

# Quantum vs classical logic: information loss

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# Quantum computing timeline

## AN EVOLUTION | Quantum computing timeline

Richard Feynman proposes building quantum computers to simulate quantum systems, following theorization by Paul Benioff

David Deutsch first describes a universal computer based on quantum physics

1st working 2-qubit quantum computer demonstrated at Oxford University

1st quantum byte created by scientists at Innsbruck (Austria)

IBM successfully tests 16-qubit quantum computer

Google Sycamore declares quantum supremacy

1981

1985

1998

2005

2017

2019

1994

Peter Shor discovers an integer factorization algorithm for quantum computing

2001

Shor's algorithm executed on a 7-qubit NMR computer at IBM's Almaden Research Center

2017

Harvard announces 51-qubit quantum computer

2020

Jiuzhang created by Chinese scientists

Sources: Indiana University, ScienceNode

The image shows a large-scale quantum computing cryostat, likely a superconducting qubit system. It consists of numerous vertical stainless steel tubes arranged in a circular pattern, each containing a qubit. The system is illuminated with red laser light, which is used for precise alignment and control. The background is dark, highlighting the intricate structure of the cryostat.

Why are  
quantum computers  
worth of such efforts?

# Bit vs qubit

0

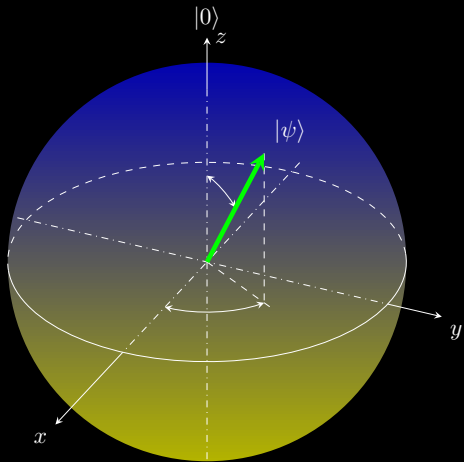


1



0 or 1

bit



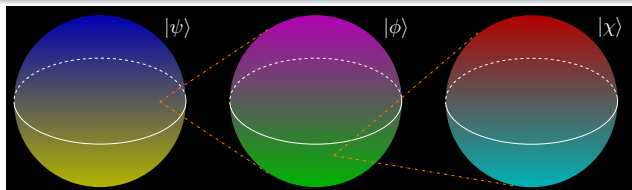
$|1\rangle$   
qubit

$$|\psi\rangle = \alpha |0\rangle + \beta |1\rangle$$
$$|\alpha|^2 + |\beta|^2 = 1$$

# Superposition & linearity

Qubit register:

$$|\psi\rangle \otimes |\phi\rangle \otimes |\chi\rangle$$



## Quantum computation

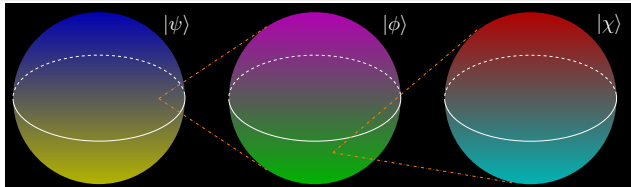
$|\text{input}\rangle$

$$|\text{output}\rangle = U |\text{input}\rangle$$

# Superposition & linearity

Qubit register:

$$|\psi\rangle \otimes |\phi\rangle \otimes |\chi\rangle$$

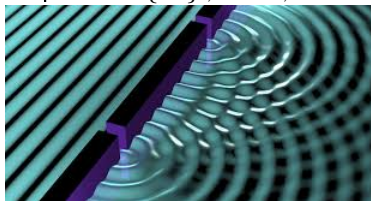


Quantum computation

$$|\text{input}\rangle = \sum_{i=0}^{2^n-1} \alpha_i |i\rangle$$

$$|\text{output}\rangle = U |\text{input}\rangle = \sum_{i=0}^{2^n-1} \alpha_i U |i\rangle$$

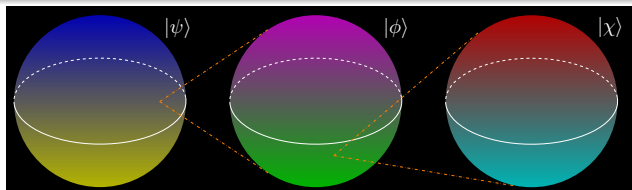
Quantum interference of amplitudes  $\{\alpha_i\}$ ,  $i = \overline{0, 2^n - 1}$



# Superposition & linearity

Qubit register:

$$|\psi\rangle \otimes |\phi\rangle \otimes |\chi\rangle$$

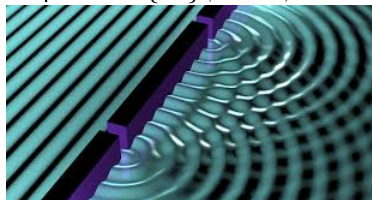


Quantum computation

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Quantum interference of amplitudes  $\{\alpha_i\}$ ,  $i = \overline{0, 2^n - 1}$



Gottesmann-Knill theorem

[Gottesman, arXiv:quant-ph/9807006]

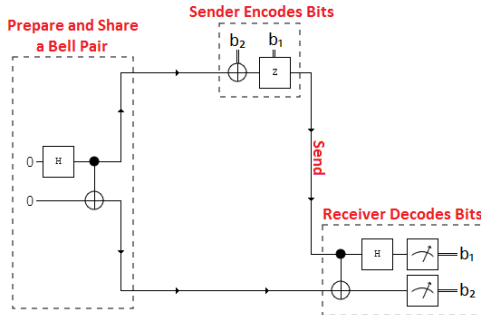
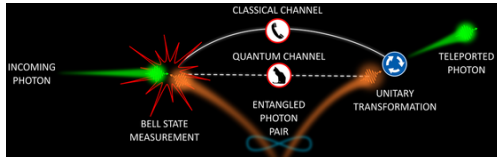
# Entanglement & communication

## Teleportation

transmit qubit  
from  $A$  to  $B$

[Bennett et al, PRL (1993)]

[Bouwmeester et al, Nature (1997)]



## Superdense coding

transmit 2 bits at the  
cost of 1 qubit

[Bennett, Wiesner, PRL (1992)]

[Mattle et al, PRL (1996)]



# Entanglement: resource or not?

- ✓ Indicates correlations impossible for arbitrary classical system (quantum nonlocality)
- ✓ Violates Bell inequalities (the Nobel Prize in Physics 2022)
- ✓ Handy in communication protocols
- ✓ If entanglement is upper-bounded  $\Rightarrow$  quantum algorithm can be efficiently simulated classically

[Jozsa, Linden, RSPA (2003)]

## Numerical analysis of Shor's algorithm

[Kendon, Munro, QIC (2006)]:

- ✗ Appearance of entanglement patterns
- ✗ No evidence of their contribution to the speed-up

## Proposition

$\Gamma_A$ : system  $S$  possesses property  $A$

[Birkhoff, Neumann, Ann.Math. (1936)]

PAST

1

0

FUTURE

## Proposition

$\Gamma_A$ : system  $S$  possesses property  $A$

[Birkhoff, Neumann, Ann.Math. (1936)]

### Classical logic

Phase space  $\mathbb{P}$

Characteristic function  $\chi_A$

splits  $\mathbb{P}$  into **domains**

### Quantum logic

Hilbert space  $\mathbb{H}$

Projection operator  $P_A$

splits  $\mathbb{H}$  into **subspaces**

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[Birkhoff, Neumann, Ann.Math. (1936)]

### Classical logic

Phase space  $\mathbb{P}$

Characteristic function  $\chi_A$

splits  $\mathbb{P}$  into **domains**

$$\Gamma_A = \text{TRUE} \Leftrightarrow \chi_A = 1$$

$$\Gamma_A = \text{FALSE} \Leftrightarrow \chi_A = 0$$

### Quantum logic

Hilbert space  $\mathbb{H}$

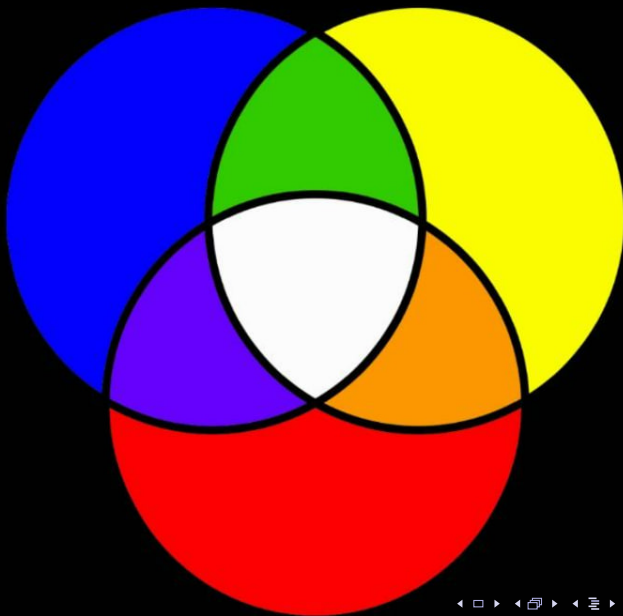
Projection operator  $P_A$

splits  $\mathbb{H}$  into **subspaces**

$$\Gamma_A = \text{TRUE} \Leftrightarrow P_A |S\rangle \neq 0$$

$$\Gamma_A = \text{FALSE} \Leftrightarrow P_A |S\rangle = 0$$

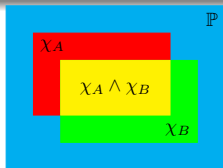
# Classical logic: subsets of $\mathbb{P}$



# Classical logic

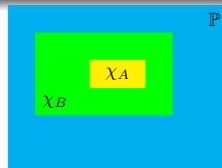
Conjunction  $\wedge$  (AND)

$$\chi_A \wedge \chi_B = \chi_A \chi_B$$



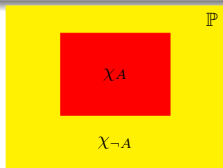
Implication  $\leq$  (ordering)

$$\chi_A \leq \chi_B \Leftrightarrow \chi_A \wedge \chi_B = \chi_A$$



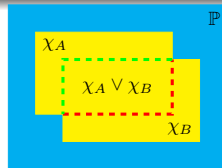
Negation  $\neg$  (NOT)

$$\chi_{\neg A} = 1 - \chi_A$$

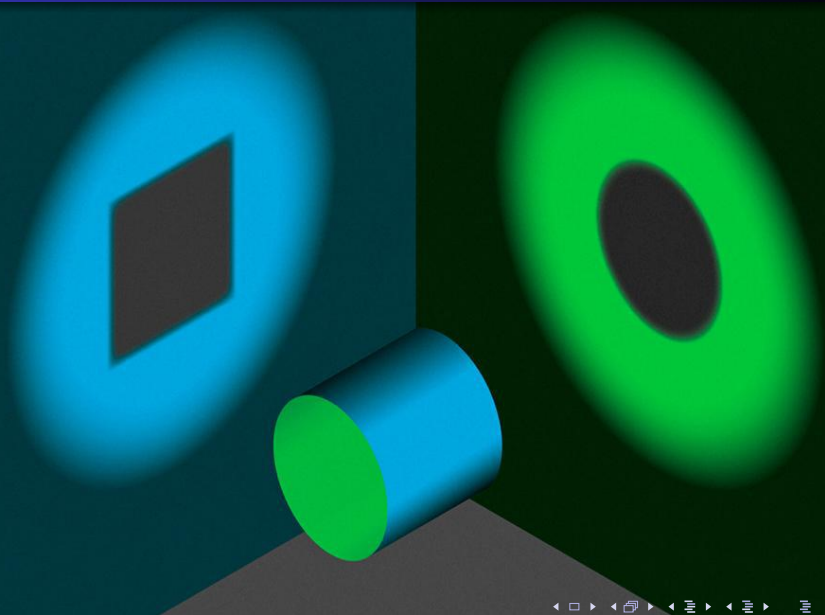


Disjunction  $\vee$  (OR)

$$\chi_A \vee \chi_B = \chi_A + \chi_B - \chi_A \wedge \chi_B$$



# Quantum logic: projection onto subspaces of $\mathbb{H}$



## Conjunction $\wedge$ (AND)

$$(P_A \wedge P_B) |S\rangle = \begin{cases} P_A P_B |S\rangle, & P_A P_B = P_B P_A \\ \lim_{n \rightarrow \infty} (P_A P_B)^n |S\rangle, & P_A P_B \neq P_B P_A \end{cases}$$

## Implication $\leq$ (ordering)

$$P_A \leq P_B \Leftrightarrow (P_A \wedge P_B) |S\rangle = P_A |S\rangle$$

## Negation $\neg$ (NOT)

$$P_{\neg A} |S\rangle = (1 - P_A) |S\rangle$$

## Disjunction $\vee$ (OR)

$$(P_A \vee P_B) |S\rangle = (P_A + P_B - P_A \wedge P_B) |S\rangle$$



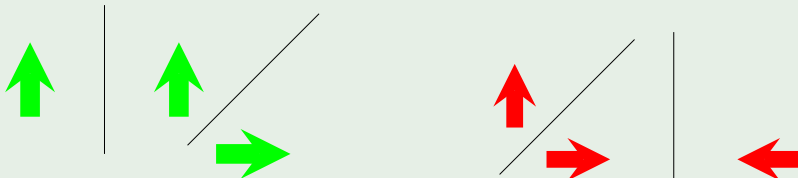
# Non-commutativity

Functions commute

$$\chi A \chi B = \chi B \chi A$$

Operators do not commute  
 $P_A P_B \neq P_B P_A$  (in general)

Example: reflections



# Non-commutativity

Functions commute

$$\chi A \chi B = \chi B \chi A$$

Operators do not commute

$$P_A P_B \neq P_B P_A \text{ (in general)}$$

Example: reflections



- ⇒ quantum logic rules are weaker than the classical ones
- ⇒ quantum and classical computation should differ

# Distributive law

Classical proposition

$$\chi_A \wedge (\chi_B \vee \chi_C) = (\chi_A \wedge \chi_B) \vee (\chi_A \wedge \chi_C)$$

Quantum proposition

$$[P_A \wedge (P_B \vee P_C)] |S\rangle \neq [(P_A \wedge P_B) \vee (P_A \wedge P_C)] |S\rangle$$

# Distributive law

Classical proposition

$$\chi_A \wedge (\chi_B \vee \chi_C) = (\chi_A \wedge \chi_B) \vee (\chi_A \wedge \chi_C)$$

Quantum proposition

$$[P_A \wedge (P_B \vee P_C)] |S\rangle \neq [(P_A \wedge P_B) \vee (P_A \wedge P_C)] |S\rangle$$

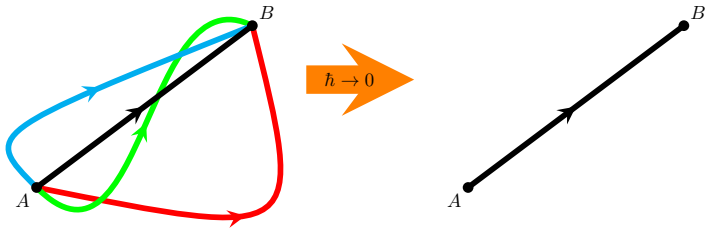
Example: spin  $s = 1/2$  projections

$P_x$  and  $P_z$  – projectors onto  $x$  and  $z$  axes correspondingly

$$[P_x \wedge \underbrace{(P_z \vee P_{-z})}_1] |S\rangle = (P_x \wedge 1) |S\rangle = P_x |S\rangle$$

$$\neq \underbrace{[(P_x \wedge P_z)]}_0 \vee \underbrace{[(P_x \wedge P_{-z})]}_0 |S\rangle \\ = (0 \vee 0) |S\rangle = 0$$

# Common basis: path integrals & $\mathbb{P}$



$$\hbar \rightarrow 0$$

- $\Rightarrow$  all possible trajectories reduce to the one
- $\Rightarrow$  information about all other trajectories is lost
- $\Leftrightarrow$  the cost of calculation should increase

$\lim_{\hbar \rightarrow 0}$

Quantum realm


$$P_A |S\rangle = |A\rangle \langle A | S\rangle$$

$$\langle A | S\rangle = \int \mathcal{D}[y] e^{iS_{|S\rangle \rightarrow |A\rangle}[y]/\hbar}$$

Classical realm

$$\lim_{\hbar \rightarrow 0} \frac{\hbar}{i} \ln \langle A | S\rangle = \chi_A S_{|S\rangle \rightarrow |A\rangle}[y]$$

$$\chi_A = \begin{cases} 1, & \delta S_{|S\rangle \rightarrow |A\rangle}[y] = 0 \\ 0, & \delta S_{|S\rangle \rightarrow |A\rangle}[y] \neq 0 \end{cases}$$


$$P_A \xrightarrow{\hbar \rightarrow 0} \chi_A$$

$P_A |S\rangle$  – pure state

$\Rightarrow$  its von Neumann entropy  $H(P_A |S\rangle) = 0$

$$P_A \xrightarrow{\hbar \rightarrow 0} \chi_A$$

$$H(\chi_A) = -\phi_A \ln \phi_A - (1 - \phi_A) \ln (1 - \phi_A) \leq \ln 2$$

$$\phi_A = \frac{\int \mathcal{D}[y] \chi_A}{\int \mathcal{D}[y]}$$

# Non-commutativity & information loss

$$P_A P_B - P_B P_A = i\hbar\Pi, \quad \Pi|\pi\rangle = \pi|\pi\rangle$$

$$|S\rangle = \sum_{\pi}^{\dim \Pi} \alpha_{\pi} |\pi\rangle$$

  $\Rightarrow$  partial trace  $\text{Tr}_{\Pi}$

additional entropy:

$$H(\chi_{\Pi}) \equiv H(P_A P_B |S\rangle) = - \sum_{\pi}^{\dim \Pi} |\alpha_{\pi}|^2 \ln |\alpha_{\pi}|^2 \leq \ln \dim \Pi$$



## Results: general expression

$$\begin{aligned} H(\mathbb{E} |S\rangle) &= \sum_{i=1}^q H(\chi_{\Gamma_{A_i}}) + H(\chi_{\Pi_1}) + \sum_{\pi_1}^{\dim \Pi_1} |\alpha_{\pi_1}|^2 H(\chi_{\Pi_2|\pi_1}) \\ &\leq (q + c) \ln 2 + \sum_{k=1}^c \ln \dim \Pi_k \end{aligned}$$

$q$  – number of qubits equipped in no conjunction

$c$  – number of conjunctions (operator products)

[TTZ, Ukr.J.Phys. (2022)]

## Results: general expression

$$H(\mathbb{E} | S \rangle) = \sum_{i=1}^q H(\chi_{\Gamma_{A_i}}) + H(\chi_{\Pi_1}) + \sum_{\pi_1}^{\dim \Pi_1} |\alpha_{\pi_1}|^2 H(\chi_{\Pi_2 | \pi_1})$$
$$\leq (q + c) \ln 2 + \sum_{k=1}^c \ln \dim \Pi_k$$

$q$  – number of qubits equipped in no conjunction

$c$  – number of conjunctions (operator products)

[TTZ, Ukr.J.Phys. (2022)]

The largest loss is generated by non-commuting expressions

## Fast Fourier Transform

$$\text{Time (FFT}_C) = \mathcal{O}(n2^n)$$

$$\text{Time (FFT}_Q) = \mathcal{O}(n^2)$$

$$H(\text{FFT}_Q) = \mathcal{O}(n2^n)$$

## Search algorithm

$$\text{Time (SA}_C) = \mathcal{O}(2^n)$$

$$\text{Time (SA}_Q) = \mathcal{O}(2^{n/2})$$

$$H(\text{SA}_Q) = \mathcal{O}(n^2 2^{n/2})$$

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$$H(\text{SA}_Q) = \mathcal{O}(n^2 2^{n/2})$$

Quantum algorithm may experience changeover under  $\lim_{\hbar \rightarrow 0}$

[Litvinov et al., in ACAMP, NATO Sci. Ser. (2002)]

# Conclusions

- ✓ The process of quantum logic reduction to its classical counterpart via  $\lim_{\hbar \rightarrow 0}$  is studied in details
- ✓ Information loss is estimated for each quantum gate from the complete set
- ✓ The largest loss is observed for non-commuting operators
- ✓ Both quantum Fast Fourier transform and Grover algorithms demonstrate significant information loss under  $\lim_{\hbar \rightarrow 0}$
- ✓ Amount of information loss  $H(\mathbb{E} | S\rangle)$  depends on  $\mathbb{E}$  due to a change in the problem the algorithm solves.

# BACKUP

$$\text{FFT}_Q = U_0 U_1 \cdots U_{n-1}$$

$$U_k = H_k C_{k,n-1} C_{k,n-2} \cdots C_{k,k+1}$$

Hadamard gate  $H_k: |q\rangle_k \rightarrow \frac{1}{\sqrt{2}} (|0\rangle_k + e^{i\pi q} |1\rangle_k)$ ,  $q = \overline{0,1}$

$$H_k = \sqrt{2} (P_{x,k} - P_{-z,k})$$

c-phase gate  $C_{k,k'}: |q\rangle_k |q'\rangle_{k'} \rightarrow e^{i\phi_{kk'} q q'} |q\rangle_k |q'\rangle_{k'}$

$$C_{k,k'} = (1 - e^{i\phi_{kk'}}) (P_{-z,k} P_{z,k'} + P_{z,k}) + e^{i\phi_{kk'}}$$

$$\phi_{kk'} = \pi/2^{k'-k}$$

# Grover search algorithm

$$\text{SA}_Q = \left\{ [2(HP_zH)^{\otimes n} - I^{\otimes n}] \otimes (P_{-z} - P_z) \right\}^{\frac{\pi}{4} 2^{n/2}} U_f$$

$$U_f: |\vec{q}\rangle \otimes |0\rangle \rightarrow |\vec{q}\rangle \otimes |f(\vec{q})\rangle, \quad \vec{q} \in \overline{0, 2^n - 1}$$