Loop Equations and Microscopic Theory of Turbulence

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Abstract

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We have found an infinite dimensional manifold of exact solutions of the Navier-Stokes loop equation for the Wilson loop in decaying Turbulence in arbitrary dimension $d > 2$. This family of solutions corresponds to a fractal curve in complex space $\mathbb{C}^d$ with random steps parametrized by Ising variables $\sigma = \pm 1$. This one-dimensional periodic Ising chain has some long-range interaction, leading to critical phenomena as its size $N \to \infty$. The Wilson loop, vorticity correlation functions, and energy dissipation rate are numerically simulated using an ensemble of $2 * 10^5$ curves, each with $N = 10^7$ steps. We found anomalous dissipation and some scaling laws with universal critical indexes, different from K41.

Abstract

Richard Feynman wrote half a century ago in his famous "Lectures in Physics"

"there is a physical problem that is common to many fields, that is very old, and that has not been solved. It is not the problem of finding new fundamental particles, but something left over from a long time ago—over a hundred years. Nobody in physics has really been able to analyze it mathematically satisfactorily in spite of its importance to the sister sciences. It is the analysis of circulating or turbulent fluids."

The turbulence problem looks deceptively simple: **find the limit of the solution of the Navier-Stokes equations when viscosity goes to zero** at fixed energy flow into the system.

$$\partial_t v_\alpha = \nu \partial_\beta \omega_{\beta\alpha} - v_\beta \omega_{\beta\alpha} - \partial_\alpha \left(p + \frac{v_\beta^2}{2} \right); \qquad (1)$$

$$\partial_\alpha v_\alpha = 0; \qquad (2)$$

In this limit, the Navier-Stokes equation tends to the Euler equation everywhere except some singular regions: vortex sheets and lines, where large Laplacian could compensate the factor of ν.

Introduction

The unsolved problem of classical turbulence **has challenged us for centuries, like a shining Himalaya peak**.

Burgers, Onzager, Heisenberg, Landau, Kolmogorov, Feynman, and many other **great scientists attempted and failed to reach the summit** but blazed the trail for the next generations.

Ultimate goal

The ultimate goal of turbulence studies is to solve the Navier-Stokes equations and determine why and how the solution covers some manifold rather than staying unique given initial data.

What are the properties of this manifold? If not Gibbs, **what is the probability distribution?**

Kolmogorov Legacy and the Ice Wall

Andrej Kolmogorov enlightened us 80 years ago by showing the path to the summit of Turbulence, and we have all tried to follow this path since then.

Kolmogorov's Ice Wall

Unfortunately, this path led us to the ice wall, which defeated all climbing attempts for the last half Century.

Some of us, in desperation, are throwing models against this wall in the hope these models will stick.

They stick for some time, but then they all slide down.

There is **no way around the Navier-Stokes equations, we have to solve them**.

The quantum correspondence or Duality

Lucky Navier-Stokes equation

Some lucky nonlinear PDEs were solved by reduction to a linear problem in a higher dimension. We claim such luck with the Navier-Stokes equation. However, we must go up to an infinite-dimensional loop space in this case.

Points in this space are closed loops in $d-$ dimensional physical space $\mathbb{R}^d$, or the set of d periodic functions $\vec{C}(\theta), \theta \in (0, 2\pi)$.

The linear problem is **quantum mechanics in loop space**.

As we have found in the 90ties, **Mig93**, this problem can be further reduced to the nonlinear ODE in 1+1 dimensions.

Loop equation

The Wilson loop average for the Turbulence

$$\Psi[C] = \left\langle \exp\left(\frac{\imath}{\nu}\oint_C v_\alpha dr_\alpha\right)\right\rangle \tag{3}$$

treated as a function of time and a functional of the periodic function $C : r_\alpha = C_\alpha(\theta);\ \theta \in (0, 2\pi)$ (not necessarily a single closed loop), satisfies the following **loop equation**

$$\imath\,\nu\partial_t\Psi[C] = \hat{H}\Psi[C]; \tag{4}$$

$$\hat{H} = \imath\,\oint d\vec{C}(\theta)\cdot\vec{W}\left[\frac{\delta}{\delta\vec{C}(\theta)}\right]; \tag{5}$$

Loop equation

From NS to the Linear problem in Loop space

This is the heart of the matter. We reduced the statistics of the nonlinear NS equation to the linear Schrödinger equation in loop space.
Moreover, the operator $\vec{W}$ has no direct dependence on the coordinate C; it only depends on the gradient in the loop space. The solution is a superposition of plane waves $\exp\left(\imath\hat{P}\cdot\hat{X}\right)$ in loop space $\hat{X} = \vec{C}(.)$.

Loop equation

The area derivative operator corresponds to vorticity

$$\hat{\omega}_{\alpha\beta} \equiv -\imath\,\nu \frac{\delta}{\delta C'_\alpha(\theta)} \int_{-\epsilon}^{\epsilon} d\xi \frac{\delta}{\delta C_\beta(\theta+\xi)} - \{\alpha \leftrightarrow \beta\}; \quad (6)$$

In addition to the loop equation, every valid loop functional $F[C]$ must satisfy the **Bianchi constraint** **MMEq79, Mig83**

$$\partial_\alpha \hat{\omega}_{\beta\gamma}(r) F[C] + \textbf{cyclic} = 0 \quad (7)$$

This constraint was analyzed in **M23PR** in the confinement region of large loops, where it was used to predict the Area law. The area derivative of the area of some smooth surface inside a large loop reduces to a local normal vector.

The origin of minimal surfaces in circulation theory

The Bianchi constraint is equivalent to the Plateau equation for a minimal surface (mean external curvature equals zero).

Dimensional Reduction

The crucial observation back in '93 was that the right side of the Loop equation, without random forcing, **dramatically simplifies in functional Fourier space**. The dynamics of the loop field can be reproduced in a simple Ansatz (plane wave in loop space):

$$\Psi[C] = \left\langle \exp\left(\frac{\imath}{\nu}\oint dC_\alpha(\theta)P_\alpha(\theta)\right)\right\rangle; \tag{8}$$

$$\partial_t \vec{P} = \vec{W}\left[\frac{-\imath}{\nu}\vec{P}'(\theta)\right]; \tag{9}$$

$$\omega_{\alpha\beta} \Rightarrow \frac{-\imath}{\nu}\Delta P_\alpha(\theta)P_\beta(\theta) - \{\alpha \leftrightarrow \beta\} \tag{10}$$

The equation becomes a PDE for $\vec{P}(\theta)$. All functional derivatives are gone!

The Bianchi constraint is **identically satisfied** with this Anzatz.

Random walk on a circle

The time dependence can be readily found, as the operator $\vec{W}$ is a third-order homogeneous functional of $\vec{P}$:

$$\vec{P}\,(t,\theta) = \sqrt{\frac{\nu}{2(t+t_0)}}\vec{F}(\theta); \tag{11}$$

Comparing terms we find the following relations for $\vec{F}(\theta)$ **(gap equations)**

$$(\Delta\vec{F})^2 = 1; \tag{12a}$$

$$\left(2\vec{F}\cdot\Delta\vec{F} - \imath\right)^2 + 1 = 4\vec{F}^2 \tag{12b}$$

These relations are **very interesting**. The complex numbers indicate **irreversibility**.
We need to find a family of solutions to this complex equation, which would lead to a **real velocity circulation**.

Random walk on a circle

How could the complex curve describe a real circulation?

This is possible if the **imaginary part of $\vec{P}(\theta)$ does not depend on θ**. Such an imaginary term will drop after integration over closed loop $\vec{C}(\theta)$.

We have found a **family of such real circulation solutions** **MB21** of our recurrent equation for arbitrary N.

$$\vec{F}_k = \frac{1}{2}\csc\left(\frac{\beta}{2}\right)\hat{\Omega}\cdot\left\{\cos(\alpha_k), \sin(\alpha_k)\vec{w}, i\cos\left(\frac{\beta}{2}\right)\right\}; \quad (13)$$

Here $\vec{w} \in \mathbb{S}^{d-3}$ is an arbitrary unit vector, and $\hat{\Omega} \in O(d)$ is an arbitrary rotation matrix.

Random walk on a circle

The loop equation is satisfied provided

$$\vec{F}_k \Rightarrow \vec{F}_{k+1}; \tag{14}$$

$$\left(\vec{F}_{k+1} - \vec{F}_k\right)^2 = 1; \tag{15}$$

$$\left(\vec{F}_{k+1}^2 - \vec{F}_k^2 - \imath\right)^2 + 1 = \left(\vec{F}_{k+1} + \vec{F}_k\right)^2; \tag{16}$$

$$\vec{F}_N = \vec{F}_0; \tag{17}$$

Random walk on a circle

The angles α_k must satisfy recurrent relation

$$\alpha_{k+1} = \alpha_k + \sigma_k \beta; \tag{18}$$

$$\alpha_N = \alpha_0 = 0; \tag{19}$$

$$\sigma_k^2 = 1 \tag{20}$$

This sequence with arbitrary signs $\sigma_k = \pm 1$ solves recurrent equation (14).

Random walk on a circle

The **closure condition** requires certain relations between these numbers.

The main condition is that β must be a **rational fraction of** 2π:

$$\beta = \frac{p}{q} * (2\pi); 0 < p < q < N; \tag{21}$$

$$(N - q) \pmod 2 = 0 \tag{22}$$

Random walk on a circle

In that case, the **periodic solution** for α_k will correspond to the following set of σ_k

$$\sigma = \{1, \dots, 1, -1, \dots, -1\}_{perm}; \tag{23}$$

This array has N_+ positive values and N_- negative values where

$$N_\pm = \frac{N \pm q}{2}; \tag{24}$$

$$N_+ + N_- = N; \tag{25}$$

$$N_+ - N_- = q; \tag{26}$$

Duality of turbulence

This simple statistical model describes a dual geometric theory of decaying Turbulence, similar to ADS/CFT duality. There are no approximations; this model **exactly solves** the loop equations.

Correlation functions

In this section, we only consider the **three-dimensional space** we live in.

The simplest observable quantities we can extract from the loop functional are the **vorticity correlation functions** **M23PR**, corresponding to the loop C backtracking between two points in space $\vec{r}_1 = 0, \vec{r}_2 = \vec{r}$, see Fig.19.

The **vorticity operators** are inserted at these two points.

Correlation functions

Correlation functions

The correlation function reduces to the following **average over the ensemble of our random curves in complex space**

$$\left\langle \vec{\omega}(\vec{0}) \cdot \vec{\omega}(\vec{r}) \right\rangle = \frac{1}{4(t+t_0)^2} \sum_{0 \le n < m < N} \left\langle \frac{\vec{\omega}_m \cdot \vec{\omega}_n}{(N(N-1)/2)} \exp\left(\frac{\imath \vec{r} \cdot \left(\vec{S}_{n,m} - \vec{S}_{m,n} \right)}{2\sqrt{\nu(t+t_0)}} \right) \right\rangle; \quad (27)$$

$$\vec{S}_{m,n} = \frac{\sum_m^n \vec{F}_k}{n - m \pmod N}; \quad (28)$$

The averaging $\langle \dots \rangle$ in these formulas involves **group integration** $\int_{O(3)} d\Omega$ with $\vec{F}_k \Rightarrow \hat{\Omega} \cdot \vec{F}_k$.

Correlation functions

The imaginary part of our solution (13) does not depend on the point on a circle. Therefore it contributes a constant term into $\vec{S}_{m,n}$ which **cancels in the difference** $\vec{S}_{n,m} - \vec{S}_{m,n}$ in the exponential, as it should.

The resulting integral over rotation matrix $\Omega \in O(3)$ is a particular case of the famous **Itzykson-Zuber-Harish-Chandra integral**. There is a **simple analytical formula**

$$\int_{O(3)} \frac{d\Omega}{|O(3)|} \exp\left(\imath \vec{r} \cdot \hat{\Omega} \cdot \vec{s}\right) = \frac{\sin\left(|\vec{r}||\vec{s}|\right)}{|\vec{r}||\vec{s}|} \tag{29}$$

Correlation functions

The numerical simulation of this correlation function **does not require significant computer resources**. This is like a simulation of a **one-dimensional Ising model with long-range forces**.

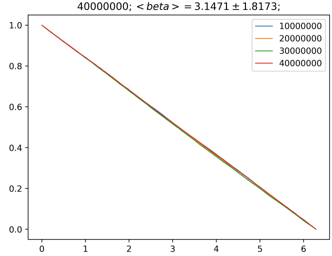
We simulated $N = 10^7$ points on a fractal curve with the **random fraction** in $\beta = 2\pi \frac{p}{q}$ and **random signs** σ_k adding up to a multiple of q, with q being the same parity as N.

We took $T = 2 * 10^5$ random data samples for $\beta, \vec{\omega}_n \cdot \vec{\omega}_m, |\Delta \vec{S}|$, where $\Delta \vec{S} = \vec{S}_{n,m} - \vec{S}_{m,n}$ with randomly chosen $0 \leq n < m < N$.

We collected **large statistics** needed for the correlation function.

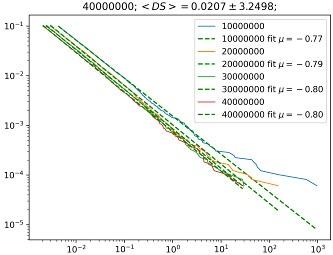
Correlation functions

The distributions of β for various N are all approximately linear and matching.



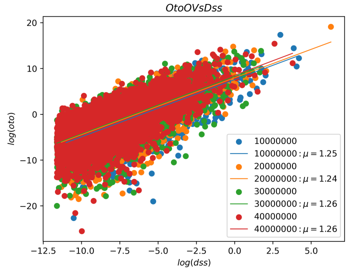
Correlation functions

The distribution of the length of the difference of average $\vec{F}$ vectors in a log-log scale for various N



Correlation functions

The cloud of points $\Delta S, \vec{\omega}_m \cdot \vec{\omega}_n$, in a log-log scale for various N.



Correlation functions

The **distribution of the difference** $\Delta S = \left|\vec{S}_{n,m} - \vec{S}_{m,n}\right|$ on Fig.24 is **a power law**

$$W\left(\Delta S > x\right) \propto x^{-\beta}; \tag{30}$$
$$\beta \approx 0.8 \tag{31}$$

We measured the dependence between $\Delta S, \vec{\omega}_m \cdot \vec{\omega}_n$ in a scattered log-log plot (Fig.25). The results are consistent for four different values of $N = 10M \ldots 40M$:

$$\vec{\omega}_m \cdot \vec{\omega}_n \propto (\Delta S)^{\mu}; \tag{32}$$
$$\mu \approx 1.25 \tag{33}$$

Our complex curve is a fractal, indeed!

We did not tune any parameters in our statistical model, just increased its size N. Still, there are fractal scaling laws.

Correlation functions

After that, the **correlation can be computed analytically** using these scaling laws (with $\rho = \frac{r}{\sqrt{2\nu t}}$)

$$\left\langle \vec{\omega}(\vec{0}) \cdot \vec{\omega}(\vec{r}) \right\rangle \propto \frac{1}{t^2} \int_0^\infty ds \frac{\sin(\rho s)}{\rho s} s^{\mu-\beta-1} =$$
$$-\frac{1}{t^2} \rho^{\beta-\mu} \cos\left(\frac{1}{2}\pi\left(\beta-\mu\right)\right) \Gamma\left(-\beta+\mu-1\right)$$
$$\propto \frac{1}{t^2} \left(\frac{\nu t}{r^2}\right)^{0.225} \tag{34}$$

Anomalous dissipation

Remarkably, this vorticity correlation function diverges at $\vec{r} \to 0$, corresponding to anomalous dissipation.

Correlation functions

This scaling index $-2 * 0.225 \approx -0.45$ is **far from** $-\frac{4}{3}$, but it only applies to decaying turbulence.

The simulation took less than an hour on the Linux workstation for $N = 1M, 2M, 3M, 4M$. We can go for $N = 1Bn$ on a cluster, as our algorithm complexity grows linearly with N.

The experimental data and simulations indicate a power decay of the dissipation rate $\mathcal{E} \sim t^{-1-n}$ with $n \approx 1.2$ in contrast with our $n \approx 0.78$.

No match with experiment so far

We cannot explain this discrepancy: perhaps our asymptotic regime has not yet been observed. It is also premature to compare our simulations on a personal workstation with an experiment: the large-scale cluster simulation is needed.

Discussion

We have presented an **exact solution of the Navier-Stokes loop equations** for the Wilson loop in decaying Turbulence, reducing fluctuating velocity field in d dimensions to the fractal curve in $2d$ dimensions.

Is it THE solution? Time will tell

Our solution is universal, rotational, and translational invariant. It has all the expected properties of isotropic decaying Turbulence.