Quantum vs classical logic: information loss

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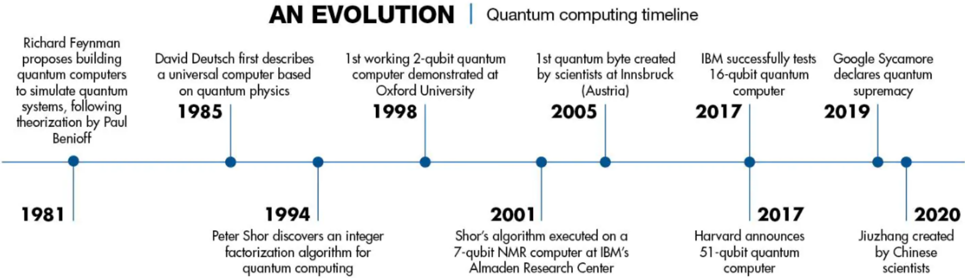
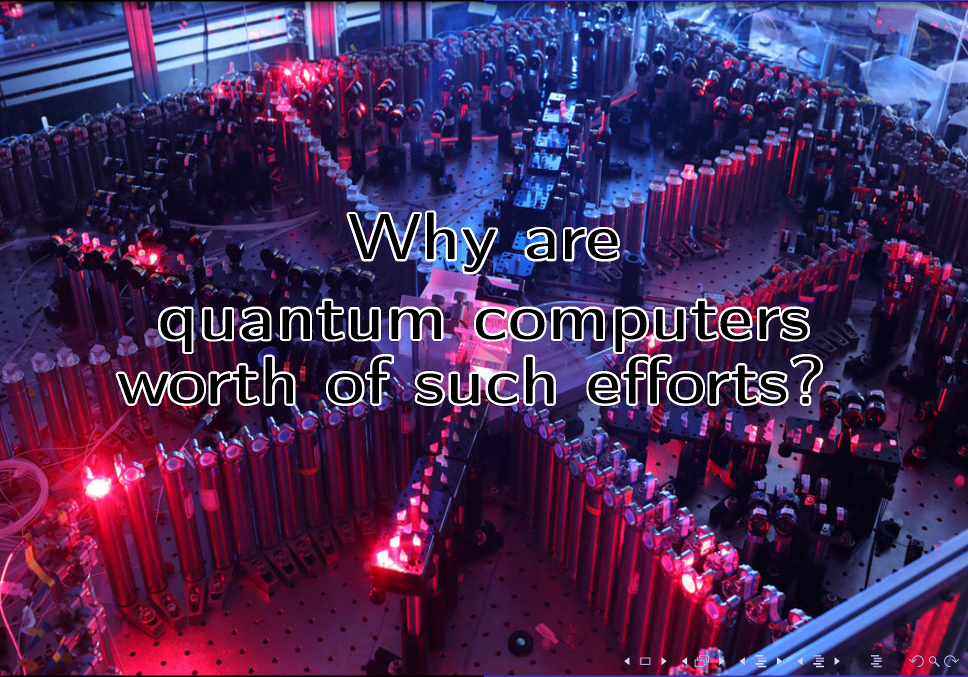
Quantum computing timeline

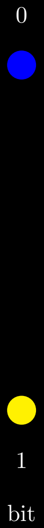


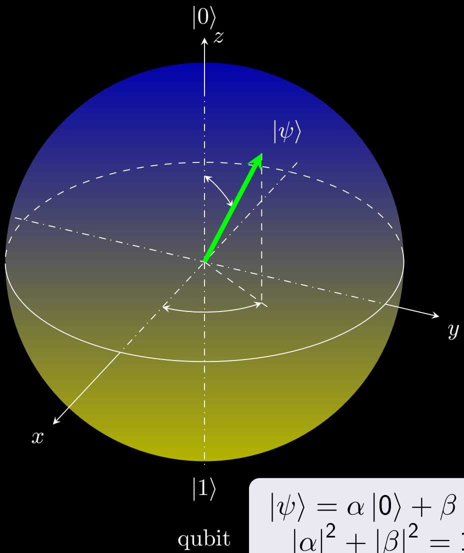
Sources: Indiana University, ScienceNode



Why are quantum computers worth of such efforts?

Bit vs qubit





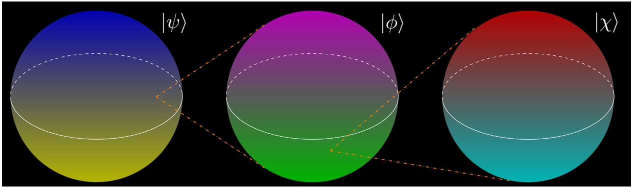
0 or 1

$$|\psi\rangle = \alpha\,|0\rangle + \beta\,|1\rangle$$
$$|\alpha|^2 + |\beta|^2 = 1$$

Superposition & linearity

Qubit register:

$$|\psi\rangle \otimes |\phi\rangle \otimes |\chi\rangle$$



Quantum computation

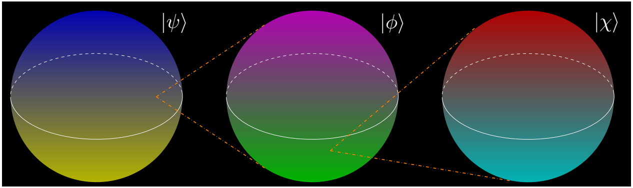
$$|\text{input}\rangle$$

$$|\text{output}\rangle = U\,|\text{input}\rangle$$

Superposition & linearity

Qubit register:

$$|\psi\rangle \otimes |\phi\rangle \otimes |\chi\rangle$$

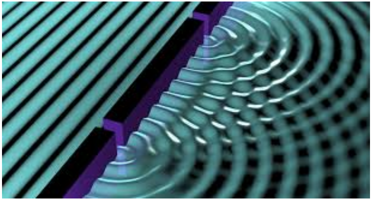


Quantum computation

$$|\mathrm{input}\rangle = \sum_{i=0}^{2^n-1} \alpha_i |i\rangle$$

$$|\mathrm{output}\rangle = U|\mathrm{input}\rangle = \sum_{i=0}^{2^n-1} \alpha_i U |i\rangle$$

Quantum interference of amplitudes $\{\alpha_i\}, i = \overline{0, 2^n - 1}$



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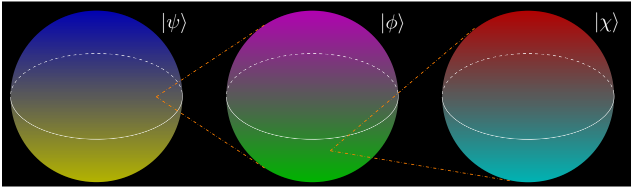

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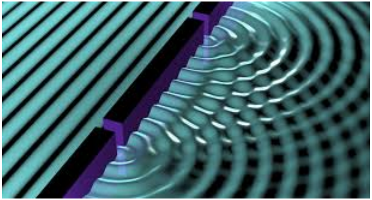


Gottesmann-Knill theorem

[Gottesman, arXiv:quant-ph/9807006]

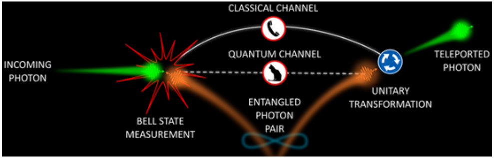
Entanglement & communication

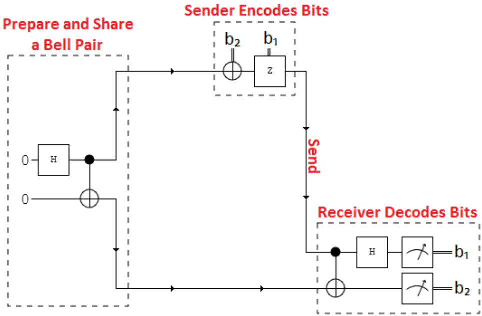
Teleportation

transmit qubit
from A to B

[Bennett et al, PRL (1993)]

[Bouwmeester et al, Nature (1997)]





Superdense coding

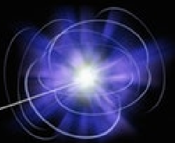
transmit 2 bits at the cost of 1 qubit

[Bennett, Wiesner, PRL (1992)]

[Mattle et al, PRL (1996)]

Entanglement: resource or not?

- ✓ Indicates correlations impossible for arbitrary classical system (quantum nonlocality)
- ✓ Violates Bell inequalities (the Nobel Prize in Physics 2022)
- ✓ Handy in communication protocols
- ✓ If entanglement is upper-bounded $\Rightarrow$ quantum algorithm can be efficiently simulated classically [Jozsa, Linden, RSPA (2003)]



Numerical analysis of Shor's algorithm [Kendon, Munro, QIC (2006)]:

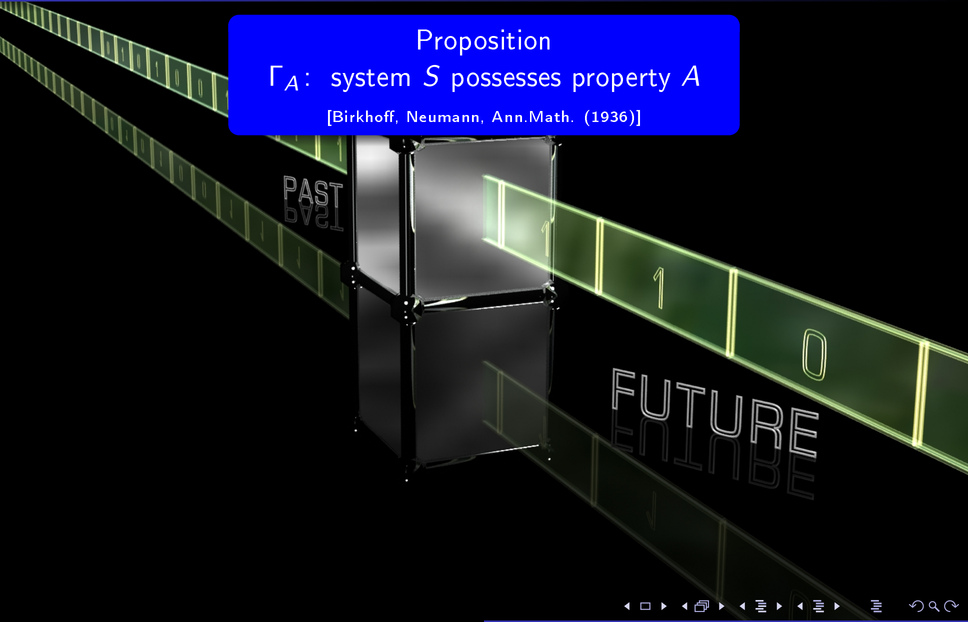
- ✗ Appearance of entanglement patterns
- ✗ No evidence of their contribution to the speed-up

Logic

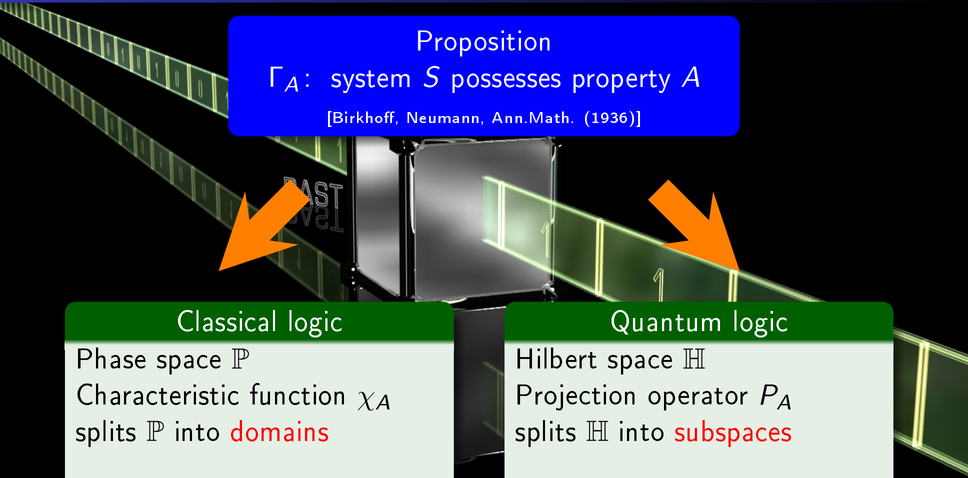
Proposition

$\Gamma_A:$ system S possesses property A

[Birkhoff, Neumann, Ann.Math. (1936)]



Logic



Proposition

Γ_A : system S possesses property A

[Birkhoff, Neumann, Ann.Math. (1936)]

Classical logic

Phase space $\mathbb{P}$

Characteristic function χ_A

splits $\mathbb{P}$ into domains

Quantum logic

Hilbert space $\mathbb{H}$

Projection operator P_A

splits $\mathbb{H}$ into subspaces

Logic

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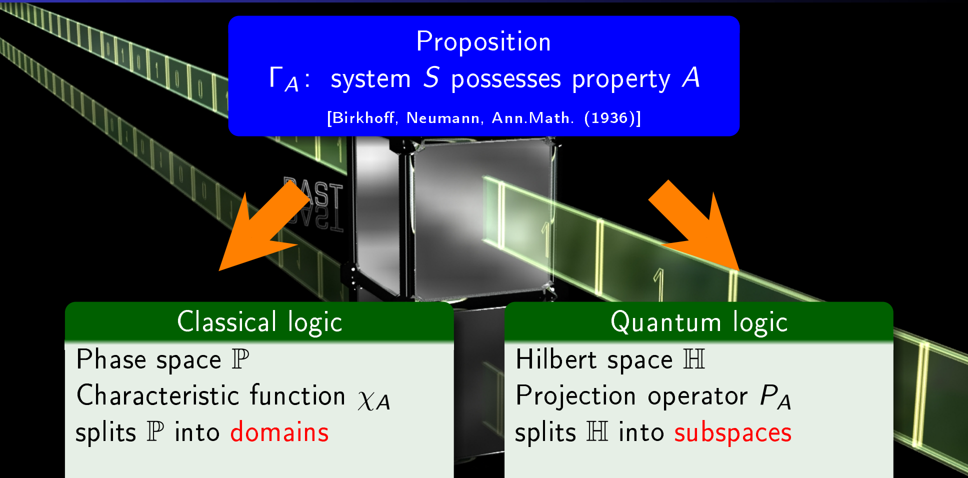


Classical logic

Phase space $\mathbb{P}$

Characteristic function χ_A

splits $\mathbb{P}$ into domains

$$\Gamma_A = \text{TRUE} \Leftrightarrow \chi_A = 1$$
$$\Gamma_A = \text{FALSE} \Leftrightarrow \chi_A = 0$$

Quantum logic

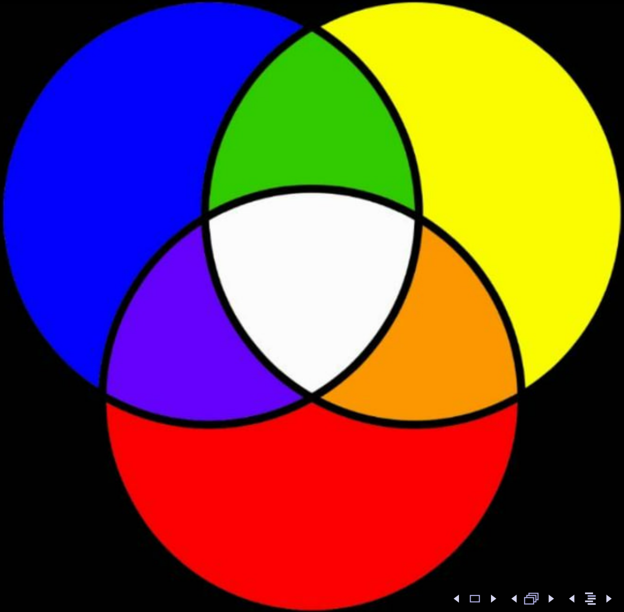
Hilbert space $\mathbb{H}$

Projection operator P_A

splits $\mathbb{H}$ into subspaces

$$\Gamma_A = \text{TRUE} \Leftrightarrow P_A\,|S\rangle \neq 0$$
$$\Gamma_A = \text{FALSE} \Leftrightarrow P_A\,|S\rangle = 0$$

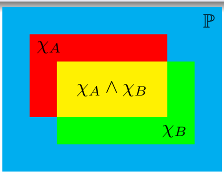
Classical logic: subsets of $\mathbb{P}$



Classical logic

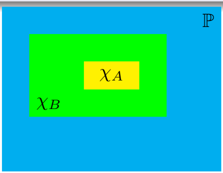
Conjunction $\wedge$ (AND)

$$\chi_A \wedge \chi_B = \chi_A \chi_B$$



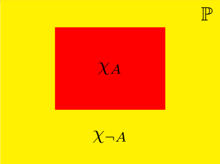
Implication $\leq$ (ordering)

$$\chi_A \leq \chi_B \Leftrightarrow \chi_A \wedge \chi_B = \chi_A$$



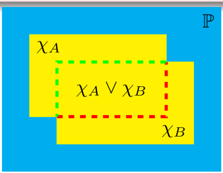
Negation $\neg$ (NOT)

$$\chi_{\neg A} = 1 - \chi_A$$

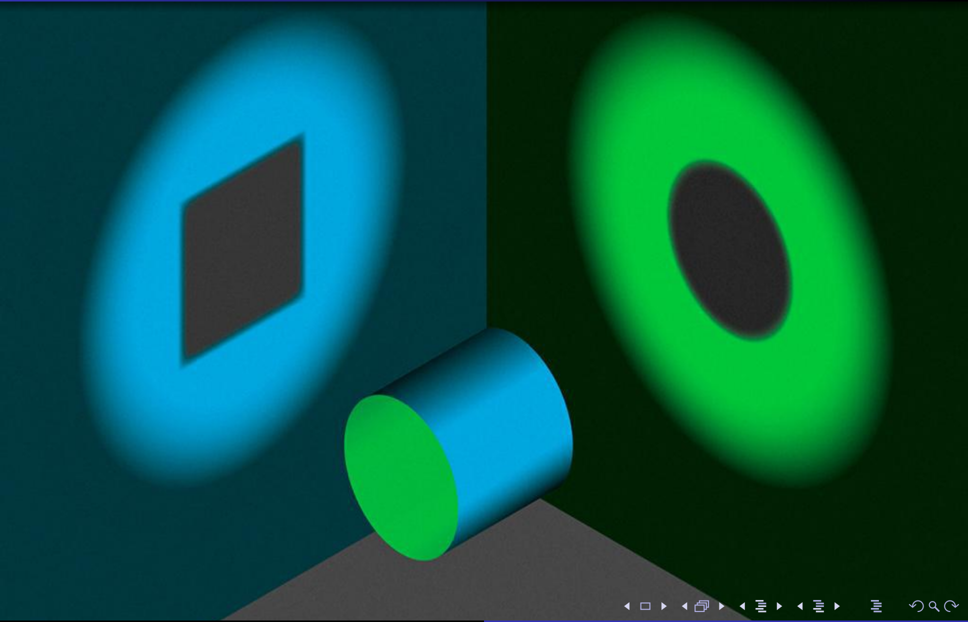


Disjunction $\vee$ (OR)

$$\chi_A \vee \chi_B = \chi_A + \chi_B - \chi_A \wedge \chi_B$$



Quantum logic: projection onto subspaces of $\mathbb{H}$



Quantum logic

Conjunction $\wedge$ (AND)

$$(P_A \wedge P_B)\,|S\rangle = \begin{cases} P_A P_B\,|S\rangle\,, & P_A P_B = P_B P_A \\ \lim_{n\to\infty} (P_A P_B)^n\,|S\rangle\,, & P_A P_B \neq P_B P_A \end{cases}$$

Implication $\leq$ (ordering)

$$P_A \leq P_B \Leftrightarrow (P_A \wedge P_B)\,|S\rangle = P_A\,|S\rangle$$

Negation $\neg$ (NOT)

$$P_{\neg A}\,|S\rangle = (1 - P_A)\,|S\rangle$$

Disjunction $\vee$ (OR)

$$(P_A \vee P_B)\,|S\rangle = (P_A + P_B - P_A \wedge P_B)\,|S\rangle$$

Non-commutativity

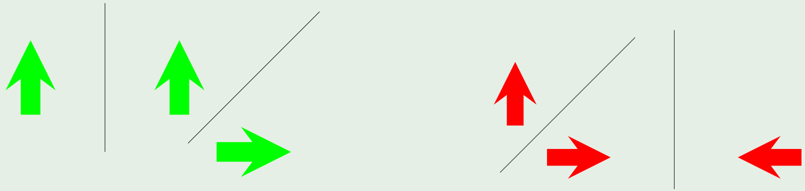
Functions commute

$\chi_A \chi_B = \chi_B \chi_A$

Operators do not commute

$P_A P_B \neq P_B P_A$ (in general)

Example: reflections



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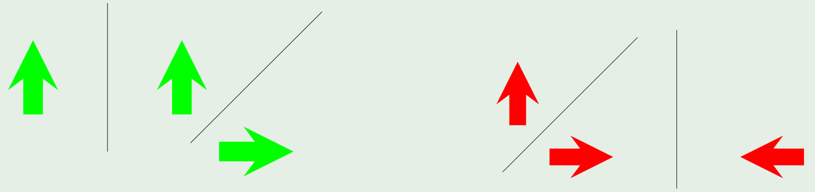

$\Rightarrow$ quantum logic rules are weaker than the classical ones

$\Rightarrow$ quantum and classical computation should differ

Distributive law

Classical proposition

$$\chi_A \wedge (\chi_B \vee \chi_C) = (\chi_A \wedge \chi_B) \vee (\chi_A \wedge \chi_C)$$

Quantum proposition

$$[P_A \wedge (P_B \vee P_C)] |S\rangle \neq [(P_A \wedge P_B) \vee (P_A \wedge P_C)] |S\rangle$$

Distributive law

Classical proposition

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Quantum proposition

$$[P_A \wedge (P_B \vee P_C)] \left|S\right\rangle \neq [(P_A \wedge P_B) \vee (P_A \wedge P_C)] \left|S\right\rangle$$

Example: spin $s = 1/2$ projections

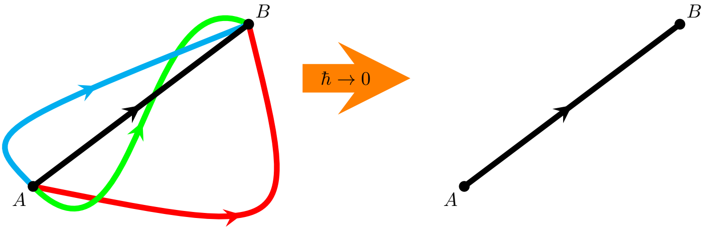
P_x and P_z – projectors onto x and z axes correpondingly

$$\Big[P_x \wedge \underbrace{(P_z \vee P_{\neg z})}_{1}\Big] \left|S\right\rangle = (P_x \wedge 1) \left|S\right\rangle = P_x \left|S\right\rangle$$

$$\neq \Big[\underbrace{(P_x \wedge P_z)}_{0} \vee \underbrace{(P_x \wedge P_{\neg z})}_{0}\Big] \left|S\right\rangle$$

$$= (0 \vee 0) \left|S\right\rangle = 0$$

Common basis: path integrals & $\mathbb{P}$



$\hbar \to 0$

$\Rightarrow$ all possible trajectories reduce to the one
$\Rightarrow$ information about all other trajectories is lost
$\Leftrightarrow$ the cost of calculation should increase

$\lim_{\hbar\to 0}$

Quantum realm

$$P_A \left|S\right\rangle = \left|A\right\rangle \left\langle A \,|\, S\right\rangle$$

$$\left\langle A \,|\, S\right\rangle = \int \mathcal{D}\left[y\right] e^{iS_{|S\rangle\to|A\rangle}\left[y\right]/\hbar}$$

Classical realm

$$\lim_{\hbar\to 0} \frac{\hbar}{i} \ln \left\langle A \,|\, S\right\rangle = \chi_A S_{|S\rangle\to|A\rangle}\left[y\right]$$

$$\chi_A = \begin{cases} 1, & \delta S_{|S\rangle\to|A\rangle}\left[y\right] = 0 \\ 0, & \delta S_{|S\rangle\to|A\rangle}\left[y\right] \neq 0 \end{cases}$$

$$P_A \overset{\hbar\to 0}{\dashrightarrow} \chi_A$$

Information loss

$P_A \left| S \right\rangle$ – pure state
$\Rightarrow$ its von Neumann entropy $H\left(P_A \left| S \right\rangle\right) = 0$

$$P_A \overset{\hbar \to 0}{\dashrightarrow} \chi_A$$

$$H\left(\chi_A\right) = -\phi_A \ln \phi_A - \left(1 - \phi_A\right) \ln \left(1 - \phi_A\right) \leq \ln 2$$

$$\phi_A = \frac{\int \mathcal{D}[y] \chi_A}{\int \mathcal{D}[y]}$$

Non-commutativity & information loss

$$P_A P_B - P_B P_A = i\hbar\Pi, \quad \Pi\,|\pi\rangle = \pi\,|\pi\rangle$$

$$|S\rangle = \sum_{\pi}^{\dim\Pi} \alpha_\pi\,|\pi\rangle$$

$\xrightarrow{\hbar\to 0}$ $\Rightarrow$ partial trace Tr_Π

additional entropy:

$$H(\chi_\Pi) \equiv H(P_A P_B\,|S\rangle) = -\sum_{\pi}^{\dim\Pi} |\alpha_\pi|^2 \ln|\alpha_\pi|^2 \le \ln\dim\Pi$$

Results: general expression

$$H\left(\mathbb{E}\,|S\rangle\right) = \sum_{i=1}^{q} H\left(\chi_{\Gamma_{A_i}}\right) + H\left(\chi_{\Pi_1}\right) + \sum_{\pi_1}^{\dim \Pi_1} |\alpha_{\pi_1}|^2\, H\left(\chi_{\Pi_2|\pi_1}\right)$$

$$\leq (q+c)\ln 2 + \sum_{k=1}^{c} \ln \dim \Pi_k$$

q – number of qubits equipped in no conjunction

c – number of conjunctions (operator products)

[TTZ, Ukr.J.Phys. (2022)]

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The largest loss is generated by non-commuting expressions

Results: examples

Fast Fourier Transform

$$\mathrm{Time}\left(\mathrm{FFT_C}\right) = \mathcal{O}\left(n2^n\right)$$

$$\mathrm{Time}\left(\mathrm{FFT_Q}\right) = \mathcal{O}\left(n^2\right)$$

$$H\left(\mathrm{FFT_Q}\right) = \mathcal{O}\left(n2^n\right)$$

Search algorithm

$$\mathrm{Time}\left(\mathrm{SA_C}\right) = \mathcal{O}\left(2^n\right)$$

$$\mathrm{Time}\left(\mathrm{SA_Q}\right) = \mathcal{O}\left(2^{n/2}\right)$$

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Quantum algorithm may experience changeover under $\lim_{\hbar\to 0}$

[Litvinov et al., in ACAMP, NATO Sci. Ser. (2002)]

Conclusions

- ✓ The process of quantum logic reduction to its classical counterpart via $\lim_{\hbar\to 0}$ is studied in details
- ✓ Information loss is estimated for each quantum gate from the complete set
- ✓ The largest loss is observed for non-commuting operators
- ✓ Both quantum Fast Fourier transform and Grover algorithms demonstrate significant information loss under $\lim_{\hbar\to 0}$
- ✓ Amount of information loss $H(\mathbb{E}\,|S\rangle)$ depends on $\mathbb{E}$ due to a change in the problem the algorithm solves.

BACKUP

Fast Fourier Transform

$$\mathrm{FFT_Q} = U_0 U_1 \cdots U_{n-1}$$
$$U_k = H_k C_{k,n-1} C_{k,n-2} \cdots C_{k,k+1}$$

$$\text{Hadamard gate } H_k: \ |q\rangle_k \to \frac{1}{\sqrt{2}} \left(|0\rangle_k + e^{i\pi q} |1\rangle_k \right), \quad q = \overline{0,1}$$
$$H_k = \sqrt{2} \left(P_{x,k} - P_{\neg z,k} \right)$$

$$\text{c-phase gate } C_{k,k'}: \ |q\rangle_k \left|q'\right\rangle_{k'} \to e^{i\phi_{kk'} q q'} |q\rangle_k \left|q'\right\rangle_{k'}$$
$$C_{k,k'} = \left(1 - e^{i\phi_{kk'}}\right) \left(P_{\neg z,k} P_{z,k'} + P_{z,k} \right) + e^{i\phi_{kk'}}$$
$$\phi_{kk'} = \pi / 2^{k'-k}$$

Grover search algorithm

$$\mathrm{SA_Q} = \left\{ \left[2\left(HP_zH\right)^{\otimes n} - I^{\otimes n} \right] \otimes \left(P_{\neg z} - P_z\right) \right\}^{\frac{\pi}{4}2^{n/2}} U_f$$

$$U_f : \left|\overrightarrow{q}\right\rangle \otimes |0\rangle \to \left|\overrightarrow{q}\right\rangle \otimes \left|f\left(\overrightarrow{q}\right)\right\rangle, \quad \overrightarrow{q} \in \overline{0, 2^n - 1}$$