A hadron-structure-oriented approach to TMD phenomenology

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Based on:

- J.O. Gonzalez, TCR, N. Sato, Phys.Rev.D 106 (2022) 3, 034002
- J.O. Gonzalez, T. Rainaldi, TCR, arXiv:2303.04921 [hep-ph]

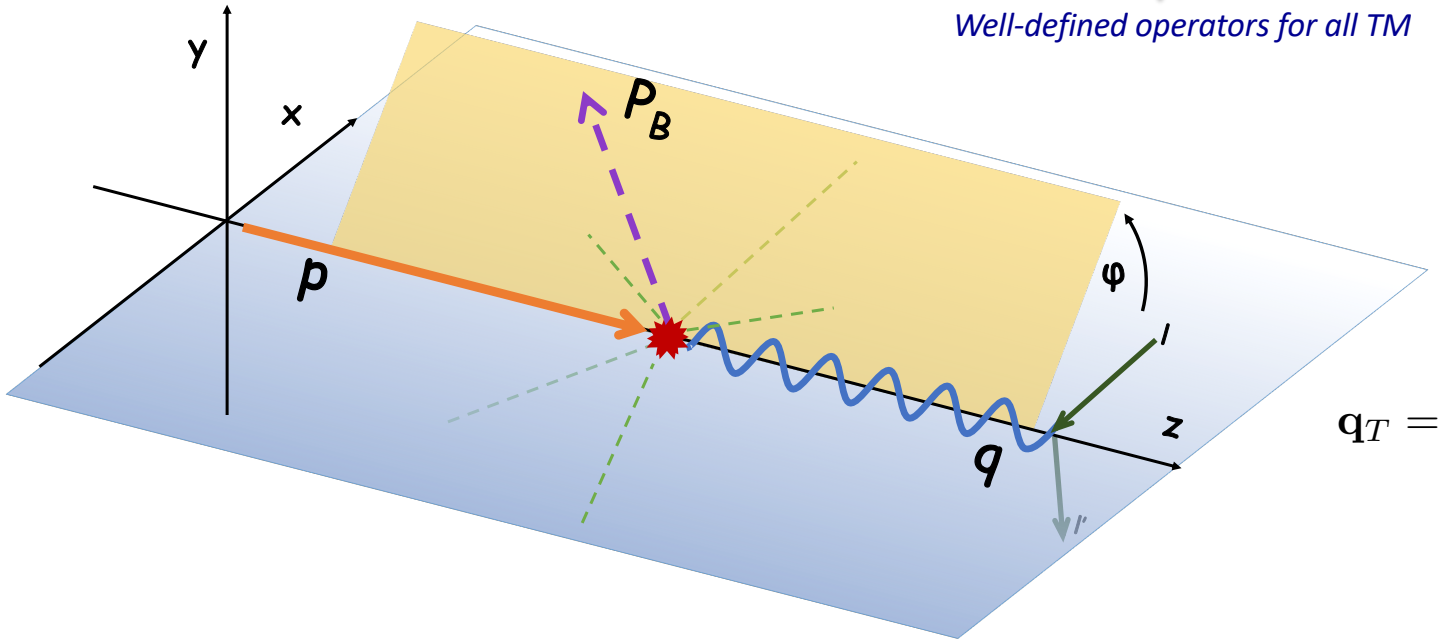
DIS 2023, March 30, 2023

TMD factorization & SIDIS $\left(for\ \frac{q_T}{Q} \ll 1\right)$

$$\frac{\mathrm{d}\sigma}{\mathrm{d}Q\,\mathrm{d}^2\mathbf{q}_T\,\mathrm{d}x\,\mathrm{d}z} = H(Q/\mu_Q)\int \mathrm{d}^2\mathbf{k}_{1T}\mathrm{d}^2\mathbf{k}_{2T}\, f_{j/p}(x,\mathbf{k}_{1T};\mu_Q,Q^2)D_{h/j}(z,z\mathbf{k}_{2T};\mu_Q,Q^2)\delta^{(2)}(\mathbf{q}_T+\mathbf{k}_{1T}-\mathbf{k}_{2T})$$

Well-defined operators for all TM

+ power suppressed



$$\mathbf{q}_T = -\frac{\mathbf{P}_{BT}}{z}$$

$$\frac{\mathrm{d}\sigma}{\mathrm{d}Q\,\mathrm{d}^2\mathbf{q}_T\,\mathrm{d}x\,\mathrm{d}z} = H(Q/\mu_Q)\int \frac{\mathrm{d}^2\mathbf{b}_T}{(2\pi)^2}e^{-i\mathbf{b}_T\cdot\mathbf{q}_T}\,\tilde{f}_{j/p}(x,\mathbf{b}_T;\mu_Q,Q^2)\tilde{D}_{h/j}(z,\mathbf{b}_T;\mu_Q,Q^2)$$

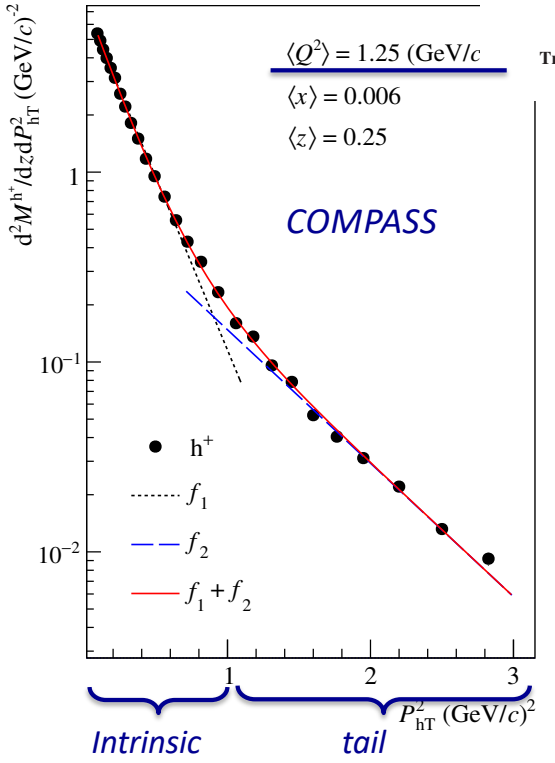
+ power suppressed

Nonperturbative structures in pheno

*"While significant effort has been put into the study of W(b) **at large b** [36, 42, 43, 44], none … adequately describe the observed Z boson distribution without introducing free parameters."*

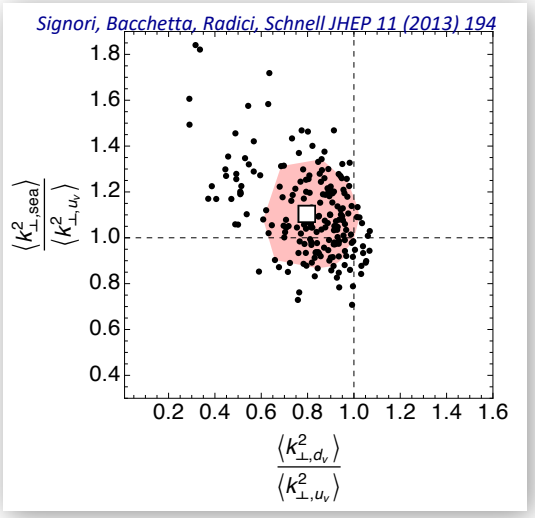
PHYSICAL REVIEW D **97,** 032006 (2018)

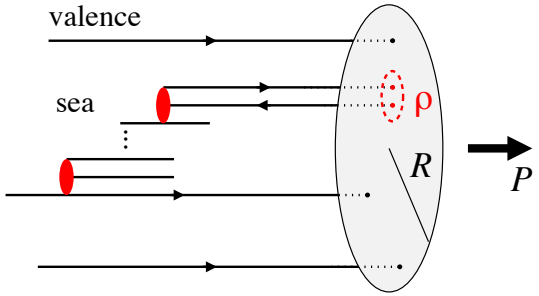
Transverse-momentum-dependent multiplicities of charged hadrons in muon-deuteron deep inelastic scattering



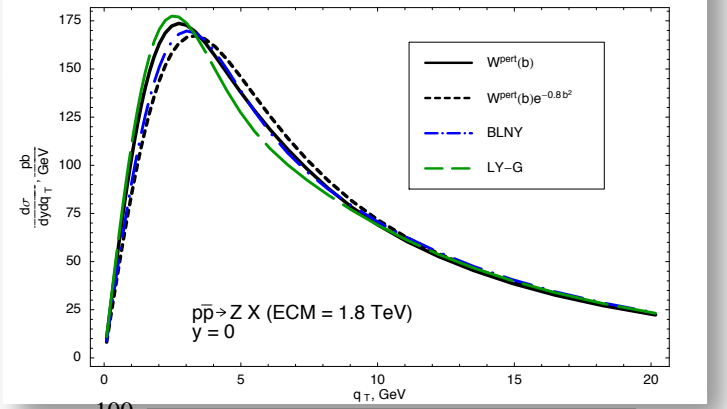
"… the two exponential functions in our parametrizations F_1 can be attributed to two completely different underlying physics mechanisms that overlap in the region $P_{hT} \simeq 1$ GeV^2."

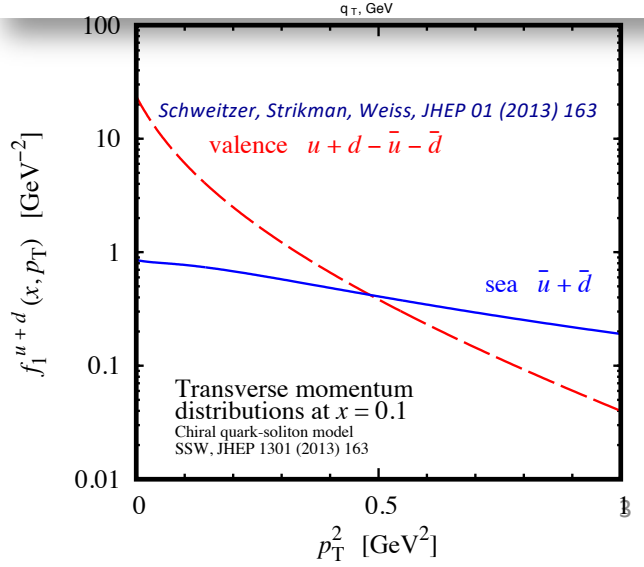
Signori, Bacchetta, Radici, Schnell JHEP 11 (2013) 194





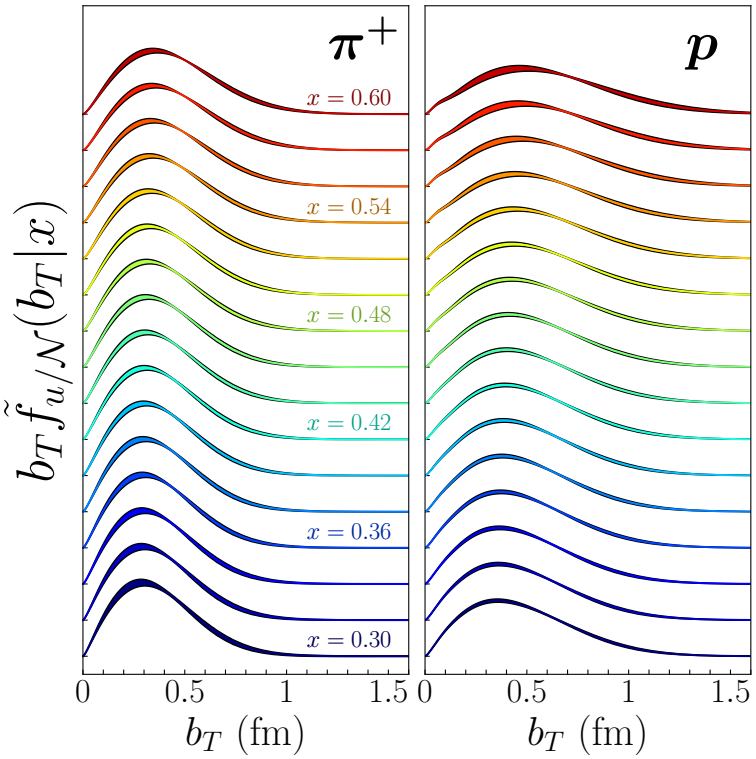
P. Nadolsky, (2004) Theory of W and Z Production





Nonperturbative structures in pheno

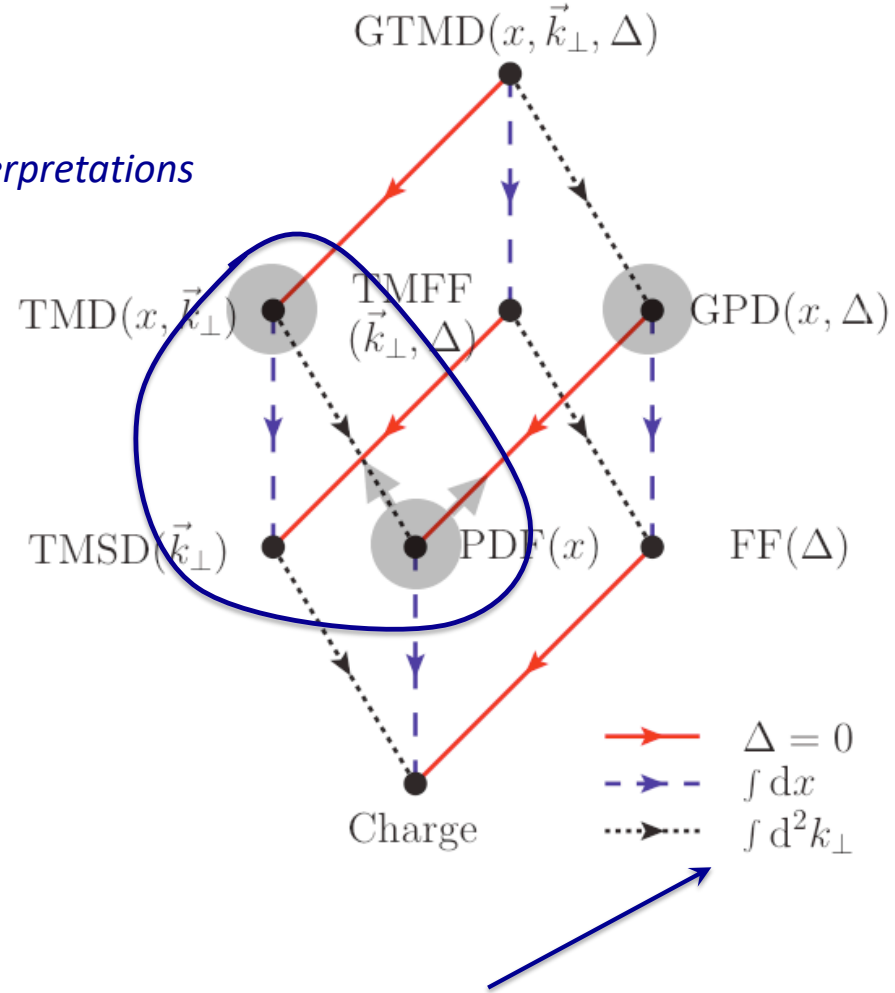
Barry, et al (2302.01192)



??

"Importantly, we have checked that the differences between the proton and pion ⟨bT |x⟩ are completely due to the nonperturbative TMD structure, independent of the collinear PDFs."

Interpretations



These integrals are often divergent

Conventional organization

$$\frac{\partial \ln \tilde{f}_{j/p}(x, b_T; \mu, \zeta)}{\partial \ln \sqrt{\zeta}} = \tilde{K}(b_T; \mu)$$

$$\frac{\mathrm{d} \ln \tilde{f}_{j/p}(x, b_T; \mu, \zeta)}{\mathrm{d} \ln \mu} = \gamma(\alpha_s(\mu); \zeta/\mu^2)$$

$$\frac{\mathrm{d}\tilde{K}(b_T; \mu)}{\mathrm{d} \ln \mu} = -\gamma_K(\alpha_s(\mu))$$

$$\tilde{f}_{j/p}(x, b_T; \mu, \zeta) = \int_x^1 \frac{\mathrm{d}\xi}{\xi} \tilde{C}_{j/k}(x/\xi, b_T; \zeta, \mu, \alpha_s(\mu)) f_{k/p}(\xi; \mu) + O\left(b_T \Lambda_{\mathrm{QCD}}\right)$$

Or

$$f_{j/p}(x, k_T; \mu, \zeta) = \frac{1}{k_T^2}\left[\int_x^1 \frac{\mathrm{d}\xi}{\xi} C_{j/k}(x/\xi, k_T; \zeta, \mu, \alpha_s(\mu)) f_{k/p}(\xi; \mu) + O\left(\frac{\Lambda_{\mathrm{QCD}}}{k_T}\right)\right]$$

$$\tilde{K}(b_T; \mu) = \tilde{K}_{\mathrm{pert}}\left(b_T \mu, \alpha_s(\mu)\right) + O\left(b_T \Lambda_{\mathrm{QCD}}\right)$$

Steps:

1) Solve evolution equations to relate overall SIDIS hard scale ($\mu_Q = Q$) to input scale ($\mu_{Q_0} = Q_0$)
2) How to use small $b_T \ll 1/\Lambda_{QCD}$ collinear factorization?
   - Partition small ($b_T < b_{\max}$) & large ($b_T > b_{\max}$) regions with a b_*
   - Define hard scale $\mu_{b_*} \sim 1/b_*$
3) Evolve again to connect Q_0 to μ_{b_*}
4) **Place remaining NP parts in an exponent:**

$$-g_{j/p}(x, b_{\mathrm{T}}) \equiv \ln\left(\frac{\tilde{f}_{j/p}(x, \boldsymbol{b}_{\mathrm{T}}; \mu_{Q_0}, Q_0^2)}{\tilde{f}_{j/p}(x, \boldsymbol{b}_*; \mu_{Q_0}, Q_0^2)}\right)$$

$$g_K(b_{\mathrm{T}}) \equiv \tilde{K}(b_*; \mu) - \tilde{K}(b_{\mathrm{T}}; \mu)$$

5) **Ansatz for g-functions**
6) **Perform small-b_T expansions & drop $O(\Lambda_{QCD}\, b_{max})$ errors**

Conventional organization

$$\int \mathrm{d}^2\mathbf{k}_{1T}\mathrm{d}^2\mathbf{k}_{2T} f_{j/p}(x,\mathbf{k}_{1T};\mu_Q,Q^2) D_{h/j}(z,z\mathbf{k}_{2T};\mu_Q,Q^2)\delta^{(2)}(\mathbf{q}_T+\mathbf{k}_{1T}-\mathbf{k}_{2T})$$

$$\int \frac{\mathrm{d}^2\boldsymbol{b}_\mathrm{T}}{(2\pi)^2}\; e^{-i\boldsymbol{q}_\mathrm{T}\cdot\boldsymbol{b}_\mathrm{T}}\; \tilde{f}^{\mathrm{OPE}}_{j/p}(x_\mathrm{bj},\boldsymbol{b}_*;\mu_{b_*},\mu^2_{b_*})\; \tilde{D}^{\mathrm{OPE}}_{h/j}(z_\mathrm{h},\boldsymbol{b}_*;\mu_{b_*},\mu^2_{b_*})$$
$$\times \exp\left\{2\int_{\mu_{b_*}}^{\mu_Q}\frac{d\mu'}{\mu'}\left[\gamma(\alpha_s(\mu');1)-\ln\frac{Q}{\mu'}\gamma_K(\alpha_s(\mu'))\right]+\ln\frac{Q^2}{\mu^2_{b_*}}\tilde{K}(b_*;\mu_{b_*})\right\}$$
$$\times \exp\left\{-g_{j/p}(x_\mathrm{bj},b_\mathrm{T})-g_{h/j}(z_\mathrm{h},b_\mathrm{T})-g_K(b_\mathrm{T})\ln\left(\frac{Q^2}{Q_0^2}\right)\right\}\; +O\left(b_{\max}\Lambda_{\mathrm{QCD}}\right)$$

$$\tilde{f}_{j/p}(x,b_T;\mu,\zeta)=\underbrace{\int_x^1\frac{\mathrm{d}\xi}{\xi}\tilde{C}_{j/k}(x/\xi,b_T;\zeta,\mu,\alpha_s(\mu))f_{k/p}(\xi;\mu)}_{\tilde{f}^{\mathrm{OPE}}_{j/p}(x,b_T;\mu,\zeta)}+O\left(b_T\Lambda_{\mathrm{QCD}}\right)$$

Steps:

1) Solve evolution equations to relate overall SIDIS hard scale ($\mu_Q = Q$) to input scale ($\mu_{Q_0} = Q_0$)
2) How to use small $b_T \ll 1/\Lambda_{QCD}$ collinear factorization?
   - Partition small ($b_T < b_{\max}$) & large ($b_T > b_{\max}$) regions with a b_*
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5) **Ansatz for g-functions**
6) **Perform small-b_T expansions & drop $O(\Lambda_{QCD}\, b_{max})$ errors**

Check consistency

- $\int \mathrm{d}^2\mathbf{k}_{1T}\mathrm{d}^2\mathbf{k}_{2T} f_{j/p}(x,\mathbf{k}_{1T};\mu_Q,Q^2) D_{h/j}(z,z\mathbf{k}_{2T};\mu_Q,Q^2)\delta^{(2)}(\mathbf{q}_T+\mathbf{k}_{1T}-\mathbf{k}_{2T})$ is uniquely determined by its operator definition

- At $q_T \approx Q$

$$= \frac{1}{q_{\mathrm{T}}^2}\left[\mathcal{C}\left(q_{\mathrm{T}}/\mu_Q,\alpha_s\left(\mu_Q\right)\right)\otimes f_{j/p}(x;\mu_Q)\otimes d_{h/j}(z;\mu_Q)+O\left(\frac{\Lambda_{\mathrm{QCD}}}{q_{\mathrm{T}}}\right)\right]$$

$q_T \sim Q, Q \to \infty$ *asymptote*

- For TMD pdfs & ffs

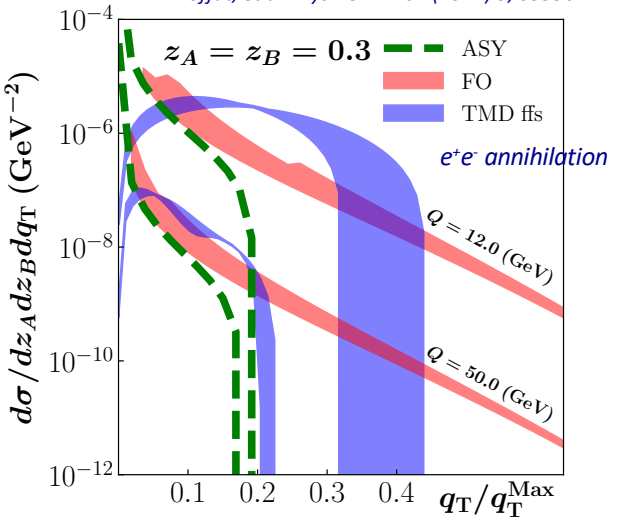
$$f_{j/p}(x,k_T\approx Q;\mu_Q,Q^2)=\frac{1}{k_T^2}\left[\int_x^1\frac{\mathrm{d}\xi}{\xi}C_{j/k}(x/\xi,k_T/Q,\alpha_s(Q))f_{k/p}(\xi;Q)+O\left(\frac{\Lambda_{\mathrm{QCD}}}{k_T}\right)\right]$$

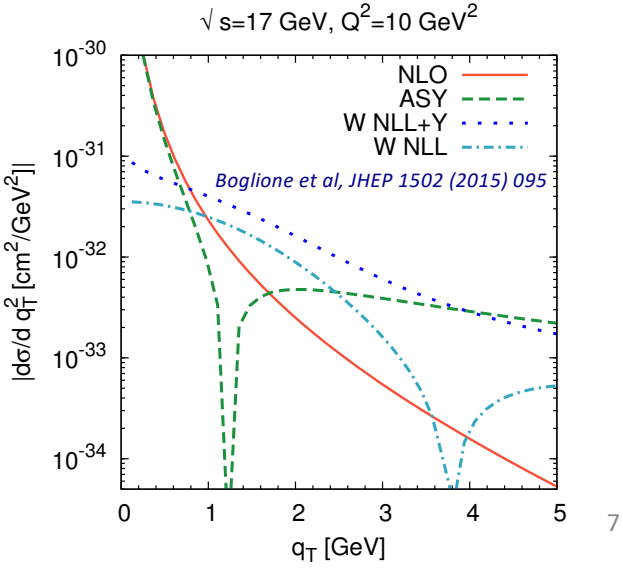
&

$$\int \mathrm{d}^2\mathbf{k}_T f_{j/p}(x,k_T;\mu_Q,Q^2)\approx f_{j/p}(x;\mu_Q)$$

- $\dfrac{\mathrm{d}}{\mathrm{d}b_{\max}}\left(\dfrac{\mathrm{d}\sigma}{\mathrm{d}^2\boldsymbol{q}_{\mathrm{T}}\ldots}\right)=0$ *for.* $b_{\max} \ll 1/\Lambda_{QCD}$

E. Moffat, et al Phys.Rev.D 104 (2021) 5, 059904



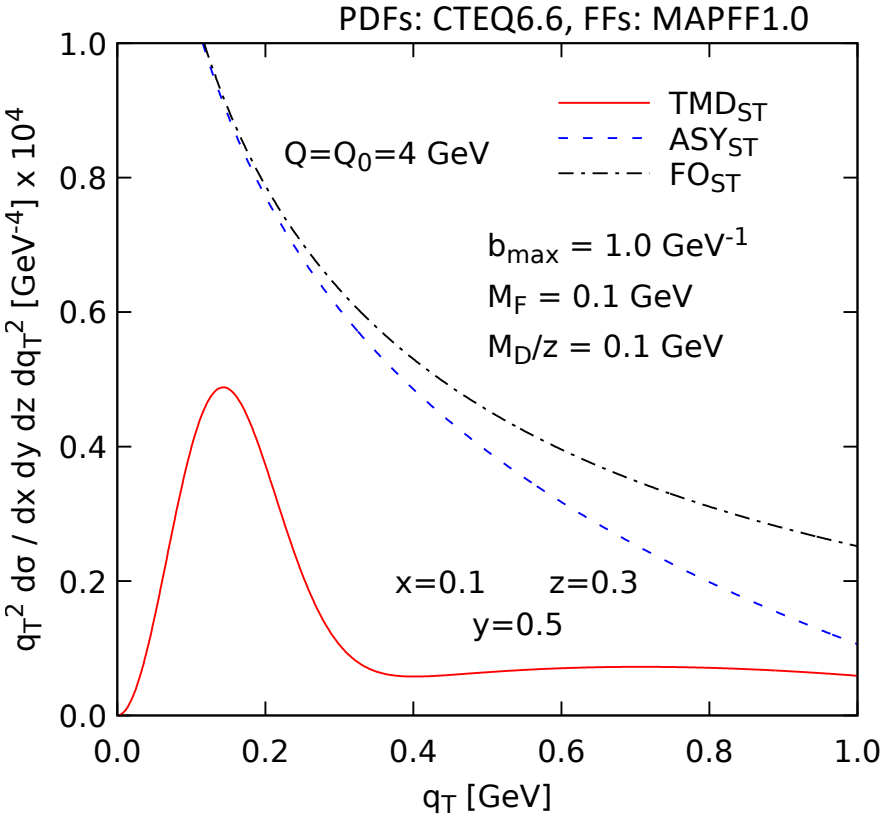


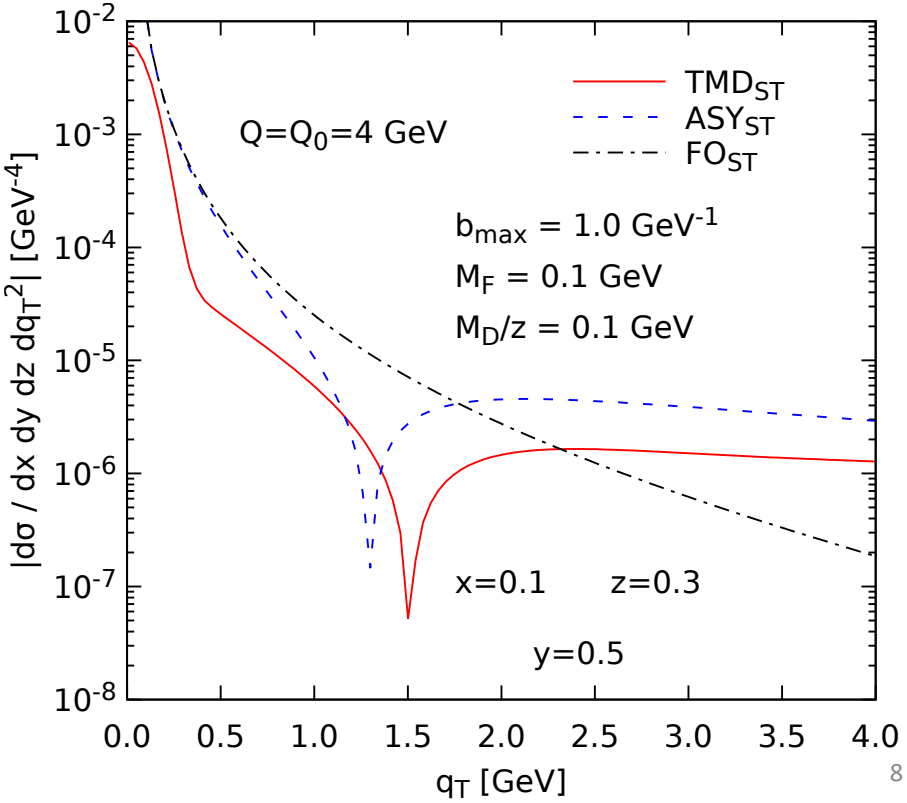
Conventional organization & problems

$$g_{j/p}(x, b_{\rm T}) = \frac{1}{4} M_F^2 \, b_{\rm T}^2$$

$$g_{h/j}(z, b_{\rm T}) = \frac{1}{4\, z^2} M_D^2 \, b_{\rm T}^2$$

(An example typical of conventional approach)





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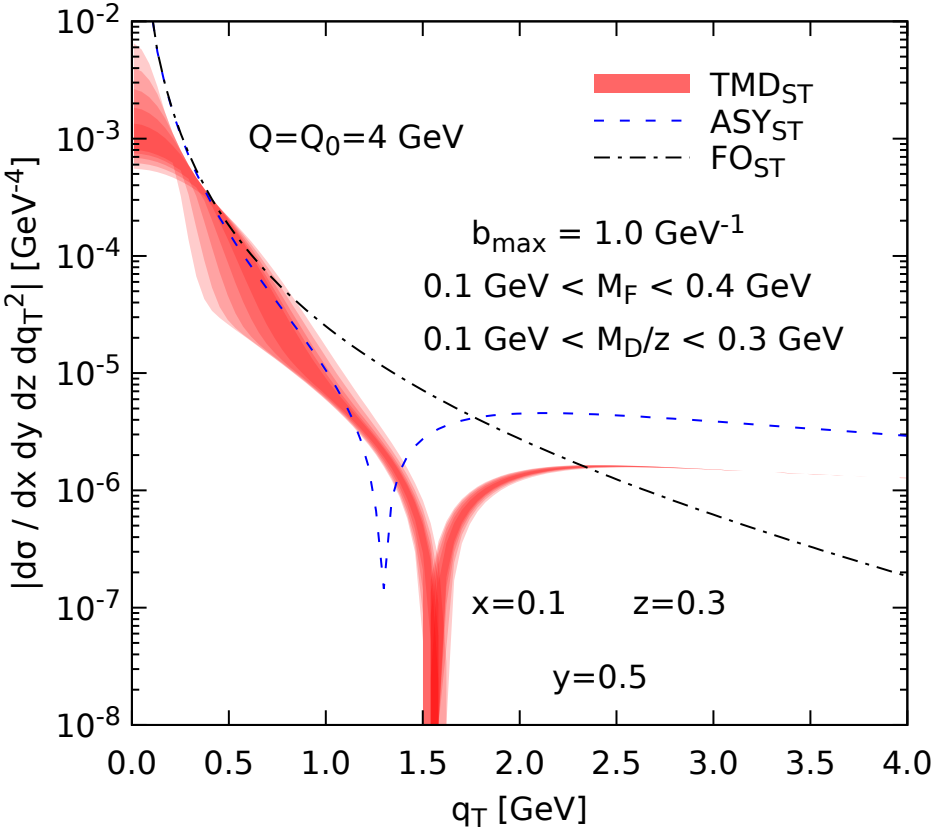

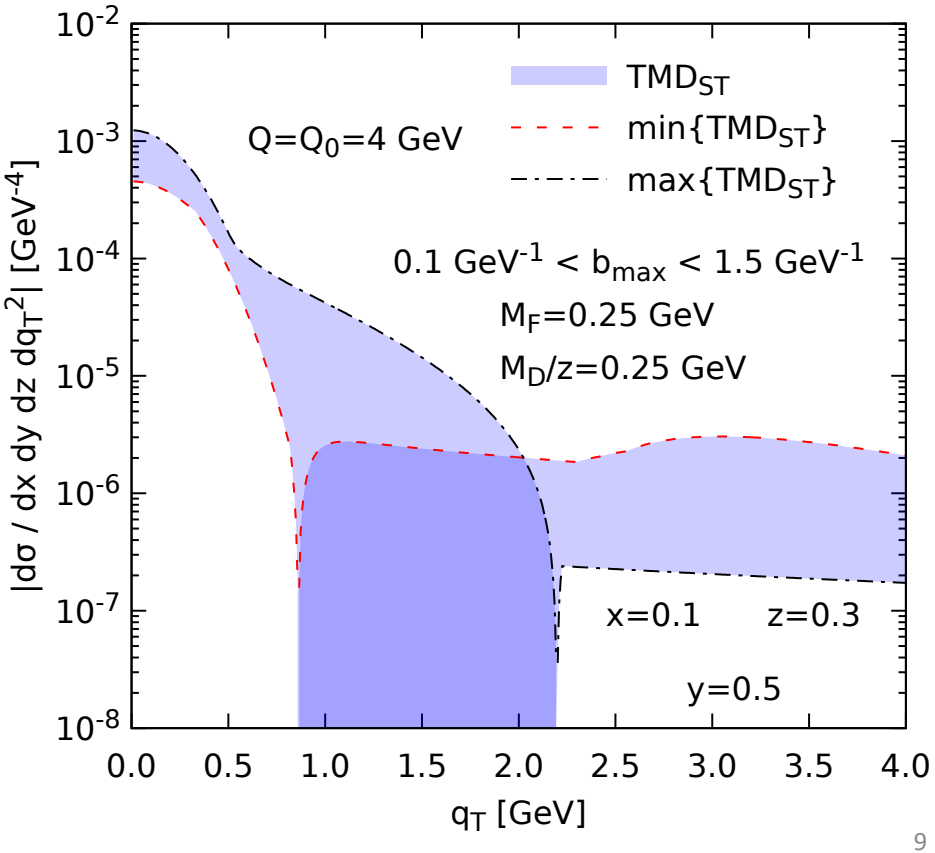

$$\frac{\mathrm{d}\sigma}{\mathrm{d}Q\,\mathrm{d}^2\mathbf{q}_T\,\mathrm{d}x\,\mathrm{d}z} = H(Q/\mu_Q)\underbrace{\int \mathrm{d}^2\mathbf{k}_{1T}\mathrm{d}^2\mathbf{k}_{2T} f_{j/p}(x,\mathbf{k}_{1T};\mu_Q,Q^2) D_{h/j}(z,z\mathbf{k}_{2T};\mu_Q,Q^2)\delta^{(2)}(\mathbf{q}_T+\mathbf{k}_{1T}-\mathbf{k}_{2T})}_{\text{Well-defined operators for all TM}} + \text{power suppressed}$$

Diagnosis

1) Consistency tests will generally fail for g-function ansatz unless constraints are imposed

2) No region at input scale $Q = Q_0$ where $\Lambda_{QCD} \ll q_T \ll Q_0$

3) Backwards evolution…
No large, perturbative $\ln\frac{Q_0}{q_T}$.

4) $\int \mathrm{d}^2\mathbf{k}_T f_{j/p}(x,k_T;\mu_Q,Q^2) \approx f_{j/p}(x;\mu_Q)$
Very badly violated at moderate scales

See T. Rainaldi talk, 4:00 pm WG1, for examples in toy model

$$\frac{\mathrm{d}\sigma}{\mathrm{d}Q\,\mathrm{d}^2\mathbf{q}_T\,\mathrm{d}x\,\mathrm{d}z} =$$

$$H(Q/\mu_Q)\int \frac{\mathrm{d}^2\boldsymbol{b}_\mathrm{T}}{(2\pi)^2}\, e^{-i\boldsymbol{q}_\mathrm{T}\cdot\boldsymbol{b}_\mathrm{T}}\, \tilde{f}_{j/p}(x,\boldsymbol{b}_\mathrm{T};\mu_{Q_0},Q_0^2)\, \tilde{D}_{h/j}(z,\boldsymbol{b}_\mathrm{T};\mu_{Q_0},Q_0^2)$$

$$\times \exp\left\{\tilde{K}(b_\mathrm{T};\mu_{Q_0})\ln\left(\frac{Q^2}{Q_0^2}\right) + \int_{\mu_{Q_0}}^{\mu_Q}\frac{\mathrm{d}\mu'}{\mu'}\left[2\gamma(\alpha_s(\mu');1) - \ln\frac{Q^2}{\mu'^2}\gamma_K(\alpha_s(\mu'))\right]\right\}$$

$$\frac{\mathrm{d}\sigma}{\mathrm{d}Q\,\mathrm{d}^2\mathbf{q}_T\,\mathrm{d}x\,\mathrm{d}z} = H(Q/\mu_Q)\underbrace{\int \mathrm{d}^2\mathbf{k}_{1T}\mathrm{d}^2\mathbf{k}_{2T} f_{j/p}(x,\mathbf{k}_{1T};\mu_Q,Q^2) D_{h/j}(z,z\mathbf{k}_{2T};\mu_Q,Q^2)\delta^{(2)}(\mathbf{q}_T+\mathbf{k}_{1T}-\mathbf{k}_{2T})}_{\text{Well-defined operators for all TM}} + \text{power suppressed}$$

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$$\frac{\mathrm{d}\sigma}{\mathrm{d}Q\,\mathrm{d}^2\mathbf{q}_T\,\mathrm{d}x\,\mathrm{d}z} =$$

$$H(Q/\mu_Q)\int \frac{\mathrm{d}^2\boldsymbol{b}_\mathrm{T}}{(2\pi)^2}\, e^{-i\boldsymbol{q}_\mathrm{T}\cdot\boldsymbol{b}_\mathrm{T}}\, \tilde{f}_{j/p}(x,\boldsymbol{b}_\mathrm{T};\mu_{Q_0},Q_0^2)\, \tilde{D}_{h/j}(z,\boldsymbol{b}_\mathrm{T};\mu_{Q_0},Q_0^2)$$

$$\times \exp\left\{\tilde{K}(b_\mathrm{T};\mu_{Q_0})\ln\left(\frac{Q^2}{Q_0^2}\right) + \int_{\mu_{Q_0}}^{\mu_Q}\frac{\mathrm{d}\mu'}{\mu'}\left[2\gamma(\alpha_s(\mu');1) - \ln\frac{Q^2}{\mu'^2}\gamma_K(\alpha_s(\mu'))\right]\right\}$$

$$\tilde{D}_{h/j}(z,\boldsymbol{b}_\mathrm{T};\mu_{Q_0},Q_0^2) = \tilde{D}_{\mathrm{inpt},h/j}(z,\boldsymbol{b}_\mathrm{T};\mu_{\overline{Q}_0},\overline{Q}_0^2)E(\overline{Q}_0/Q_0,b_\mathrm{T})$$

$$\tilde{f}_{i/p}(x,\boldsymbol{b}_\mathrm{T};\mu_{Q_0},Q_0^2) = \tilde{f}_{\mathrm{inpt},i/p}(x,\boldsymbol{b}_\mathrm{T};\mu_{\overline{Q}_0},\overline{Q}_0^2)E(\overline{Q}_0/Q_0,b_\mathrm{T})$$

$$\lim_{b_\mathrm{T}\to 0}\overline{Q}_0 \sim 1/b_\mathrm{T}$$

Instead, characterize the full behavior of TMD functions at the input scale

A hadron-structure-oriented reorganization

1) Use the uniquely determined TMDs for all k_T

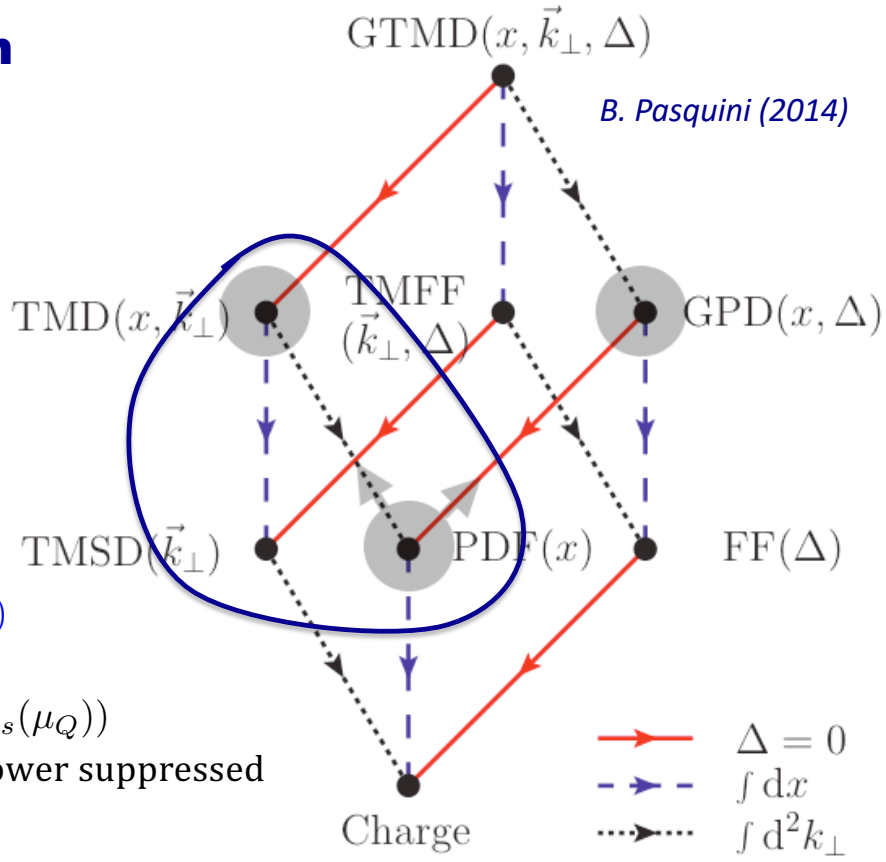
2) Smoothly transform directly between nonperturbative TM dependence at small TM ($k_T \approx \Lambda_{QCD}$) & perturbative (collinear) TM at large TM ($k_T \approx Q$)

3) (Approximate) probability interpretation

- Parton model: $\int \mathrm{d}^2\mathbf{k}_T f_{j/p}(x, k_T; \mu_Q, Q^2) = f_{j/p}(x; \mu_Q)$
- QCD: $\pi \int^{\mu_Q^2} \mathrm{d}k_T^2 f_{j/p}(x, k_T; \mu_Q, Q^2) = f_{j/p}(x; \mu_Q) + O\left(\alpha_s(\mu_Q)\right)$ + power suppressed

4) All should apply at input scale, Q_0

5) Pheno: Must be simple to swap one model/parametrization for another



A hadron-structure-oriented reorganization

- Recipe: Transform a model TMD parametrization into an evolved parametrization at other scales:
  - See Sec. VI of Phys.Rev.D 106 (2022) 3, 034002

- No b_{max}, no b_*

- HSO approach is equivalent to standard TMD factorization, CSS, etc, just with constraints on the g-functions

- Straightforward to translate between standard treatment and HSO.
  - See Sec. IX of Phys.Rev.D 106 (2022) 3, 034002

An $O(\alpha_s)$ example

$$f_{\mathrm{inpt},i/p}(x,\boldsymbol{k}_{\mathrm{T}};\mu_{Q_0},Q_0^2) = \frac{1}{2\pi}\frac{1}{k_{\mathrm{T}}^2+m_{f_{i,p}}^2}\left[A_{i/p}^f(x;\mu_{Q_0}) + B_{i/p}^f(x;\mu_{Q_0})\ln\frac{Q_0^2}{k_{\mathrm{T}}^2+m_{f_{i,p}}^2}\right] + \frac{1}{2\pi}\frac{1}{k_{\mathrm{T}}^2+m_{f_{g,p}}^2}A_{i/p}^{f,g}(x;\mu_{Q_0})$$
$$+ C_{i/p}^f\, f_{\mathrm{core},i/p}(x,\boldsymbol{k}_{\mathrm{T}};Q_0^2)$$

$$D_{\mathrm{inpt},h/j}(z,z\boldsymbol{k}_{\mathrm{T}};\mu_{Q_0},Q_0^2) = \frac{1}{2\pi z^2}\frac{1}{k_{\mathrm{T}}^2+m_{D_{h,j}}^2}\left[A_{h/j}^D(z;\mu_{Q_0}) + B_{h/j}^D(z;\mu_{Q_0})\ln\frac{Q_0^2}{k_{\mathrm{T}}^2+m_{D_{h,j}}^2}\right] + \frac{1}{2\pi z^2}\frac{1}{k_{\mathrm{T}}^2+m_{D_{h,g}}^2}A_{h/j}^{D,g}(z;\mu_{Q_0})$$
$$+ C_{h/j}^D\, D_{\mathrm{core},h/j}(z,z\boldsymbol{k}_{\mathrm{T}};Q_0^2)$$

- C^f & C^D constrained by:

$$f_{i/p}^c(x;\mu_{Q_0}) \equiv 2\pi\int_0^{\mu_{Q_0}}\mathrm{d}k_{\mathrm{T}}\,k_{\mathrm{T}}f_{i/p}(x,\boldsymbol{k}_{\mathrm{T}};\mu_{Q_0},Q_0^2) = f_{i/p}(x;\mu_{Q_0}) + \mathcal{H}_{i/i'}\otimes f_{i'/p} + \mathrm{p.s.}$$
$$d_{h/j}^c(z;\mu_{Q_0}) \equiv 2\pi z^2\int_0^{\mu_{Q_0}}\mathrm{d}k_{\mathrm{T}}\,k_{\mathrm{T}}D_{h/j}(z,z\boldsymbol{k}_{\mathrm{T}};\mu_{Q_0},Q_0^2) = d_{h/j}(z;\mu_{Q_0}) + \mathcal{H}_{j'/j}\otimes d_{h/j'} + \mathrm{p.s.}$$

An $O(\alpha_s)$ example

- Parametrizing the very small transverse momentum

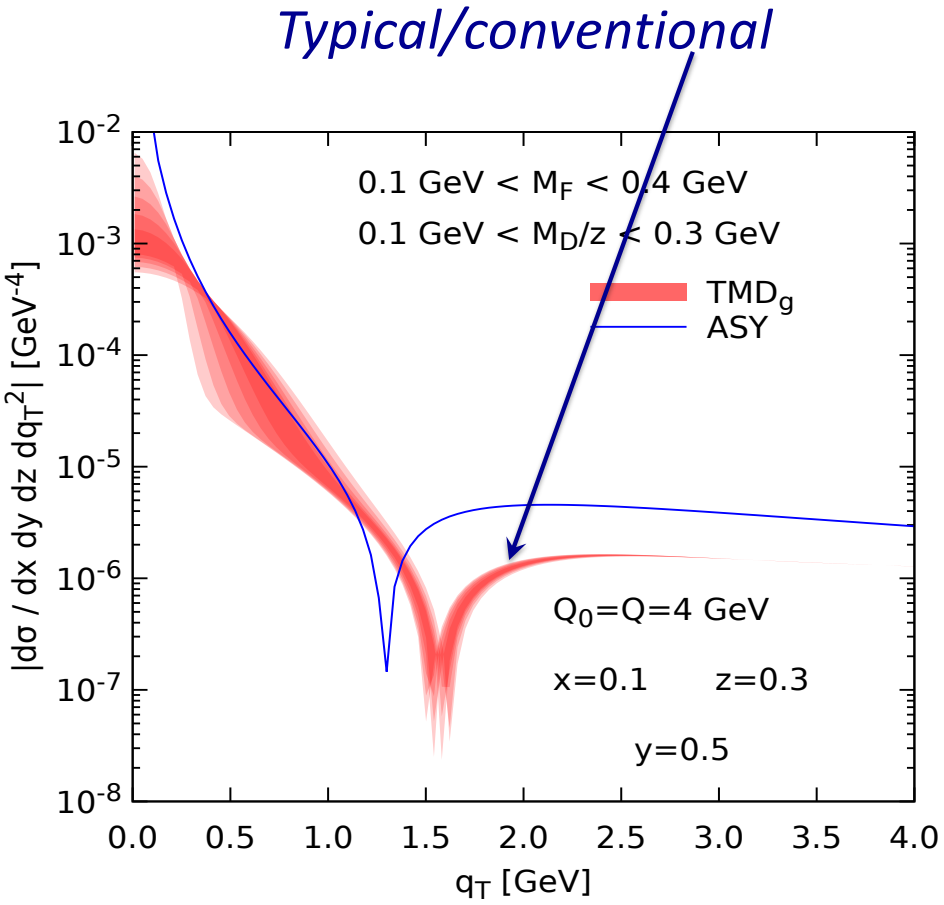
A. Gaussian model

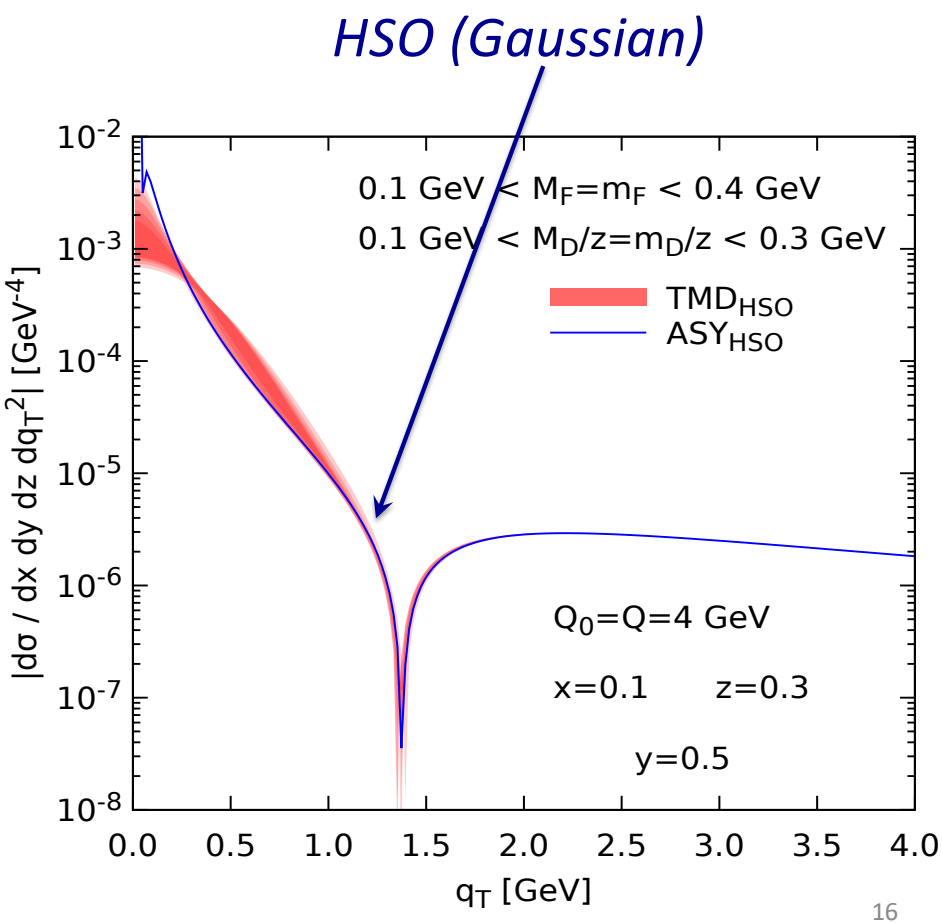
$$f^{\text{Gauss}}_{\text{core},i/p}(x,\boldsymbol{k}_{\text{T}};Q_0^2)=\frac{e^{-k_{\text{T}}^2/M_{\text{F}}^2}}{\pi M_{\text{F}}^2}\,,\qquad D^{\text{Gauss}}_{\text{core},h/j}(z,z\boldsymbol{k}_{\text{T}};Q_0^2)=\frac{e^{-z^2\,k_{\text{T}}^2/M_{\text{D}}^2}}{\pi M_{\text{D}}^2}$$

B. Spectator model

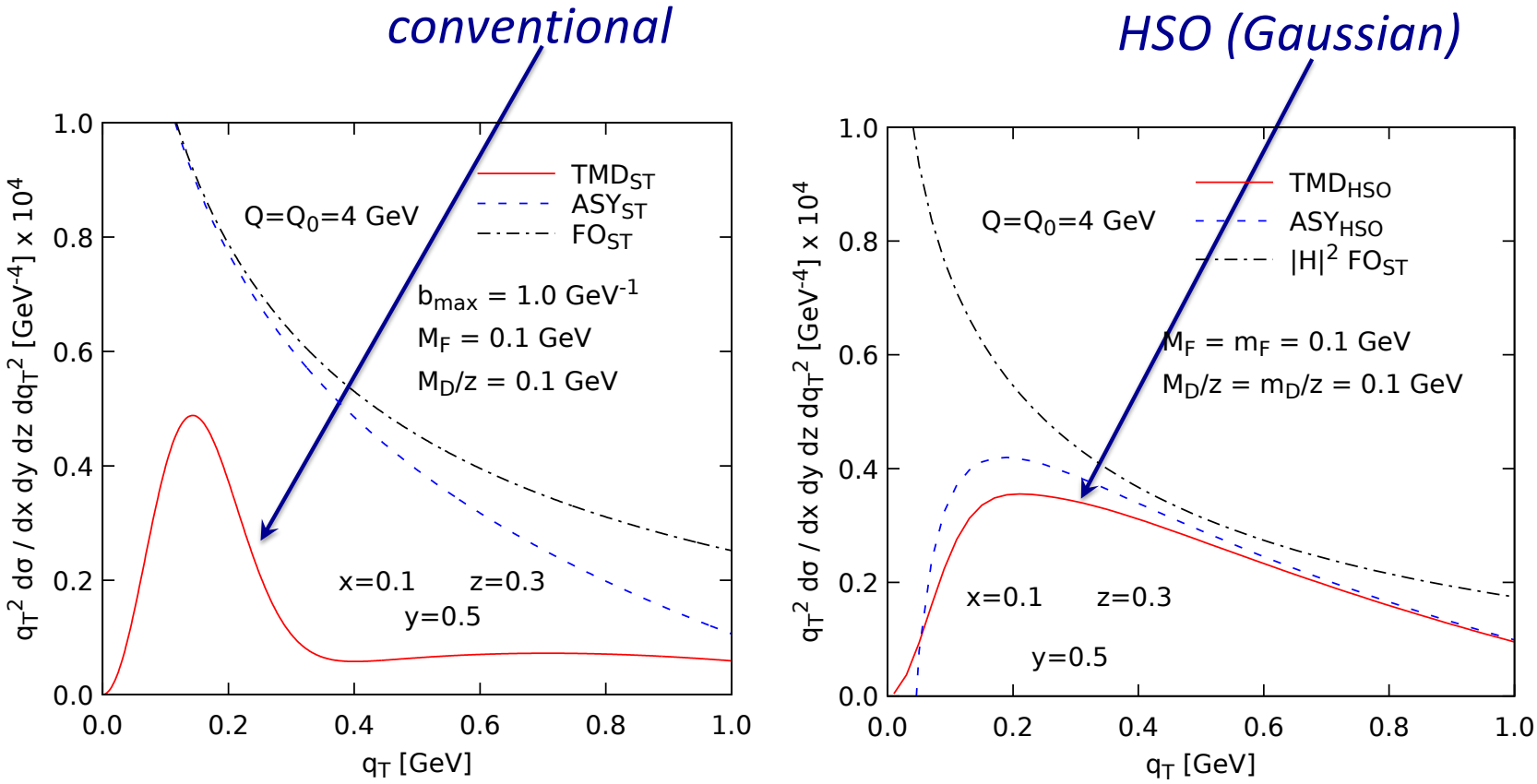
$$f^{\text{Spect}}_{\text{core},i/p}(x,\boldsymbol{k}_{\text{T}};Q_0^2)=\frac{6M_{0\text{F}}^6}{\pi\,(2M_{\text{F}}^2+M_{0\text{F}}^2)}\,\frac{M_{\text{F}}^2+k_{\text{T}}^2}{(M_{0\text{F}}^2+k_{\text{T}}^2)^{\,4}}\,,\quad D^{\text{Spect}}_{\text{core},h/j}(z,z\boldsymbol{k}_{\text{T}};Q_0^2)=\frac{2M_{0\text{D}}^4}{\pi\,(M_{\text{D}}^2+M_{0\text{D}}^2)}\,\frac{M_{\text{D}}^2+k_{\text{T}}^2z^2}{(M_{0\text{D}}^2+k_{\text{T}}^2z^2)^{\,3}}$$

Compare standard with constrained





Compare standard with constrained



Summary

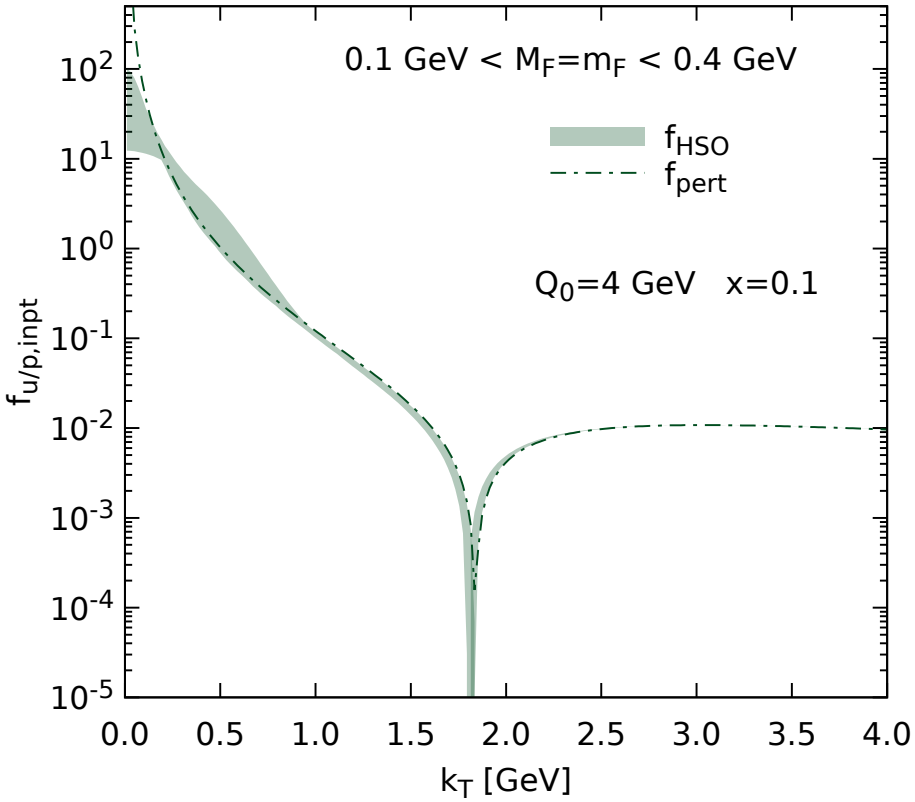
- Switching to a hadron-structure-oriented approach to pheno with TMD factorization improves consistency in the large transverse behavior of TMD correlation functions

- Necessary for understanding the shapes of nonperturbative distributions, separating perturbative and nonpeturbative parts, etc

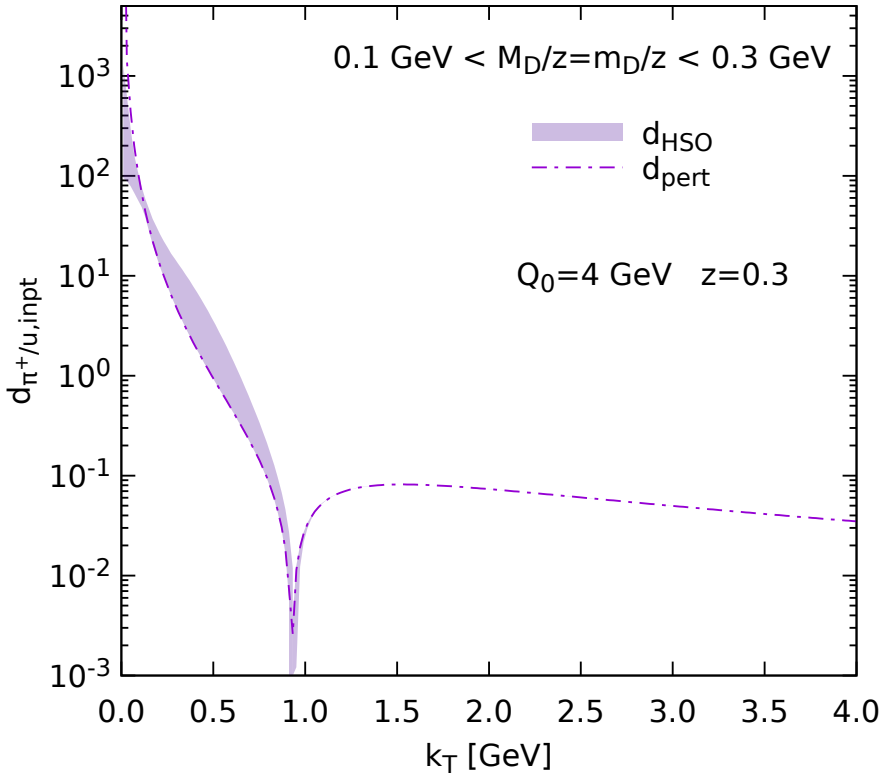
- Next:
  - Applications
  - Higher orders
  - Incorporating NP calculations (lattice, EFTs, models etc)
  - Spin dependent TMDs

Backup

up-quark TMD pdfs



π^+ from up-quark TMD ff



HSO (Gaussian)

Backup

$$A^f_{i/p}(x;\mu_{Q_0}) \equiv \sum_{ii'} \delta_{i'i} \frac{\alpha_s(\mu_{Q_0})}{\pi} \left\{ \left[(P_{i'i} \otimes f_{i'/p})(x;\mu_{Q_0}) \right] - \frac{3C_F}{2} f_{i'/p}(x;\mu_{Q_0}) \right\} ,$$

$$B^f_{i/p}(x;\mu_{Q_0}) \equiv \sum_{i'i} \delta_{i'i} \frac{\alpha_s(\mu_{Q_0}) C_F}{\pi} f_{i'/p}(x;\mu_{Q_0}) ,$$

$$A^{f,g}_{i/p}(x;\mu_{Q_0}) \equiv \frac{\alpha_s(\mu_{Q_0})}{\pi} \left[(P_{ig} \otimes f_{g/p})(x;\mu_{Q_0}) \right] ,$$

$$C^f_{i/p} \equiv \frac{1}{N^f_{i/p}} \Bigg[f_{i/p}(x;\mu_{Q_0}) - A^f_{i/p}(x;\mu_{Q_0}) \ln\left(\frac{\mu_{Q_0}}{m_{f_{i,p}}}\right) - B^f_{i/p}(x;\mu_{Q_0}) \ln\left(\frac{\mu_{Q_0}}{m_{f_{i,p}}}\right) \ln\left(\frac{Q_0^2}{\mu_{Q_0} m_{f_{i,p}}}\right) ,$$

$$- A^{f,g}_{i/p}(x;\mu_{Q_0}) \ln\left(\frac{\mu_{Q_0}}{m_{f_{g,p}}}\right) + \frac{\alpha_s(\mu_{Q_0})}{2\pi} \left\{ \sum_{ii'} \delta_{i'i} [\mathcal{C}^{i/i'}_\Delta \otimes f_{i'/p}](x;\mu_{Q_0}) + [\mathcal{C}^{i/g}_\Delta \otimes f_{g/p}](x;\mu_{Q_0}) \right\} \Bigg] .$$

$$P_{ig}(x) = T_F \left[x^2 + (1-x)^2 \right] ,$$

$$\mathcal{C}^{i/i}_\Delta(x) = C_F(1-x) - C_F \frac{\pi^2}{12} \delta(1-x) ,$$

$$\mathcal{C}^{g/p}_\Delta(x) = 2T_F x(1-x) ,$$

$$N^f_{i/p} \equiv 2\pi \int_0^\infty dk_{\mathrm{T}} k_{\mathrm{T}} \, f_{\mathrm{core},i/p}(x, \boldsymbol{k}_{\mathrm{T}}; Q_0^2)$$

Backup

$$A^D_{h/j}(z;\mu_{Q_0}) \equiv \sum_{jj'} \delta_{j'j} \frac{\alpha_s(\mu_{Q_0})}{\pi} \left\{ \left[(P_{jj'} \otimes d_{h/j'})(z;\mu_{Q_0}) \right] - \frac{3C_F}{2} d_{h/j'}(z;\mu_{Q_0}) \right\} ,$$

$$B^D_{h/j}(z;\mu_{Q_0}) \equiv \sum_{jj'} \delta_{j'j} \frac{\alpha_s(\mu_{Q_0}) C_F}{\pi} d_{h/j'}(z;\mu_{Q_0}) ,$$

$$A^{D,g}_{h/j}(z;\mu_{Q_0}) \equiv \frac{\alpha_s(\mu_{Q_0})}{\pi} \left[(P_{gj} \otimes d_{h/g})(z;\mu_{Q_0}) \right] ,$$

$$C^D_{h/j} \equiv \frac{1}{N^D_{h/j}} \Bigg[d_{h/j}(z;\mu_{Q_0}) - A^D_{h/j}(z;\mu_{Q_0}) \ln\left(\frac{\mu_{Q_0}}{m_{D_{h,j}}}\right) - B^D_{h/j}(z;\mu_{Q_0}) \ln\left(\frac{\mu_{Q_0}}{m_{D_{h,j}}}\right) \ln\left(\frac{Q_0^2}{\mu_{Q_0} m_{D_{h,j}}}\right) ,$$

$$- A^{D,g}_{h/j}(z;\mu_{Q_0}) \ln\left(\frac{\mu_{Q_0}}{m_{D_{h,g}}}\right) + \frac{\alpha_s(\mu_{Q_0})}{2\pi} \left\{ \sum_{jj'} \delta_{j'j} [\mathcal{C}^{j'/j}_\Delta \otimes d_{h/j'}](z;\mu_{Q_0}) + [\mathcal{C}^{g/j}_\Delta \otimes d_{h/g}](z;\mu_{Q_0}) \right\} \Bigg] .$$

$$P_{qq}(z) = P_{\bar{q}\bar{q}}(z) = C_F \left[\frac{1+z^2}{(1-z)_+} + \frac{3}{2}\delta(1-z) \right] ,$$

$$P_{gq}(z) = C_F \frac{1+(1-z)^2}{z} ,$$

$$\mathcal{C}^{q/q}_\Delta(z) = 2P_{qq}(z) \ln z + C_F(1-z) - C_F \frac{\pi^2}{12} \delta(1-z) ,$$

$$\mathcal{C}^{g/q}_\Delta(z) = 2P_{gq}(z) \ln z + C_F z ,$$

$$N^D_{h/j} \equiv 2\pi z^2 \int_0^\infty dk_{\rm T} k_{\rm T} D_{{\rm core},h/j}(z, z\boldsymbol{k}_{\rm T}; Q_0^2) .$$

Backup

$$
\begin{aligned}
f^{c}_{\mathrm{inpt},i/p}(x;\mu_{Q_0}) =& 2\pi \int_0^{\mu_{Q_0}} \mathrm{d}k_{\mathrm{T}}\, k_{\mathrm{T}} f_{\mathrm{inpt},i/p}(x,\boldsymbol{k}_{\mathrm{T}};\mu_{Q_0},Q_0^2) = \\
& C^{f}_{i/p}\, f^{c}_{\mathrm{core},i/p}(x;\mu_{Q_0}) + \frac{1}{2} A^{f,g}_{i/p}(x;\mu_{Q_0}) \ln\left(1+\frac{\mu^2_{Q_0}}{m^2_{f_{g,p}}}\right) \\
& + \frac{1}{2} A^{f}_{i/p}(x;\mu_{Q_0}) \ln\left(1+\frac{\mu^2_{Q_0}}{m^2_{f_{i,p}}}\right) + \frac{1}{4} B^{f}_{i/p}(x;\mu_{Q_0}) \left[\ln^2\left(\frac{m^2_{f_{i,p}}}{Q_0^2}\right) - \ln^2\left(\frac{\mu^2_{Q_0}+m^2_{f_{i,p}}}{Q_0^2}\right)\right] \\
& = f_{i/p}(x;\mu_{Q_0}) + O\left(\alpha_s(\mu_0), \frac{m^2}{Q_0^2}\right) ,
\end{aligned}
$$

$$
\begin{aligned}
d^{c}_{\mathrm{inpt},h/j}(z;\mu_{Q_0}) =& 2\pi z^2 \int_0^{\mu_{Q_0}} \mathrm{d}k_{\mathrm{T}}\, k_{\mathrm{T}} D_{\mathrm{inpt},h/j}(z,z\boldsymbol{k}_{\mathrm{T}};\mu_{Q_0},Q_0^2) = \\
& C^{D}_{h/j}\, d^{c}_{\mathrm{core},h/j}(z;\mu_{Q_0}) + \frac{1}{2} A^{D,g}_{h/j}(z;\mu_{Q_0}) \ln\left(1+\frac{\mu^2_{Q_0}}{m^2_{D_{h,g}}}\right) \\
& + \frac{1}{2} A^{D}_{h/j}(z;\mu_{Q_0}) \ln\left(1+\frac{\mu^2_{Q_0}}{m^2_{D_{h,j}}}\right) + \frac{1}{4} B^{D}_{h/j}(z;\mu_{Q_0}) \left[\ln^2\left(\frac{m^2_{D_{h,j}}}{Q_0^2}\right) - \ln^2\left(\frac{\mu^2_{Q_0}+m^2_{D_{h,j}}}{Q_0^2}\right)\right] \\
& = d_{h/j}(z;\mu_{Q_0}) + O\left(\alpha_s(\mu_0), \frac{m^2}{Q_0^2}\right) ,
\end{aligned}
$$