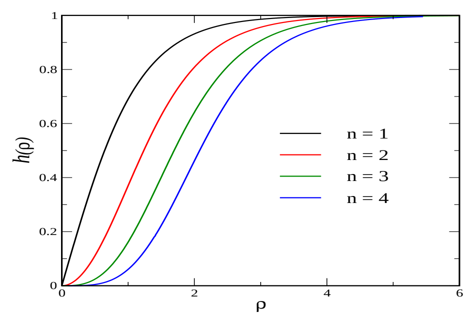
$$\Phi_S = vh(\rho) \qquad \text{and} \qquad \boldsymbol{A}_S = nv\hat{\boldsymbol{\varphi}}\frac{g(\rho)}{\rho} \qquad\qquad (\rho = evr)$$

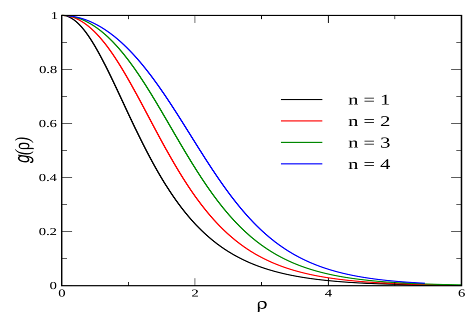
⋆ boundary conditions (in singular gauge)

$$h(0) = 0\,, \quad g(0) = 1 \qquad \text{and} \qquad \lim_{\rho\to\infty} h(\rho) = 1\,, \quad \lim_{\rho\to\infty} g(\rho) = 0$$

⋆ numerical results (in units of v^2)

	$E_{\rm cl}$			
M_H/M_A	$n=1$	$n=2$	$n=3$	
0.8	5.73	11.15	16.47	type I
1.0	6.28	12.57	18.85	BPS
1.2	6.78	13.88	21.08	type II





Higgs Fluctuations about ANO Vortex

⋆ reveals singularities & paves way to renormalization

⋆ singular gauge

-) vortex profiles approach vacuum configuration far away from the center ✓

-) singularity at center prevents Fourier transform ϟ

(needed for Feynman diagrams)

⋆ background potential outside realm for textbook proof on analyticity of Jost function

⋆ Jost function not gauge invariant (in contrast to phase shift)

quadratic divergence not excluded; encoded in $\nu(t) \longrightarrow \text{const.}$ as $t \longrightarrow \infty$

constant maybe dropped, as only $\frac{d\nu(t)}{dt}$ entered in the first place (Krein formula)

⋆ fluctuation equation for $\phi \sim v h(\rho) + K_{|\ell|}(t\rho)\eta_\ell(\rho)$

$$\frac{1}{\rho}\partial_\rho \rho \partial_\rho \eta_\ell = 2t Z_\ell(t\rho)\partial_\rho \eta_\ell + \frac{1}{\rho^2}\left[g^2(\rho) - 2\ell g(\rho)\right]\eta_\ell + V_H(\rho)\,\eta_\ell\,, \qquad Z_\ell(z) = \frac{K_{|\ell|+1}(z)}{K_{|\ell|}(z)} - \frac{|\ell|}{z}$$

⋆ standard fluctuation potential: $V_H(\rho) = 3(h^2(\rho) - 1)$

⋆ boundary conditions: $\lim_{\rho\to\infty} \eta_\ell(\rho) = 1$ and $\lim_{\rho\to\infty} \frac{d}{d\rho}\eta_\ell(\rho) = 0$

⋆ immediate access to Jost function: $\nu_\ell(t) = \lim_{\rho\to 0} \ln\left(\eta_\ell(\rho)\right)$

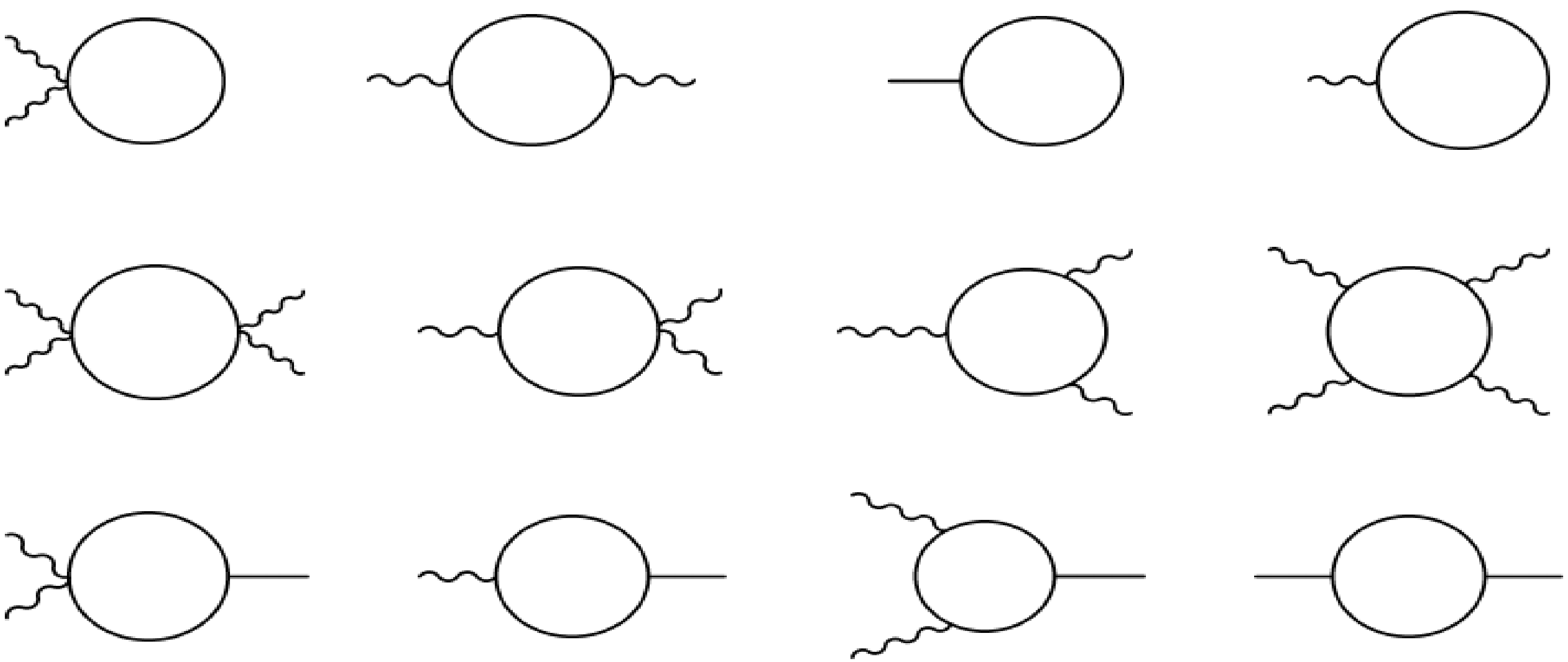
⋆ Born series by iteration: $\nu_\ell(t) = 1 + \nu_\ell^{(1)}(t) + \nu_\ell^{(2)}(t) + \ldots$

⋆ singularity at center from $\lim_{\rho\to 0} g(\rho) = 1$?

⋆ superficially divergent one-loop diagrams (from expansion of the effective action)

gauge invariance not manifest $\Longrightarrow$ need to consider all of them

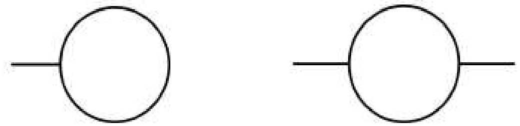
vertices: $\boldsymbol{A}_S^2$, $\boldsymbol{A}_S \cdot \boldsymbol{\nabla}$, V_H



straight lines: scalar field; curly lines: gauge field

external lines: Fourier transform of vortex profiles

⋆ solely external scalar fields:



standard treatment of scalar potential V_H: produces $\eta_\ell^{(1)}$ and $\eta_\ell^{(2)}$ in Born series
(iterate wave-equation with $g(\rho) = 0$)

⋆ without gauge invariance: potentially quadratically divergent diagrams with two gauge fields

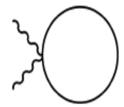
gauge-variant, *e.g.* sharp cut-off regularization

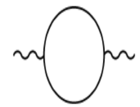
$$\frac{\Omega_D}{D} \frac{\Lambda^D}{\Lambda^2 + m^2} \int d^4x \, A_\mu(x) A^\mu(x) + \mathcal{O}\left(\Lambda^{D-2}\right)$$

has the same $\rho \to 0$ singularity as the Born approximation from the singular potential

$$\int d^4x A_\mu(x) A^\mu(x) \quad \longrightarrow \quad 2\pi n^2 T L \int_{\rho_{\min}}^{\infty} d\rho \, \frac{g^2(\rho)}{\rho}$$

⋆ Born series for singular terms

- $\eta_\ell^{(3)}$: first order for $\left(\frac{g}{\rho}\right)^2$ 

- $\eta_\ell^{(4)} + \eta_\ell^{(5)} - \frac{1}{2}\left(\eta_\ell^{(4)}\right)^2$: first and second order for $\frac{\ell g}{\rho^2}$ 

⋆ numerically confirmed

$$\lim_{L\to\infty} \sum_{\ell=-L}^{L} \left\{ \eta_\ell^{(3)} + \eta_\ell^{(4)} + \eta_\ell^{(5)} - \frac{1}{2}\left(\eta_\ell^{(4)}\right)^2 \right\}\Bigg|_{\rho=\rho_{\rm min}} \longrightarrow \int_{\rho_{\rm min}}^{\infty} \frac{d\rho}{\rho}\, g^2(\rho) + \mathcal{O}\left(\frac{1}{t}\right) \quad \text{as} \quad \rho_{\rm min} \to 0$$

⋆ alternatively, remove ultraviolet divergences that emerge from V_H and expect

$$\nu_L(t)_L = \sum_{\ell=-L}^{L} \left\{ \ln\left(\eta_\ell\right) - \eta_\ell^{(1)} - \eta_\ell^{(2)} + \frac{1}{2}\left(\eta_\ell^{(1)}\right)^2 \right\}\Bigg|_{\rho=\rho_{\rm min}} - \int_{\rho_{\rm min}}^{\infty} \frac{d\rho}{\rho}\, g^2(\rho) \quad \longrightarrow \quad 0$$

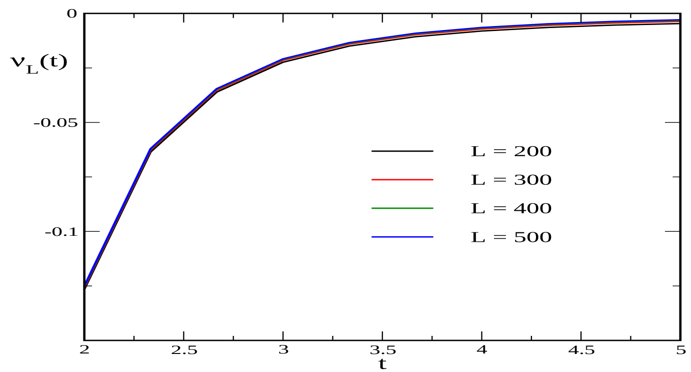
for $L, t \to \infty$ and $\rho_{\rm min} \to 0$

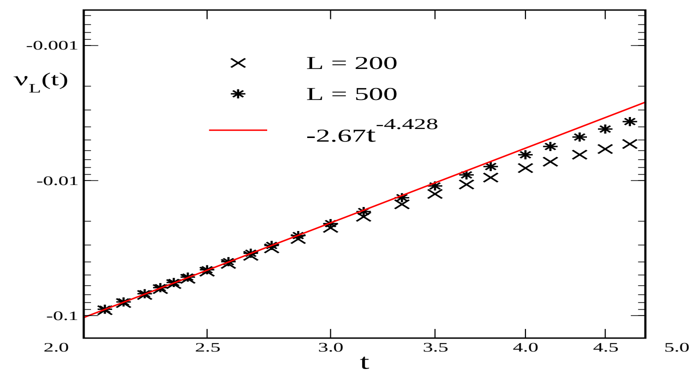
⋆ limit $L \to \infty$ conceptually problematic for singular background

(non-singular profiles work just fine)

what is large L; perhaps $L \propto \langle\rho\rangle t$?

⋆ typical test profile functions (kink & BPS for gauge field)





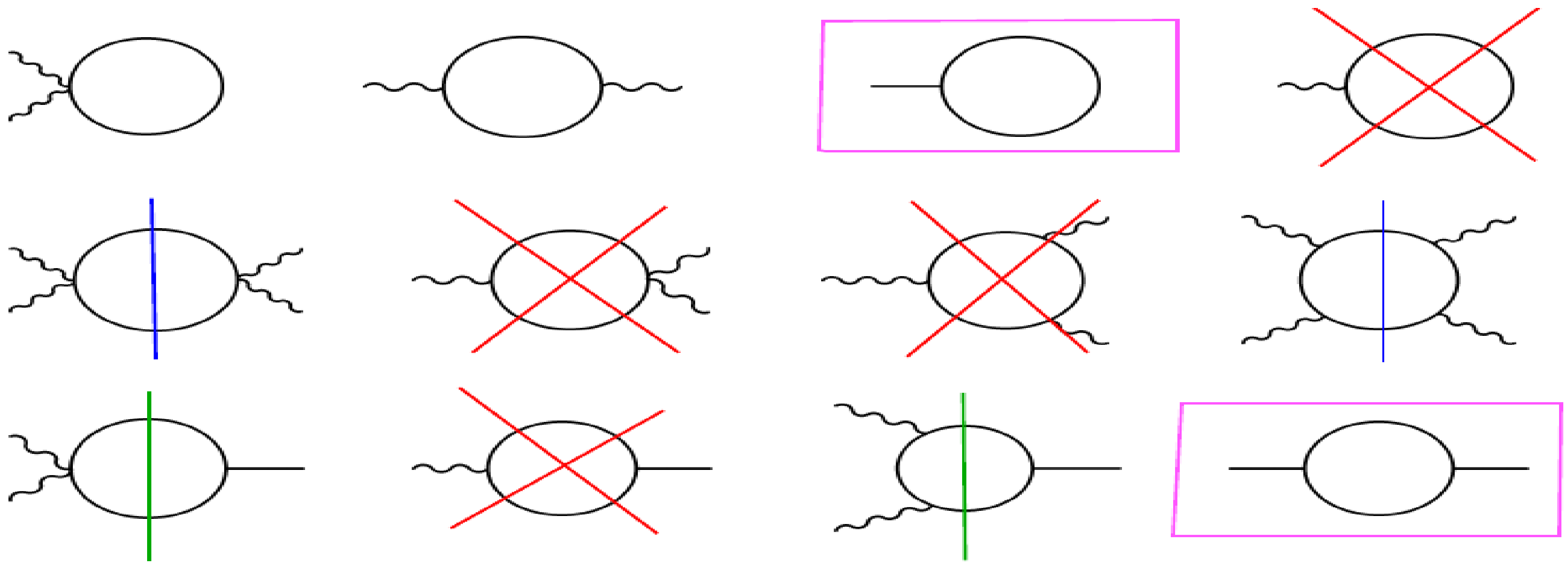
⋆ indeed seems to converge to zero, though large L needed already at moderate t

⋆ but wrong power law: suggests $\int dt\, t\, [\nu(t)]_L \leq \infty$? (should be log. divergent)

already an issue in the context of fermions coupled to a QED vortex

(Pasipoularides, hep-th/0012031; Graham *et al.*, hep-th/0410171)

⋆ gauge invariant treatment of Feynman diagrams (dimensional regularization)



-) red crosses vanish identically (odd numbers of gauge profile insertions)

-) blue and green markers: ultraviolet divergences cancel

-) pink boxes: already dealt with via $\eta_\ell^{(1)}$ and $\eta_\ell^{(2)}$

⋆ two gauge field insertions: 

⋆ contribution to VPE per unit length in dimensional regularization ($D = 4 - 2\epsilon$)

$$E^{(A)}_{\rm VPE}\Big|_{\rm div.} = \frac{1}{12\epsilon(4\pi)^2}\int d^2x\, F_{\mu\nu}F^{\mu\nu}$$

⋆ standard wavefunction renormalization for gauge field

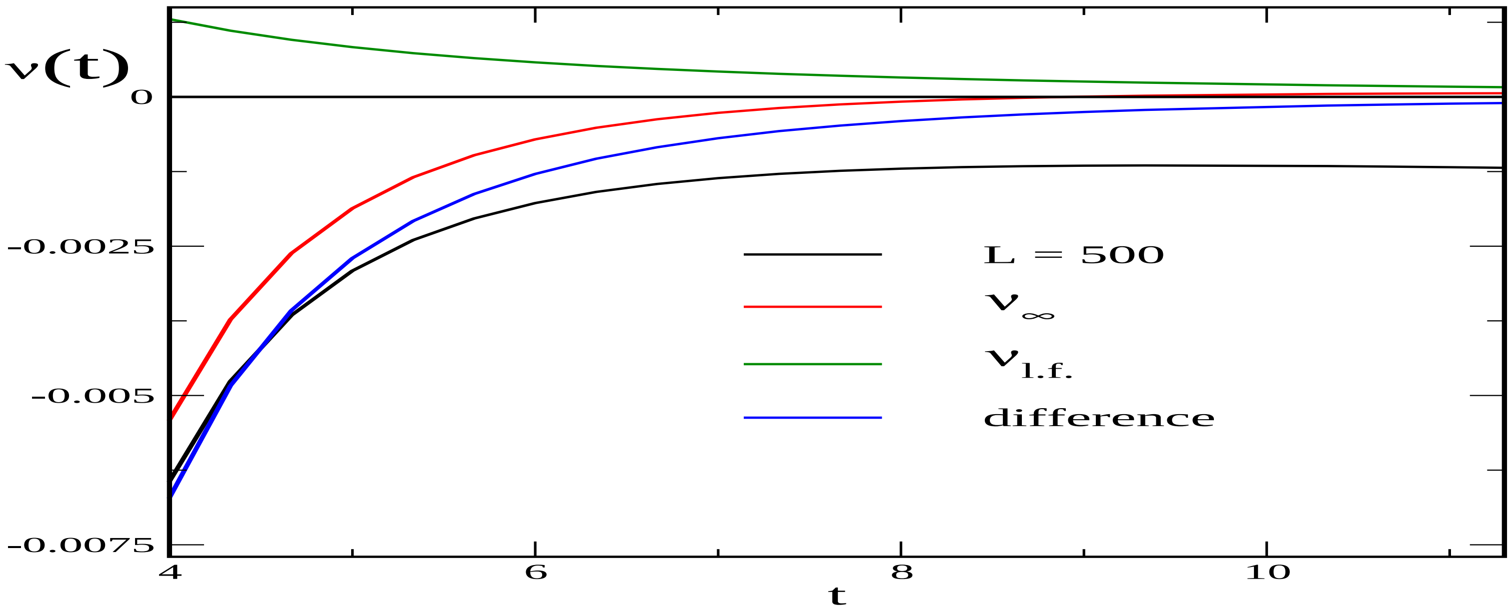
⋆ scattering data analog of this logarithmic ultraviolet divergence:

$$E^{(A)}_{\rm VPE}\Big|_{\rm div.} = \frac{1}{96\pi^2}\left[\int d^2x\, F_{\mu\nu}F^{\mu\nu}\right]\int \frac{l^2 dl}{\sqrt{l^2+M^2}^3}\Bigg|_{\rm div.} \qquad (M \text{ arbitrary})$$

implies $\quad [\nu(t)]_L \quad \longrightarrow \quad \nu_{\rm l.f.}(t) = \frac{n^2}{t^2}\frac{1}{12}\int_0^\infty \frac{d\rho}{\rho}\, g'^2(\rho) > 0$

⋆ extrapolation needed

$$[\nu(t)]_L = \nu_\infty(t) + \frac{c_1(t)}{L} + \frac{c_2(t)}{L^2} + \ldots \qquad \text{as} \qquad \rho_{\min} \to 0$$



⋆ finally, correct asymptotic behavior but not yet renormalized

⋆ standard procedure not applicable because Born series is ill-defined

⋆ fake boson trick: generate scattering data with the correct asymptotic behavior and that can be identified with a log. divergent Feynman diagram

$\overline{\nu}^{(2)}(t)$: summed second order Jost function from boson fluctuations scattering of a scalar potential $V_f(\rho)$

$$E_{\rm VPE} = \frac{1}{2\pi}\int_m^\infty t dt \left[\nu_\infty(t) - c_B \overline{\nu}^{(2)}(t)\right] + \quad \text{finite renormalization contributions}$$

(incl. $E^{(2)}_{\rm fb}$)

$$c_B = -\frac{n^2}{3}\frac{\int_0^\infty \frac{d\rho}{\rho}\, g'^2(\rho)}{\int_0^\infty \rho d\rho\, V_f^2(\rho)}$$

$$\left[\overline{\nu}^{(2)}(t) \to -\frac{1}{4t^2}\int_0^\infty \rho d\rho\, V_f^2(\rho) \quad \text{as} \quad t \to \infty\right]$$

VPE of ANO Vortex in BPS Scenario

⋆ full theory in the BPS case $\lambda = 2e^2$ has (simple) scattering problem

⋆ quantum fluctuations: $\Phi = \Phi_S + \eta$ and $A^\mu = A^\mu_S + a^\mu$

⋆ gauge fixing for gauge fluctuations

$$\mathcal{L}_{\rm gf} = -\frac{1}{2}\left[\partial_\mu a^\mu + {\rm i}e\left(\Phi_S \eta^* - \Phi_S^* \eta\right)\right]^2$$

⋆ Faddeev-Popov ghosts c

$$\mathcal{L}_{\rm gh} = \overline{c}\left(\partial_\mu \partial^\mu + 2e^2 |\Phi_S|^2\right) c + \quad \text{non-harmonic terms}$$

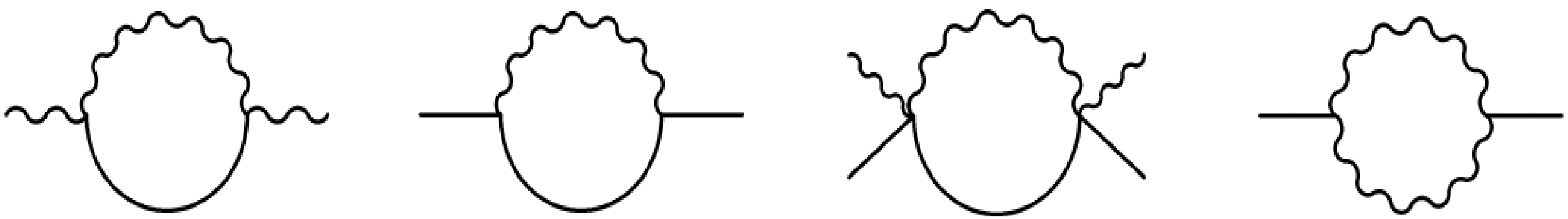
⋆ gauge fluctuations, a^μ: time-like & longitudinal ($\parallel$ vortex) decouple from transverse modes

⋆ ghost and decoupled gauge fluctuations partially cancel in $E_{\rm VPE}$!

-) full cancellation occurs in $D = 3 + 1$ (Min & Lee, hep-th/9409006)

-) in $D = 2 + 1$ only one component canceled: simple additional scalar scattering problem (adds negatively to $E_{\rm VPE}$)

⋆ additional divergent Feynman diagrams with gauge fluctuations



do not generate new singularities associated with $\rho \to 0$

⋆ four gauge invariant counterterm structures

$$C_g F_{\mu\nu}F^{\mu\nu} + C_h \left|D_\mu \Phi\right|^2 + C_0 \left(|\Phi|^2 - v^2\right) + C_V \left(|\Phi|^2 - v^2\right)^2$$

⋆ four conditions:

no-tadpole, two residues of propagators, Higgs mass

$\Longrightarrow$ gauge field mass acquires quantum corrections

its effect on $E_{\rm VPE}$ is two-loop order

Numerical Results

⋆ $D = 2 + 1$

	$n=1$	$n=2$	$n=3$	$n=4$
$E_{\rm CT} + E_{\rm FD}$	-0.2484	-0.6546	-1.0801	-1.5215
$E_{\rm scat.}$	-0.0882	-0.3408	-0.6631	-1.0205
$E_{\rm VPE}$	-0.3367	-0.9955	-1.7432	-2.5420

$$E_{\rm VPE}(n) \approx -0.3348 - 0.6314(n-1) - 0.0350(n-1)^2$$

binding energies

$$E_{\rm VPE}(n) - nE_{\rm VPE}(1) \approx -0.297(n-1) - 0.035(n-1)^2 < 0$$

$\star$ $D = 3 + 1$

	$n=1$	$n=2$	$n=3$	$n=4$
$E_{\rm CT} + E_{\rm FD}$	0.0078	-0.0054	-0.0114	-0.0157
$E_{\rm scat.}$	-0.0255	-0.0969	-0.1782	-0.2627
$E_{\rm VPE}$	-0.0177	-0.1023	-0.1896	-0.2784

$$E_{\rm VPE}(n) \approx -0.0166 - 0.0869(n-1)$$

binding energies

$$E_{\rm VPE}(n) - nE_{\rm VPE}(1) \approx -0.070(n-1) < 0$$

either case favors coalesced over isolated vortices

Summary

⋆ (almost) no word about numerical subtleties

⋆ consequences of singular vortex structure

- need to make contact with free Green's function requires singular gauge
- singular structure requires a non-dynamical *zeroth* order Born subtraction for Jost function
- *zeroth* order Born subtraction complies with gauge invariance (no quadratic divergence)
- slowly converging angular momentum sum requires extrapolation

⋆ VPE of BPS vortex approximately scales with $(1-n)$, classical energy with n

⋆ stabilization of BPS vortices with higher winding number by quantum corrections

⋆ first study of VPE in a renormalizable soliton model in different topological sectors

- relevant for binding energies in a particle interpretation of solitons
- models in $D=1+1$ lack localized static solutions with higher winding numbers
- typically soliton models in higher dimensions are not renormalizable (Skyrme, NJL, ...)

Thank you for

your attention !