EGO EUROPEAN GRAVITATIONAL OBSERVATORY

TAUP VIENNA 2023

Optimal control theory for the angular control of the full payload for AdV+ PhaseII

Manuel Pinto*, Maddalena Mantovani, Julia Casanueva

Interferometer, Sensing and Control team

*e-mail: *manuel.pinto@ego-gw.it*

TAUP 2023, August 28th-September 1st 2023

Introduction and Outline

- Proposal of a new control strategy for the angular degrees of freedom of a Fabry-Perot cavity in the presence of radiation pressure effect, for AdV+ Phase II configuration;
- Introduction of large terminal masses (marionettas and mirrors) which induces not negligible asymmetries in the opto-mechanical system;
- Impossibility of fully-decoupling all the DoFs -> hard to exploit usual techniques based on frequency domain SISO-like controllers design;

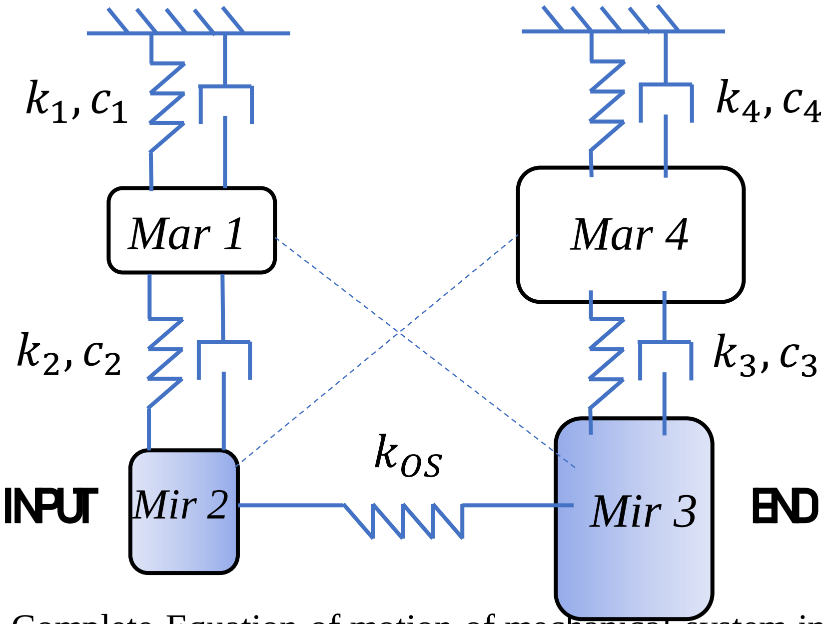
Given the premises, an approach of designing MIMO-like controllers in time-domain is investigated;

- Optimal Control theory is used instead: by the minimization of a specific cost function, direct closed-loop stability with the optimal phase margin available will be obtained.
- The present study is perfomed tackling the following steps:
    1. a State Space formulation of the opto-mechanical system is obtained
    2. design of a LQR control (namely Linear Quadratic Regulator) with an additional Integrator (i.e. LQI) will be described;
    3. to complete the control loop design architecture, the design of a state estimator, i.e. Kalman Filter, will be reported, by using realistic data of sensors noise, in order to evaluate robustness and convergence limitation of the filter.

Details of the present study are reported in the technical note *VIR-0219A-23*

State Space payload modeling

Coupled model: two double pendula (Marionette + Mirror) connected by radiation pressure effect:



The total stiffness matrix is given by the sum of the **Mechanical** and the **Optical** matrices:

$$\boldsymbol{K_{TOT}} = \boldsymbol{K_m} + \boldsymbol{K_{OS}}$$

Where $\boldsymbol{K_m}$ and $\boldsymbol{K_{OS}}$ are respectively:

$$\boldsymbol{K_m} = \begin{bmatrix} (k_1+k_2) & -k_2 & & \\ -k_2 & k_2 & & \\ & & k_3 & -k_3 \\ & & -k_3 & (k_3+k_4) \end{bmatrix} \qquad \boldsymbol{K_{OS}} = \begin{bmatrix} & & & \\ & k_{rpi} & k_p & \\ & k_p & k_{rpe} & \\ & & & \end{bmatrix}$$

Optical spring coupling given by: $\boldsymbol{K_{OS}} = \dfrac{2PL}{c(1-g_a g_b)} \cdot \begin{bmatrix} -g_b & 1 \\ 1 & -g_a \end{bmatrix}$

$$g_* = 1 - \frac{L}{R.o.C_*}$$

Stability criteria for cavities

Complete Equation of motion of mechanical system in State Space form:

$$\begin{Bmatrix} \dot\theta_1 \\ \ddot\theta_1 \\ \dot\theta_2 \\ \ddot\theta_2 \\ \dot\theta_3 \\ \ddot\theta_3 \\ \dot\theta_4 \\ \ddot\theta_4 \end{Bmatrix} = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ \left(-\frac{k_1}{I_1}-\frac{k_2}{I_1}\right) & \left(-\frac{c_1}{I_1}-\frac{c_2}{I_1}\right) & \frac{k_2}{I_1} & \frac{c_2}{I_1} & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ \frac{k_2}{I_2} & \frac{c_2}{I_2} & \left(-\frac{k_2}{I_2}-\frac{k_{rpi}}{I_2}\right) & -\frac{c_2}{I_2} & -\frac{k_p}{I_2} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & -\frac{k_p}{I_3} & 0 & \left(-\frac{k_3}{I_3}-\frac{k_{rpe}}{I_3}\right) & -\frac{c_3}{I_3} & \frac{k_3}{I_3} & \frac{c_3}{I_3} \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & \frac{k_3}{I_4} & \frac{c_3}{I_4} & \left(-\frac{k_3}{I_4}-\frac{k_4}{I_4}\right) & \left(-\frac{c_3}{I_4}-\frac{c_4}{I_4}\right) \end{bmatrix} \begin{Bmatrix} \theta_1 \\ \dot\theta_1 \\ \theta_2 \\ \dot\theta_2 \\ \theta_3 \\ \dot\theta_3 \\ \theta_4 \\ \dot\theta_4 \end{Bmatrix} + \begin{bmatrix} 0 & 0 \\ I_1^{-1} & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & I_4^{-1} \end{bmatrix} \begin{Bmatrix} T_{input} \\ T_{end} \end{Bmatrix}$$

state matrix $\boldsymbol{A}$

state vector $\boldsymbol{x}$

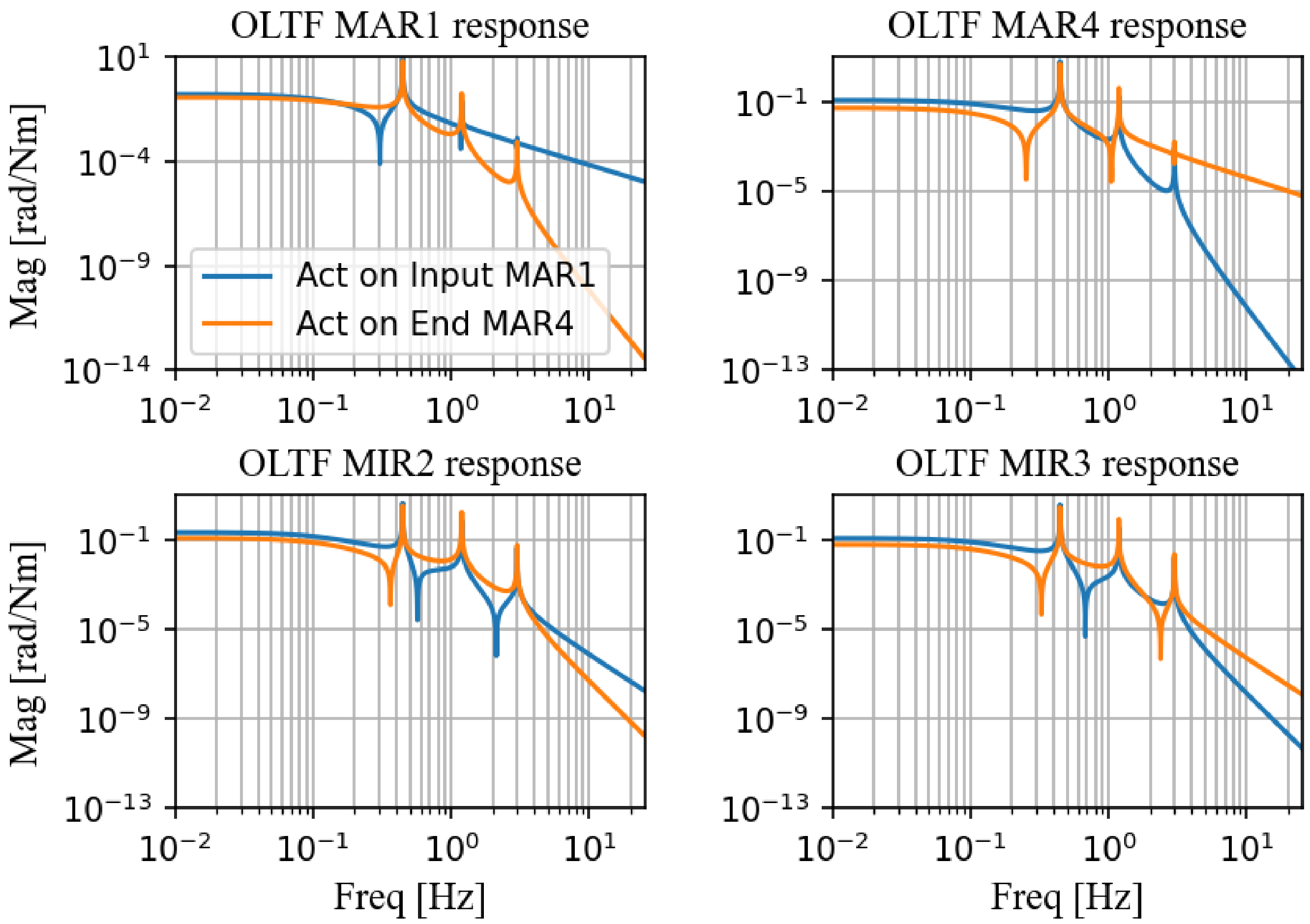
input matrix $\boldsymbol{B}$

input vector $\boldsymbol{u}$

$$\dot{\boldsymbol{x}} = \boldsymbol{A}\boldsymbol{x} + \boldsymbol{B}\boldsymbol{u}$$

Transfer Functions from State Space modeling

From the differential equation formalism (SS) it is possible to obtain the system transfer function, in mirror base:



$$\dot{\boldsymbol{x}} = \boldsymbol{A}\boldsymbol{x} + \boldsymbol{B}\boldsymbol{u}$$

$$\frac{\boldsymbol{\theta}}{\boldsymbol{T}} = (\boldsymbol{C} \cdot (s * \boldsymbol{II} - \boldsymbol{A})^{-1}\boldsymbol{B} + \boldsymbol{D})$$

where: **II** = identity matrix $N_{DoF} \times N_{DoF}$

Three principal resonance modes can be noticed: the most critical one will be the higher at 3 Hz.

System needs to be diagonalized, in order to switch from mirror coordinates to the modal coordinates to decouple the single modes of vibrations.
This is obtained by computing the eigenvalues and eigenvector matrices of the system.
Due to the asymmetry of the system, however, it is difficult to obtain a full-diagonal system.

Modal analysis (decoupling DoFs)

$$\boldsymbol{I}\ddot{\boldsymbol{\theta}} + \boldsymbol{D}\dot{\boldsymbol{\theta}} + \boldsymbol{K_{tot}}\boldsymbol{\theta} = \boldsymbol{T}(t) \;\rightarrow \boldsymbol{D} = 0$$

$$\boldsymbol{I}\ddot{\boldsymbol{\theta}} + \boldsymbol{K_{tot}}\boldsymbol{\theta} = \boldsymbol{T}(t)$$

$$det|\boldsymbol{K_{tot}} - \lambda \boldsymbol{I}| = 0$$

In first approximation we can consider high **q** (zero damping). Otherwise, we need to consider the value of Damping matrix **D** while computing the eigenvalues (**D** must be diagonalized through $\boldsymbol{U^T D U}$): see *Frazer, Duncan and Collar* method.

$$det\left\|\begin{bmatrix}(k_1+k_2) & -k_2 & & \\ -k_2 & (k_2+k_{rpi}) & k_p & \\ & k_p & (k_3+k_{rpe}) & -k_3 \\ & & -k_3 & (k_3+k_4)\end{bmatrix} - \lambda\begin{bmatrix}I_1 & & & \\ & I_2 & & \\ & & I_3 & \\ & & & I_4\end{bmatrix}\right\| = 0$$

To decouple the equation of motion we apply the eigenvalues problem($det|\boldsymbol{K_{tot}} - \lambda\boldsymbol{I}| = 0$) in order to find the **eigenvalues** matrix $\boldsymbol{\Lambda}$ (which contains the resonance frequencies ($\lambda = \omega^2$) of the system) and the **eigenvectors** matrix $\boldsymbol{U}$ (in which each column represents one mode of vibration).

However, given the asymmetry of the system, after the diagonalization process, we don't obtain anymore a pure **tilt** (+ mode) or a **shift** (- mode) of the beam for given couples of torques (eigenvectors of reduced matrix $\tilde{U}^T$).

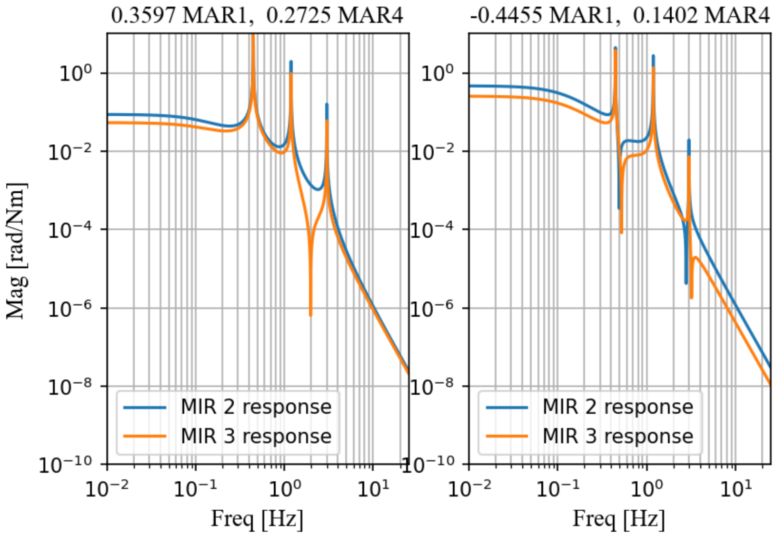
To find U ($\boldsymbol{U}_{i\times n} = \left[y^{(i,n)} \quad \varepsilon^{(i,n)} \quad \varphi^{(i,n)} \quad \xi^{(i,n)}\right]^T$), we need to solve the system:

$$\begin{cases}\left((k_1+k_2) - \lambda_i I_1\right)\cdot y^{(i,n)} - k_2\cdot\varepsilon^{(i,n)} = 0 \\ -k_2\cdot y^{(i,n)} + \left((k_2+k_{rpi}) - \lambda_i I_2\right)\cdot\varepsilon^{(i,n)} + k_p\cdot\varphi^{(i,n)} = 0 \\ k_p\cdot\varepsilon^{(i,n)} + \left((k_3+k_{rpe}) - \lambda_i I_3\right)\cdot\varphi^{(i,n)} - k_3\cdot\xi^{(i,n)} = 0 \\ -k_3\cdot\varphi^{(i,n)} + \left((k_3+k_4) - \lambda_i I_4\right)\cdot\xi^{(i,n)} = 0\end{cases}$$

$$\boldsymbol{U}_{i\times n} = \left[y^{(i,n)} \quad \varepsilon^{(i,n)} \quad \varphi^{(i,n)} \quad \xi^{(i,n)}\right]^T$$

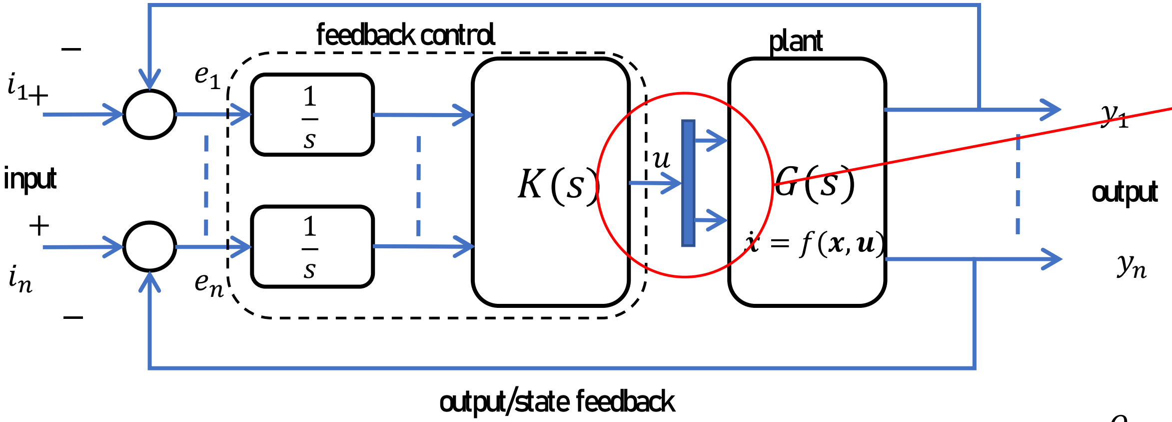
To be decoupled, the following equation **must give the identity matrix**:

$$\boldsymbol{U^T I U = II}$$



$$\tilde{U}^T T(t) = \begin{bmatrix}-0.4455 & 0.1402 \\ 0.3597 & 0.2725 \\ 0.0859 & -0.2006 \\ 0.0166 & 0.0228\end{bmatrix} T(t)$$

Control regulators design (coupled system)



different actuation are sent to the two marionettas.

Given the difficulties to decouple the Dofs, we can work with the coupled system, by conveniently creating two different actuations to the two marionettas:

$$\dot{\boldsymbol{x}} = \boldsymbol{A}\boldsymbol{x} + \boldsymbol{B}\boldsymbol{u}$$
$$\boldsymbol{y} = \boldsymbol{C}\boldsymbol{x} + \boldsymbol{N}\boldsymbol{d}$$
$$\boldsymbol{u} = -\boldsymbol{K}_{\boldsymbol{LQI}}(\boldsymbol{x} - \boldsymbol{x}_{\boldsymbol{T}})$$

$$Q = \begin{bmatrix} q_1 & & & \\ & q_2 & & \\ & & \ddots & \\ & & & q_n \end{bmatrix} \qquad x = \begin{Bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{Bmatrix}$$

Feedback control by minimizing a cost function **J**, and tuning the Q and R matrices:

$$J = \int_0^T \{\boxed{\boldsymbol{x}^T \boldsymbol{Q} \boldsymbol{x}} + \boxed{\boldsymbol{u}^T \boldsymbol{R} \boldsymbol{u}}\} dt$$

Q = Cost on the state: usually big values if we want $\boldsymbol{x}$ to stabilyze quickly without spending a lot of energy

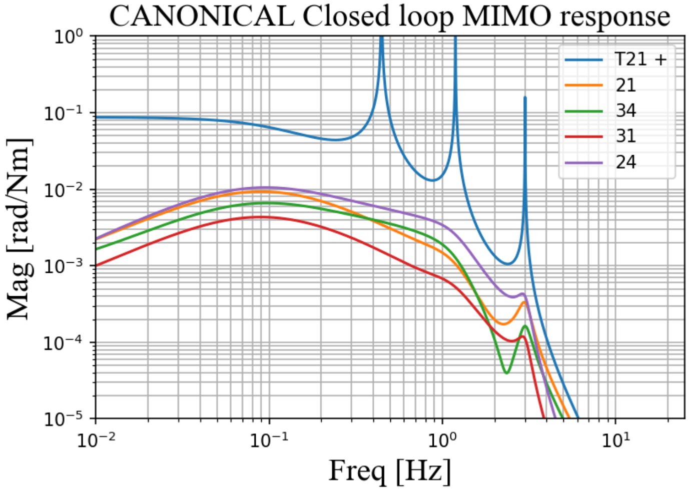
Feedback matrix is obtained by solving Riccati's equation:

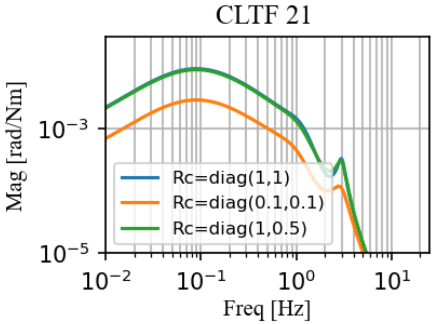
$$\boldsymbol{A}^T \boldsymbol{P} + \boldsymbol{P}\boldsymbol{A} - \boldsymbol{P}\boldsymbol{B}\boldsymbol{R}^{-1}\boldsymbol{B}^T \boldsymbol{P} + \boldsymbol{Q} = -\dot{\boldsymbol{P}}$$
$$\boldsymbol{K}_{LQI} = \boldsymbol{R}^{-1}\boldsymbol{B}^T \boldsymbol{P}$$

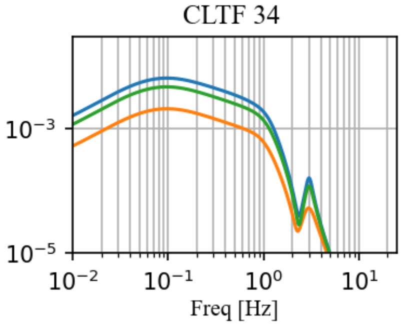
R= Cost on the actuation: big or low depending if we want to design a cheap or expensive controller. One or the other choice have consequences on the performances (e.g. reducing of the settling time - too strong step response - loss of sensitivity)

To close the loop we define a new State matrix ***Ac***, which take into account the control feedback matrix ***K_LQI***

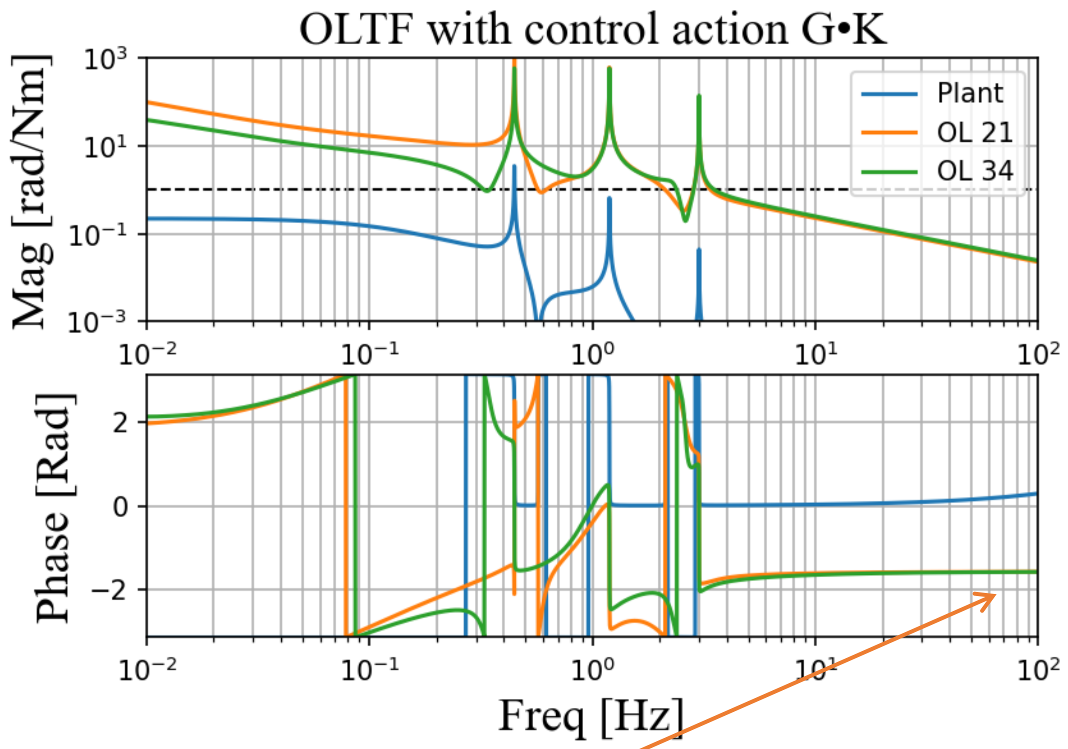
$$\boldsymbol{A}_{CL} = \widetilde{\boldsymbol{A}} - \widetilde{\boldsymbol{B}}\boldsymbol{K}_{LQI}$$

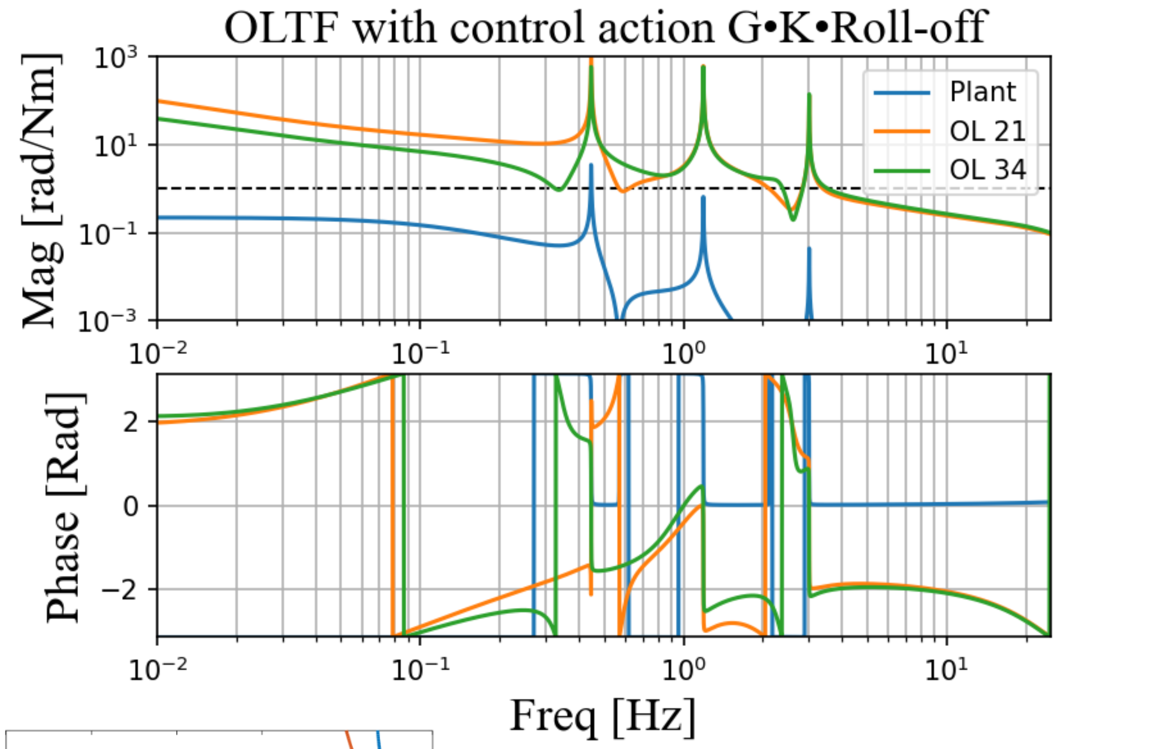




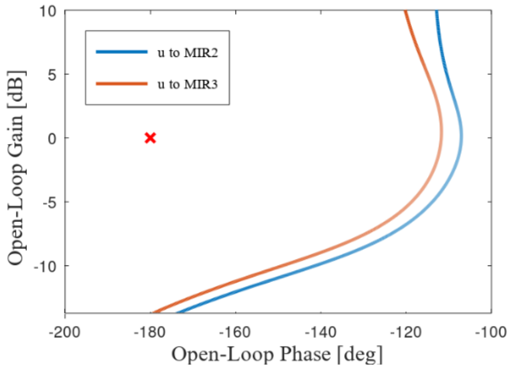


Performance of the loop: OLTF analysis



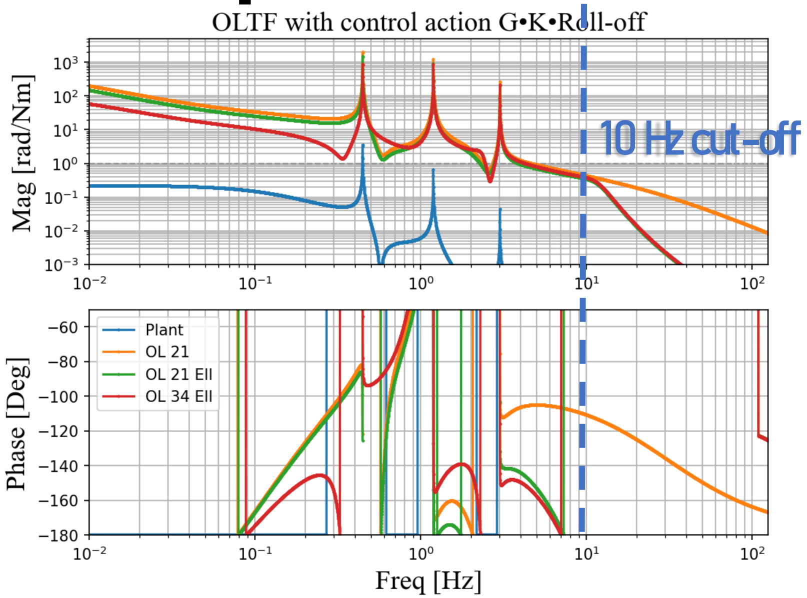


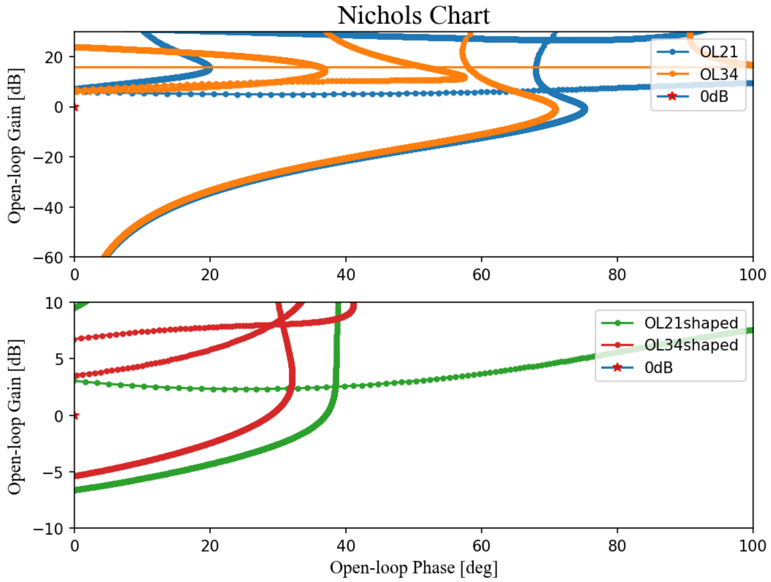
Optimal LQI techniques allow to obtain direct closed loop stability with infinite gain-margin and at least 60 deg of phase-margin. Indeed, we obtain a loop unconditionally stable up to the higher frequencies. In reality, we will have to deal with hardware limitations; thus, the bandwidth will have to be reduced.



Introduction of a Low-pass filter @25 Hz which put a cut-off frequency of the loop and yet allows 68/72 deg of PM and a loop UGF of 5 Hz.

Optimization of the loop: LF and HF shaping





We are interested not only in stabilizing the plant, but also in optimizing the **low-frequency** and the **high-frequency** regions of the loop:

- reducing the overall **rms** of the system;
- reduce the re-introduction of control **noise**.

These two goals are achieved by using dedicated structures in the control filters such as Lag Filters (aka Boost Filters), and Roll-off filters.

Since we want to reduce the noise re-introduction generally above 10 Hz, this will inevitably cause a huge constraint on the controllability of the higher resonance mode at 3 Hz.
We need a trade-off between the possibility to have the loop bandwidth above this mode, and to roll-off efficiently the higher frequencies.

Cutting the noise ahead **10 Hz** with the use of an elliptic filter (and other dedicated structures), will determine a consistent reduction of the Phase-margin of the loop from 68 to around **30-35 deg**. Consequently, the Gain-margin will be very limited.

Solution: put the bandwidth of the loop before the last resonance mode. This will imply to increase the difficulties to control the 3 Hz mode.

Kalman Filter - LQIG controller

$$\begin{cases} \dot{\hat{x}} - A\hat{x} - Bu - K_E(y - C\hat{x}) = 0 \\ \dot{L} = LA^T - LC^T K_E^T + AL - K_E CL + K_E R_S K_E^T + R_D \\ K_E = LC^T R_S^{-1} \\ u = -K_{LQI}(\hat{x} - x_T) \end{cases}$$

- N.B. remember, in canonical representaion we have:

$$\dot{v}_n(t) = e_n(t) = i_n(t) - y_n(t)$$

$$\begin{Bmatrix} \dot{x} \\ \dot{v} \end{Bmatrix} = \begin{bmatrix} A_C & 0 \\ -C_c & 0 \end{bmatrix} \begin{Bmatrix} x \\ v \end{Bmatrix} + \begin{bmatrix} B_C \\ 0 \end{bmatrix} u$$

with: $\dot{\tilde{x}} = \begin{Bmatrix} \dot{x} \\ \dot{v} \end{Bmatrix}$; $\tilde{x} = \begin{Bmatrix} x \\ v \end{Bmatrix}$

- With the estimation block, the overall closed-loop system is:

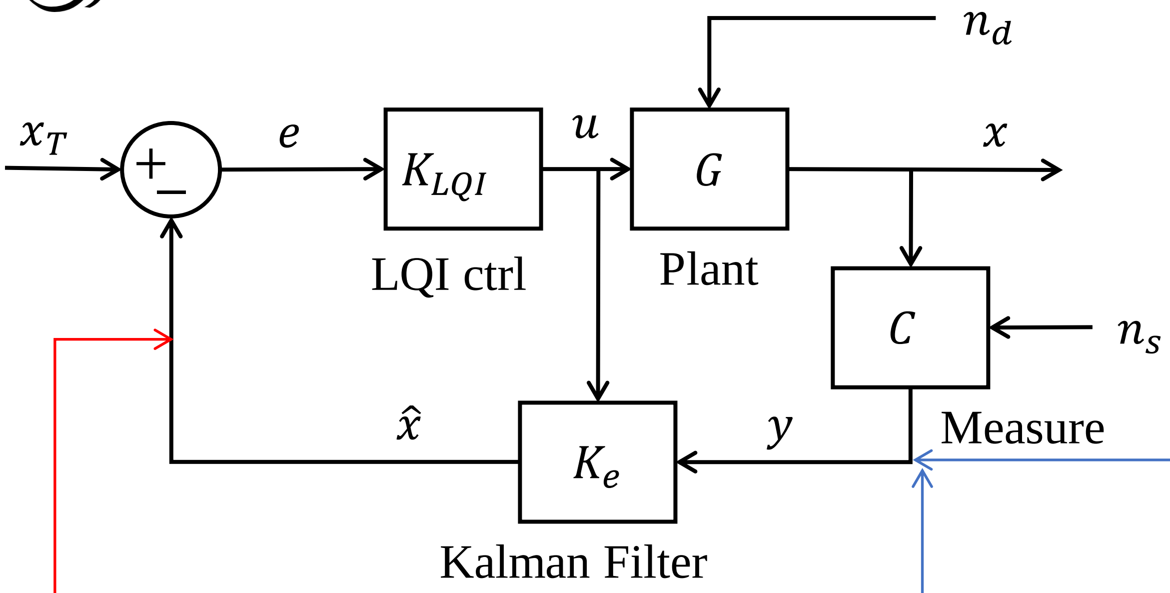
$$\begin{Bmatrix} \dot{\tilde{x}} \\ \dot{e} \end{Bmatrix} = \begin{bmatrix} \tilde{A} - \tilde{B}K_{LQI} & \tilde{B}K_{LQI} \\ 0 & \tilde{A} - K_E\tilde{C} \end{bmatrix} \begin{Bmatrix} \tilde{x} \\ e \end{Bmatrix}$$

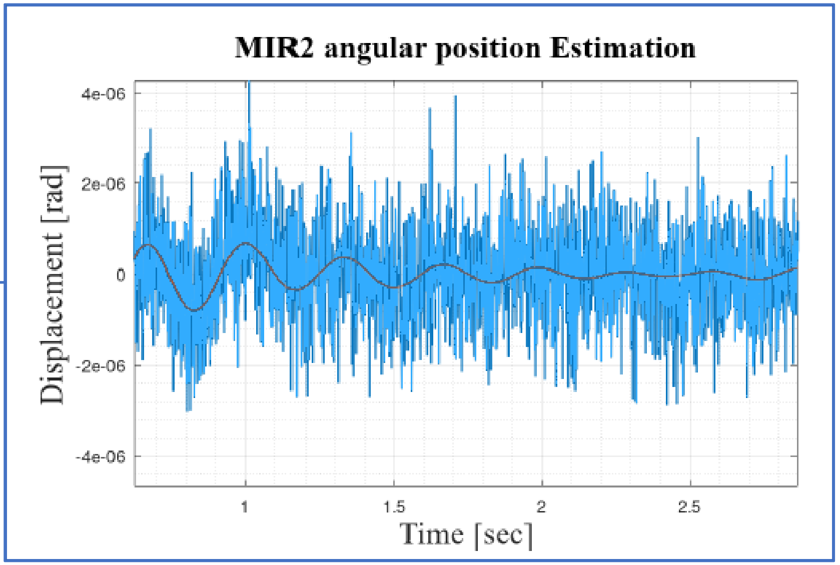
In real working condition we don't have access to the whole variables of the state vectors (e.g. velocities). Additionally, everything is spoiled by NOISE!
The Kalman filter allows an **estimation** $\hat{x}$ of the system state x by performing an **optimal blending** of the given **theoretical model** and the **available measurements.**
Such estimate is obtained through a **linear combination** of the two equations describing the process dynamics and the sensor dynamics, both affected by noises.

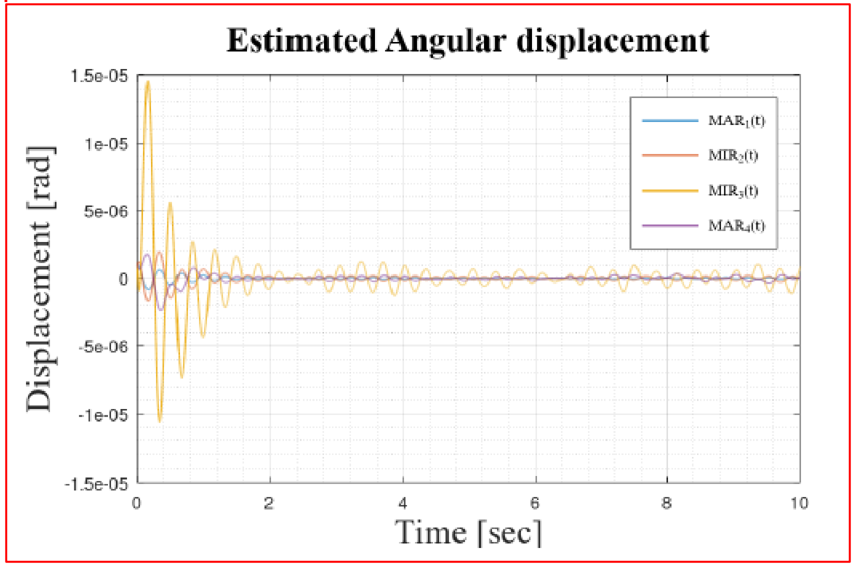
Ingredients for Kalman filter design:
- **Mechanical model** of the system;
- **Control/Input** equation;
- Set of raw **measurements**;
- Values of **covariance noise** matrices for measurements and process

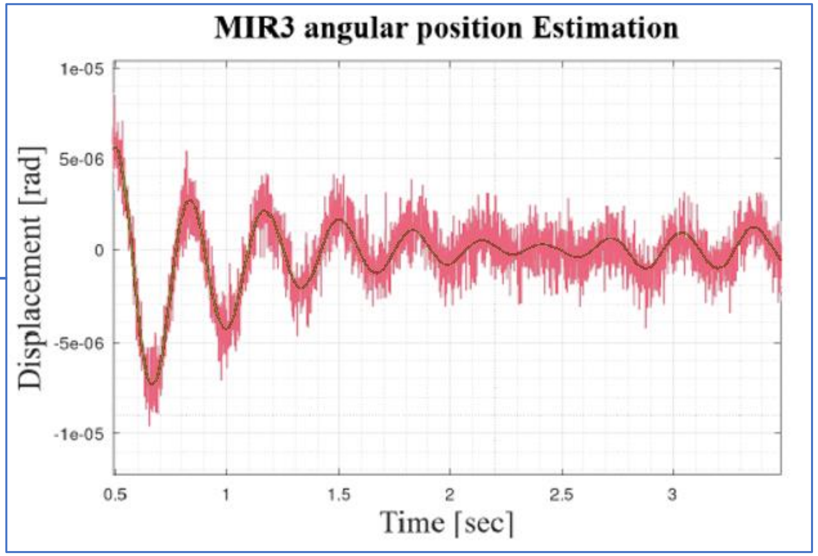
The eigenvalues of the closed-loop system are the ones of $\tilde{A} - \tilde{B}K_{LQI}$ and $\tilde{A} - K_E\tilde{C}$.
This proves the separation principle, which states that it's possible to design indipendently an optimal controller and an optimal observer (by assuming that the state is available/observable/controllable).

Simulations results









Concluding remarks

- A proposal of a new control strategy for the control of the angular degrees of freedom for a coupled opto-mechanical system has been proposed;
- The study relies on the use of the Optimal Control theory which, by the minimization of a specific cost function allows to control all the modes, obtaining in principle infinite Gain-margin with at least 60 deg of Phase-margin;
- However, hardware limitations put some constraint on the actual stability margins available, since the needs to cut-off the sensor noise from 10 Hz, forces to reduce the available Phase- margin down to 25 deg. This choice will lead to a very reduced Gain-margin.
- One solution will be to put the bandwidth below the 3 Hz resonance mode.
- The implementation of a State Estimator (i.e. Kalman filter) allows to estimate all the state variables not monitored by the sensors and to close the feedback loop with the estimated signals.
- Studies to improve the model are on-going (like introducing the possibility to study the longitudinal to angular coupling), together with the same performed study for the Pitch DoF (θ_x).

Thanks!