Electron Mass Renormalization and QED Trace Anomaly

Michael Eides

Department of Physics and Astronomy
University of Kentucky, USA

XII International Conference on New Frontiers in Physics
10 - 22 July 2023, Crete, Greece

ME, Eur.Phys.J. C **83***, 356 (2023)*

Outline

1. Energy-momentum tensor and mass
2. Standard mass renormalization
3. Mass matrix element of EMT trace
4. What is the connection between two sets of diagrams?

EMT and mass

Matrix elements of EMT

- Translational invariance: $\langle \boldsymbol{p}'|T^{\mu\nu}(x)|\boldsymbol{p}\rangle = e^{i(p'-p)\cdot x}\langle \boldsymbol{p}'|T^{\mu\nu}(0)|\boldsymbol{p}\rangle$
- $\Longrightarrow \int d^3x\langle \boldsymbol{p}'|T^{00}(x)|\boldsymbol{p}\rangle = (2\pi)^3\delta^{(3)}(\boldsymbol{p}-\boldsymbol{p}')\langle \boldsymbol{p}'|T^{00}(0)|\boldsymbol{p}\rangle$
- $H = \int d^3xT^{00}(x)$ – Hamiltonian
- Covariant normalization: $\langle \boldsymbol{p}'|\boldsymbol{p}\rangle = 2E_p(2\pi)^3\delta^{(3)}(\boldsymbol{p}-\boldsymbol{p}')$
- $\Longrightarrow \int d^3x\langle \boldsymbol{p}'|T^{00}(x)|\boldsymbol{p}\rangle = 2E_p^2(2\pi)^3\delta^{(3)}(\boldsymbol{p}-\boldsymbol{p}')$
- $\Longrightarrow \langle \boldsymbol{p}|T^{00}(0)|\boldsymbol{p}\rangle = 2E_p^2$
- Lorentz invariance: $\langle \boldsymbol{p}|T^{\mu\nu}(0)|\boldsymbol{p}\rangle = 2p^\mu p^\nu, \quad \langle \boldsymbol{p}|T^{\mu}{}_{\mu}(0)|\boldsymbol{p}\rangle = 2m^2$
- Rest frame & nonrelativistic normalization $\langle \boldsymbol{p}'|\boldsymbol{p}\rangle = (2\pi)^3\delta^{(3)}(\boldsymbol{p}-\boldsymbol{p}')$
- $\Longrightarrow \langle \mathbf{0}|T^{\mu}{}_{\mu}(0)|\mathbf{0}\rangle = m, \qquad \langle \mathbf{0}|T^{00}(0)|\mathbf{0}\rangle = m$

EMT trace anomaly

- $T_0^{\mu\nu} = [T^{\mu\nu}]_R$ – symmetric & conserved EMT tensor, it is not renormalized
- EMT trace in gauge theories (QED, QCD, etc.) is anomalous

$$T_{0\ \mu}^{\mu} = (1+\gamma_m(e_0))\bar{\psi}_0 m_0 \psi_0 + \frac{\beta(e_0)}{2e_0} F_0^2 = (1+\gamma_m(e))[\bar{\psi} m \psi]_R + \frac{\beta(e)}{2e}[F^2]_R$$

- QED: $\beta(e)/2e = \alpha/6\pi$, $\gamma_m(e) = 3\alpha/2\pi$

Reminder: standard mass renormalization

- Renormalized perturbation theory, dimensional regularization ($d = 4 - \epsilon$)

$$\mathcal{L} = -\frac{1}{4}F^2 + \bar{\psi}(i\not{\partial} - m)\psi - \mu^{\frac{\epsilon}{2}} e\bar{\psi}\not{A}\psi,$$

$$\delta\mathcal{L} = -\frac{1}{4}\delta Z_3 F^2 + \bar{\psi}(i\delta Z_2 \not{\partial} - \delta_m)\psi - \mu^{\frac{\epsilon}{2}} e\delta Z_1 \bar{\psi}\not{A}\psi,$$

$$\mathcal{L} + \delta\mathcal{L} = -\frac{1}{4}Z_3 F^2 + iZ_2\bar{\psi}\not{\partial}\psi - mZ_m\bar{\psi}\psi - eZ_1\mu^{\frac{\epsilon}{2}}\bar{\psi}\not{A}\psi$$

Reminder: standard mass renormalization

$$\mathcal{L}_0 = \mathcal{L} + \delta\mathcal{L} = -\frac{1}{4}F_0^2 + \bar{\psi}_0(i\partial\!\!\!/ - m_0)\psi_0 - e_0\bar{\psi}_0 A\!\!\!/_0\psi_0$$

- Renormalization constants

$$Z_1 = 1 + \delta Z_1, \quad Z_2 = 1 + \delta Z_2, \quad Z_3 = 1 + \delta Z_3$$

$$mZ_m = m(1 + \delta Z_m) = m + \delta_m, \quad m_0 = mZ_mZ_2^{-1}$$

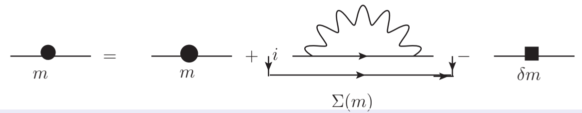
$$\delta m = m - m_0 = m - mZ_mZ_2^{-1}$$

- One-loop counterterms

$$\delta m^{(2)} \equiv m\delta Z_2 - \delta_m = \Sigma(m) = \frac{3\alpha}{4\pi}m\left[\frac{2}{\epsilon} - \gamma + \ln(4\pi) + \ln\frac{\mu^2}{m^2} + \frac{4}{3}\right]$$

$$\delta Z_2 = \Sigma'(p\!\!\!/ = m) = -\frac{\alpha}{4\pi}\left[\frac{2}{\epsilon} - \gamma + \ln(4\pi) + \ln\frac{\mu^2}{m^2} + 2\ln\frac{\lambda^2}{m^2} + 4\right]$$

- One-loop mass renormalization $\delta m^{(2)} = \Sigma(m)$



Mass matrix element of EMT trace

- $T \equiv \langle \mathbf{0} | T^{\mu}{}_{\mu}(0) | \mathbf{0} \rangle$ should be equal mass
- $T^{\mu}{}_{\mu} = (1 + \gamma_m(e_0))\bar{\psi}_0 m_0 \psi_0 + \frac{\beta(e_0)}{2e_0} F_0^2$ – EMT trace
- $\frac{\beta(e_0)}{2e_0} F_0^2$ does not contribute to the matrix element in one-loop approximation
- One-loop approximation:

$$T \approx \langle \mathbf{0} | (1 + \gamma_m)(m - \delta m^{(2)}) \bar{\psi}_0 \psi_0 | \mathbf{0} \rangle$$

$$= m + m\delta Z_2 - \delta m^{(2)} + m\gamma_m + \Gamma_m(m)$$

- $\Gamma_m(m)$ – one-loop diagram for the scalar vertex $m\bar{\psi}\psi$

$$\Gamma_m(m) = \frac{\alpha}{4\pi} m \left\{ \frac{8}{\epsilon} - 4\gamma + 2\ln\frac{\lambda^2}{m^2} + 4\ln\frac{\mu^2}{m^2} + 2 + 4\ln(4\pi) \right\}$$

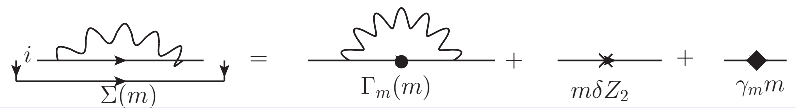
- $\Longrightarrow T = m$ by direct calculation

What is the connection between two sets of diagrams?

- Observation from two analytic expressions for mass

$$\Sigma(m) = \Gamma_m(m) + m\delta Z_2 + \gamma_m m$$

- Why this relationship holds?
- Equivalent question: Why two different sets of diagrams coincide?



Self-energy and scalar vertex

- Taylor series:

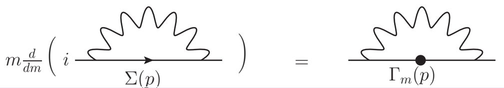
$$\Sigma(\not{p}) = \Sigma(\not{p} = m) + (\not{p} - m)\Sigma'(\not{p} = m) + O((\not{p} - m)^2)$$
$$= \delta m^{(2)} + (\not{p} - m)\delta Z_2 + O((\not{p} - m)^2)$$

Self-energy and scalar vertex

- Differentiate with respect to m at $\not p = m \Longrightarrow$

$$m\frac{d\Sigma(\not p)}{dm}_{|\not p = m} = m\frac{d\Sigma(\not p = m)}{dm} - m\Sigma'(\not p = m)$$

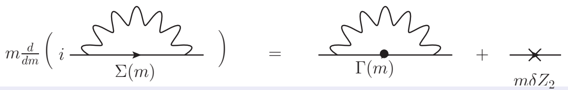
- $m\frac{d}{dm}\left(\frac{1}{\not p - m}\right) = \frac{1}{\not p - m} m \frac{1}{\not p - m} \Longrightarrow m\frac{d\Sigma(p)}{dm} = \Gamma_m(p)$



- Then

$$m\frac{d\Sigma(\not p)}{dm}_{|\not p = m} = m\frac{d\Sigma(\not p = m)}{dm} - m\Sigma'(\not p = m)$$

$$\Longrightarrow \Gamma_m(m) + m\delta Z_2 = m\frac{d\Sigma(\not p = m)}{dm}$$



Self-energy and mass anomalous dimension

- Another perspective on $m\frac{d(\Sigma(\not{p}=m))}{dm} \equiv m\frac{d(\delta m^{(2)})}{dm}$?
- But $\delta m^{(2)} = m - m_0 \approx -m\delta Z_m + m\delta Z_2$

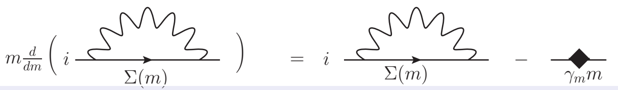
$$\Longrightarrow m\frac{d(\delta m^{(2)})}{dm} = \delta m^{(2)} + m\left(-m\frac{d\delta Z_m}{dm} + m\frac{d\delta Z_2}{dm}\right)$$

$$= \delta m^{(2)} + m\left(\mu\frac{d\delta Z_m}{d\mu} - \mu\frac{d\delta Z_2}{d\mu}\right) = \delta m^{(2)} + \mu\frac{d}{d\mu}\left(m\delta Z_m - m\delta Z_2\right)$$

$$= \delta m^{(2)} - \mu\frac{d\delta m^{(2)}}{d\mu}$$

- Recall $\gamma_m = -\mu\frac{d\delta Z_m}{d\mu} + \mu\frac{d\delta Z_2}{d\mu}$

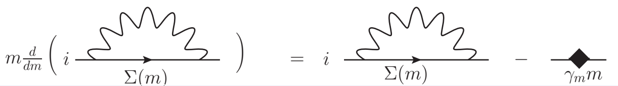
$$\Longrightarrow m\frac{d(\delta m^{(2)})}{dm} = \delta m^{(2)} - \mu\frac{d\delta m^{(2)}}{d\mu} = \delta m^{(2)} - \gamma_m m$$



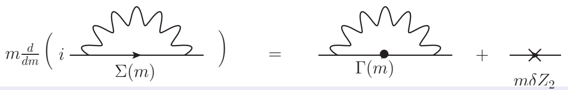
- It worked because functionally $\delta m^{(2)} = mf(m/\mu)$

Self-energy and EMT matrix element

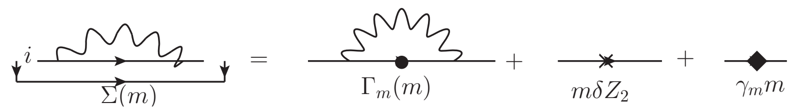
- Graphically we have shown



- and



- Combing two graphic relationships we obtain



- or analytically

$$\Sigma(m) = \Gamma_m(m) + m\delta Z_2 + \gamma_m m$$

- This explains how it happens that $\langle \mathbf{0}| T^{\mu}{}_{\mu}(0)|\mathbf{0}\rangle = m$

- We expect similar relationships to hold beyond one-loop and thus justify the equality of the standard mass renormalization and the matrix element of the EMT trace

- We expect similar relationships to hold beyond one-loop and thus justify the equality of the standard mass renormalization and the matrix element of the EMT trace

Thank you!