

# Moments of nucleon GPDs from the leading-twist expansion of the quasi-GPD matrix element

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in collaboration with: S. Bhattacharya, M. Constantinou, K. Cichy, J. Dodson,  
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# Generalized parton distributions

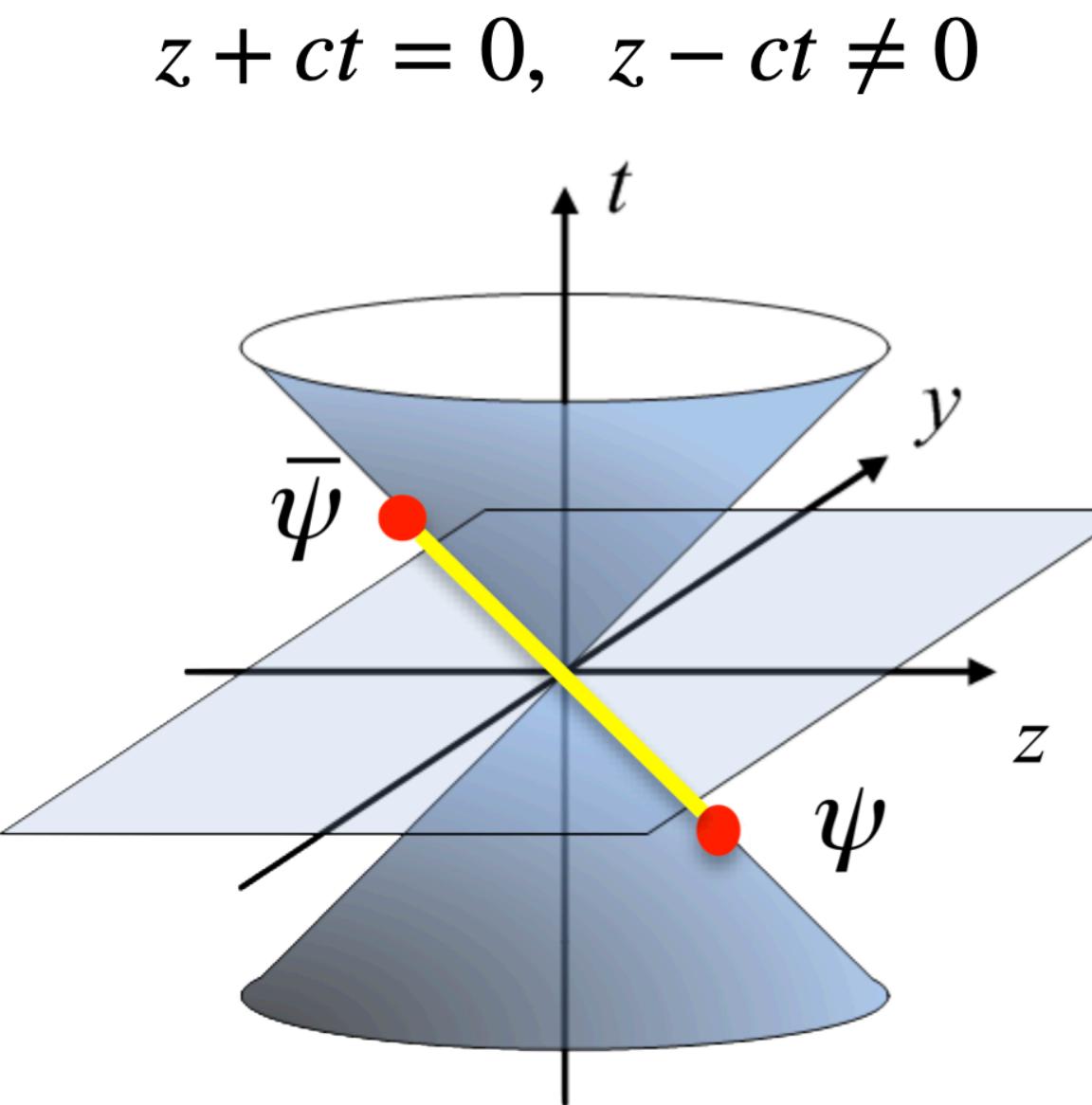
The gauge-invariant off-forward matrix elements,

$$F^\mu(z, P, \Delta) = \langle p_f | \bar{q}(-\frac{z}{2}) \gamma^\mu \mathcal{W}(-\frac{z}{2}, \frac{z}{2}) q(\frac{z}{2}) | p_i \rangle$$

Light-cone GPDs,

$$F(x, \xi, \Delta, \mu) = \int \frac{dz^-}{4\pi} e^{-ixP^+ z^-} F^\mu(z, P, \Delta)$$

- $\gamma^\mu = \gamma^+$
- $z = ln_-$ ,  $z^2 = 0$
- $\mathcal{W}(-\frac{z}{2}, \frac{z}{2}) = \mathcal{P} \exp(i \int_{-ln_-/2}^{ln_-/2} dl' A^+)$



# Generalized parton distributions

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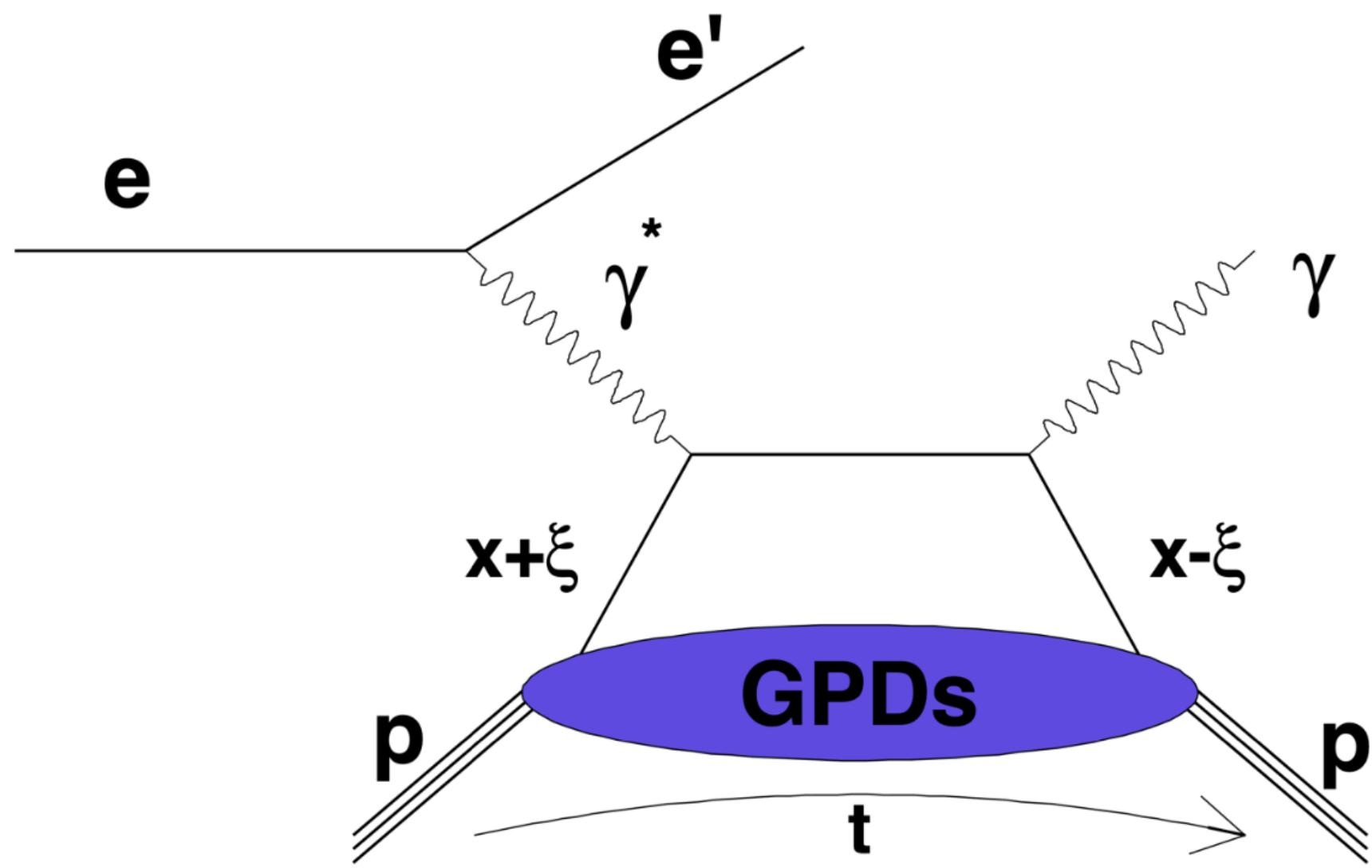
Depends on  $x$  and Lorentz invariant products of the vectors  $p_f, p_i$  (or  $P = (p_f + p_i)/2$ ,  $\Delta = p_f - p_i$ ) and  $n_-$ , conventionally to be

$$\xi = -(\Delta \cdot n_-)/(2P \cdot n_-)$$

$$t = \Delta^2$$

# Generalized parton distributions

DVCS



The golden process to study the quark GPDs is DVCS

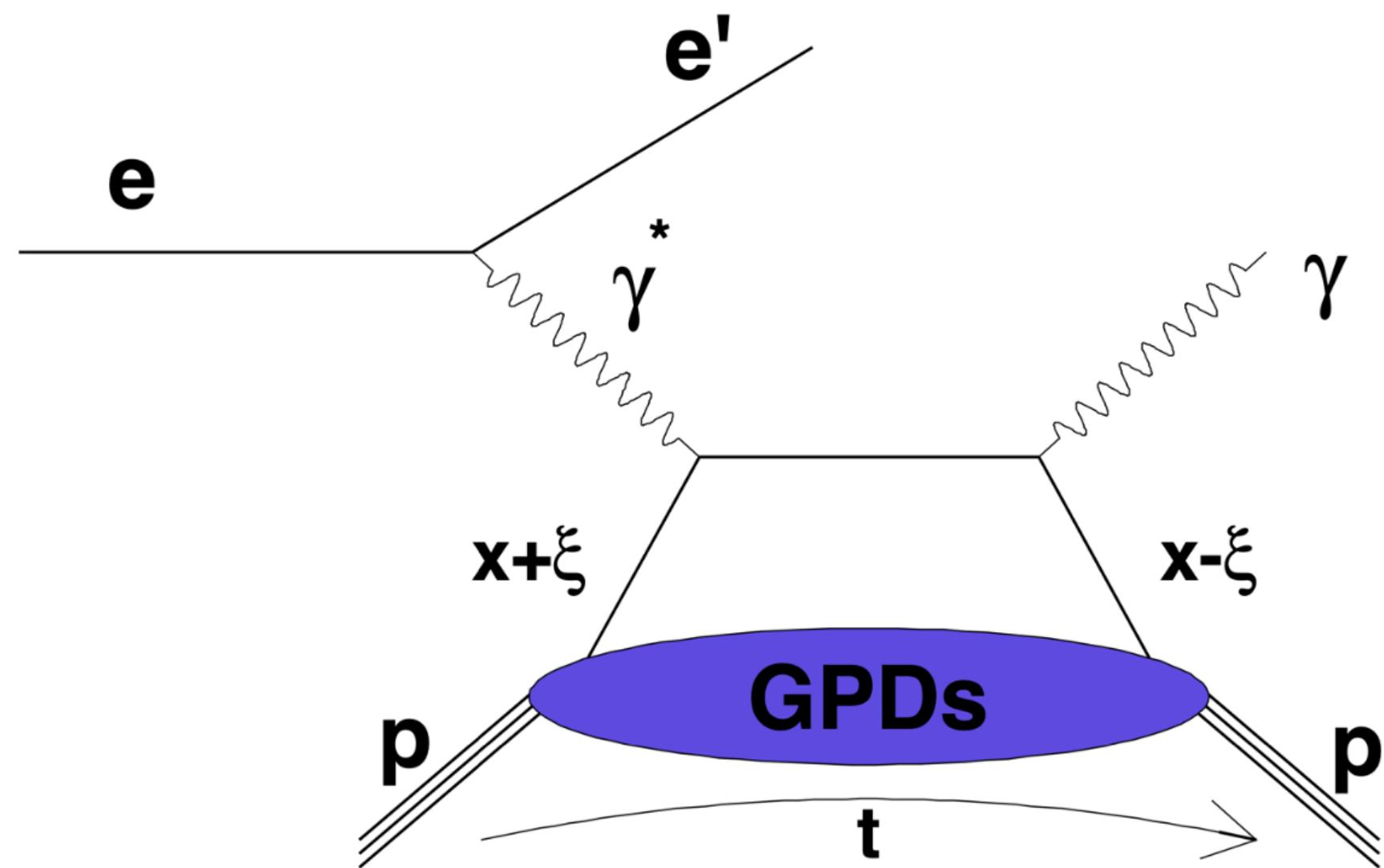
Challenging:

- observables appear at the **amplitude level**
- multi-dimensionality  $(x, \xi, t)$
- the momentum fraction  $x$  is **integrated over** (Compton Form Factors)

$$\mathcal{F}(\xi, t; Q^2) = \int_{-1}^1 dx \left[ \frac{1}{\xi - x - i\epsilon} \pm \frac{1}{\xi + x - i\epsilon} \right] F(x, \xi, t; Q^2)$$

# Generalized parton distributions

## DVCS

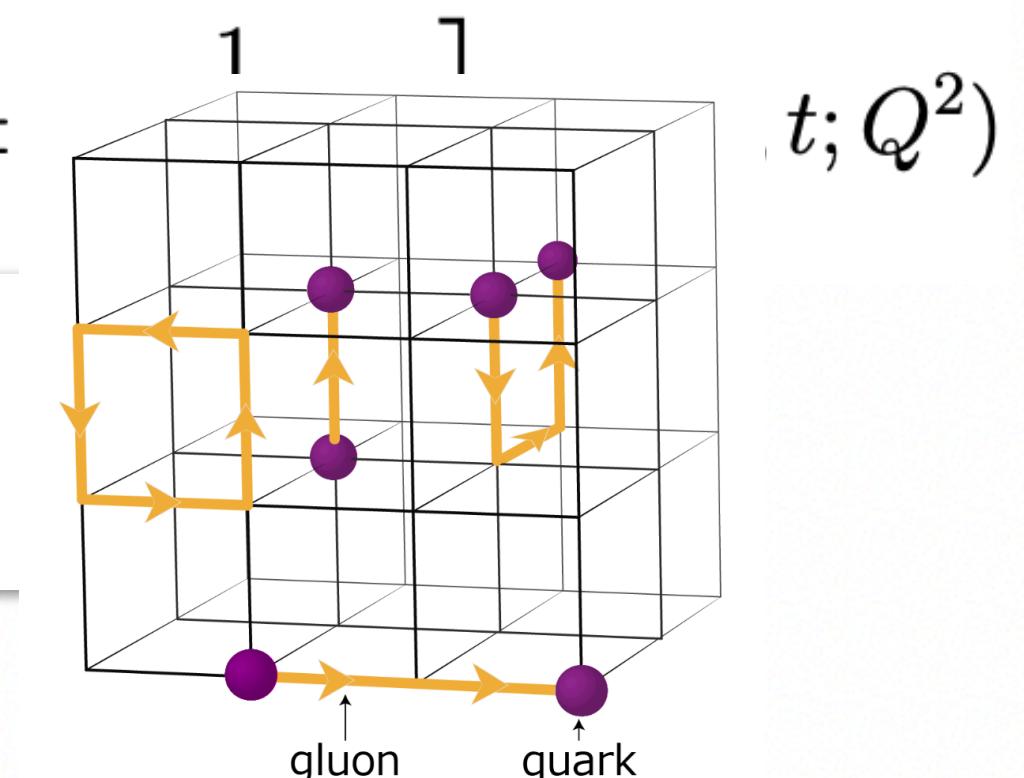


## Challenging:

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- multi-dimensionality  $(x, \xi, t)$
- the momentum fraction  $x$  is integrated over (Compton Form Factors)

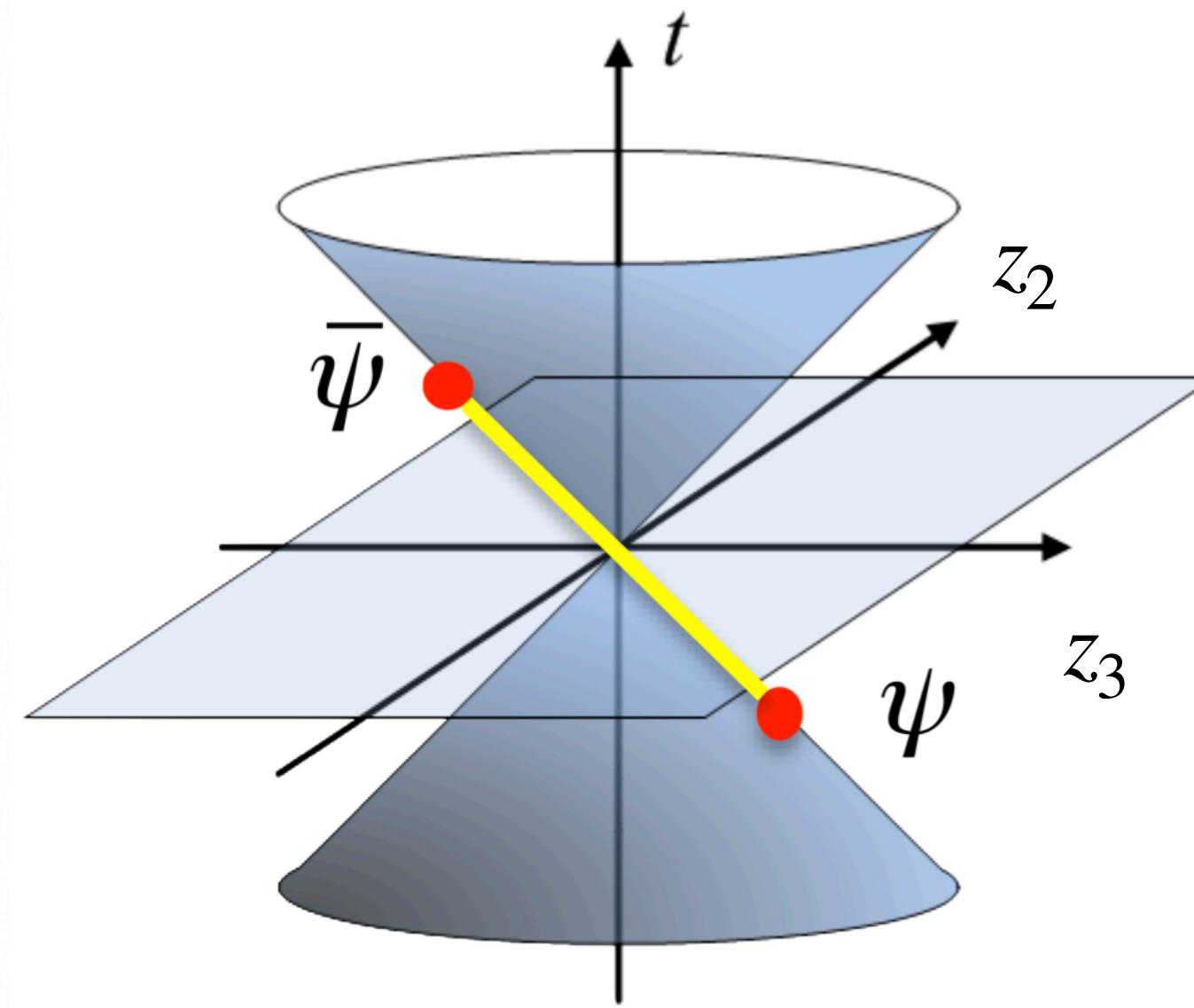
$$\mathcal{F}(\xi, t; Q^2) = \int_{-1}^1 dx \left[ \frac{1}{\xi - x - i\epsilon} \pm \right]$$

Complementary knowledge from lattice QCD is essential.



# Generalized parton distributions

$$z_3 + ct = 0, \quad z_3 - ct \neq 0$$



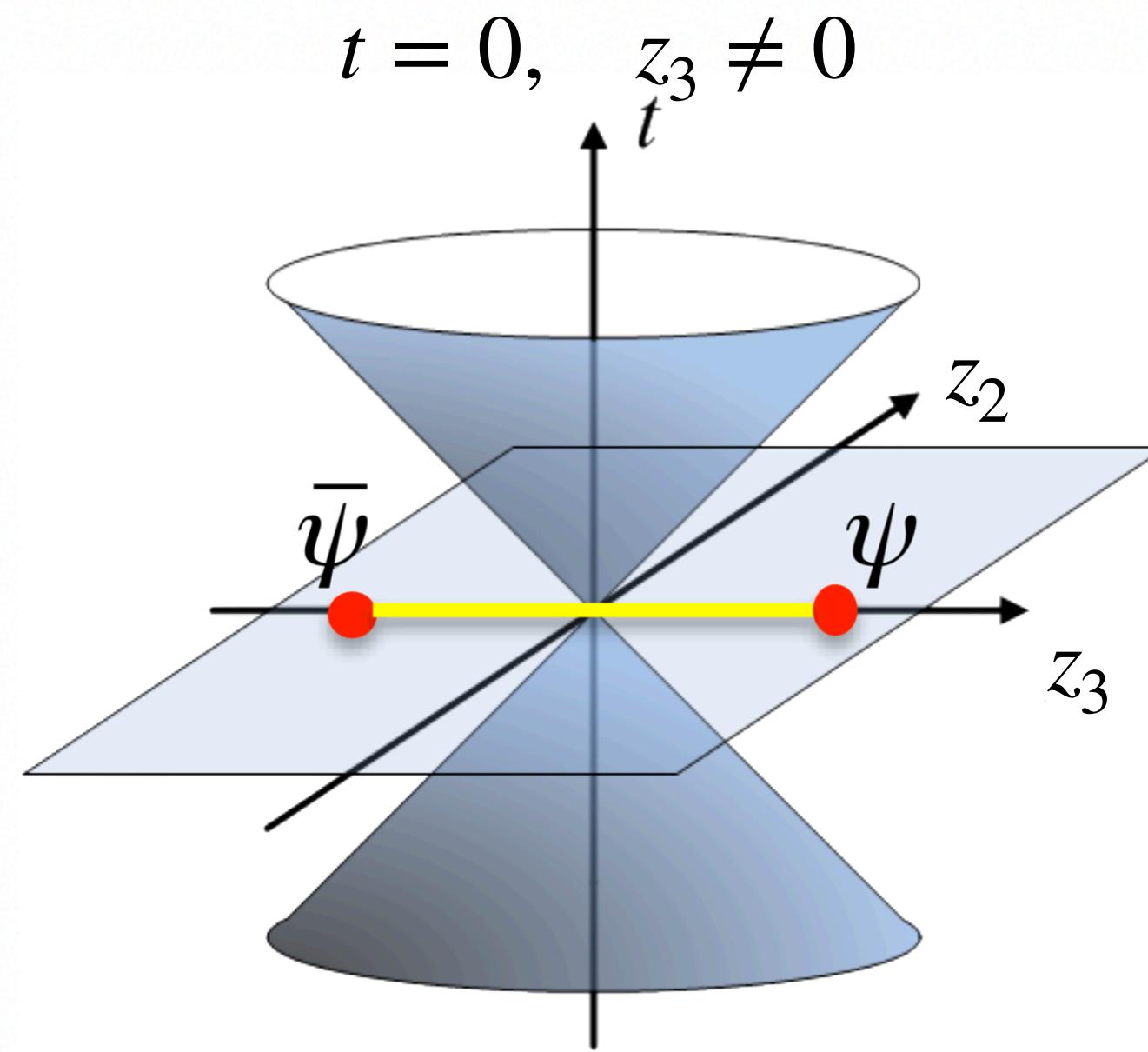
- Mellin or Gegenbauer Moments from leading-twist **local operators**.

$$\bar{q} \gamma^\sigma \overleftrightarrow{D}^{\alpha_1} \dots \overleftrightarrow{D}^{\alpha_n} q$$

ETMC, PRD 101 (2022)  
ETMC, PRD 83 (2011)

Light-cone correlation: Cannot be calculated on the lattice

# Generalized parton distributions



$$\mathcal{F}^\mu(z, P, \Delta)$$

$$= \langle p_f | \bar{q}(-\frac{z}{2}) \gamma^\mu \mathcal{W}(-\frac{z}{2}, \frac{z}{2}) q(\frac{z}{2}) | p_i \rangle$$

$$z = (0, 0, 0, z_3), z^2 = z_3^2$$

- Mellin or Gegenbauer Moments from leading-twist **local operators**.

$$\bar{q} \gamma^\sigma \overleftrightarrow{D}^{\alpha_1} \dots \overleftrightarrow{D}^{\alpha_n} q$$

ETMC, PRD 101 (2022)  
ETMC, PRD 83 (2011)

- **Large-momentum effective theory:**  $x$ -space matching of **quasi-PDF**.

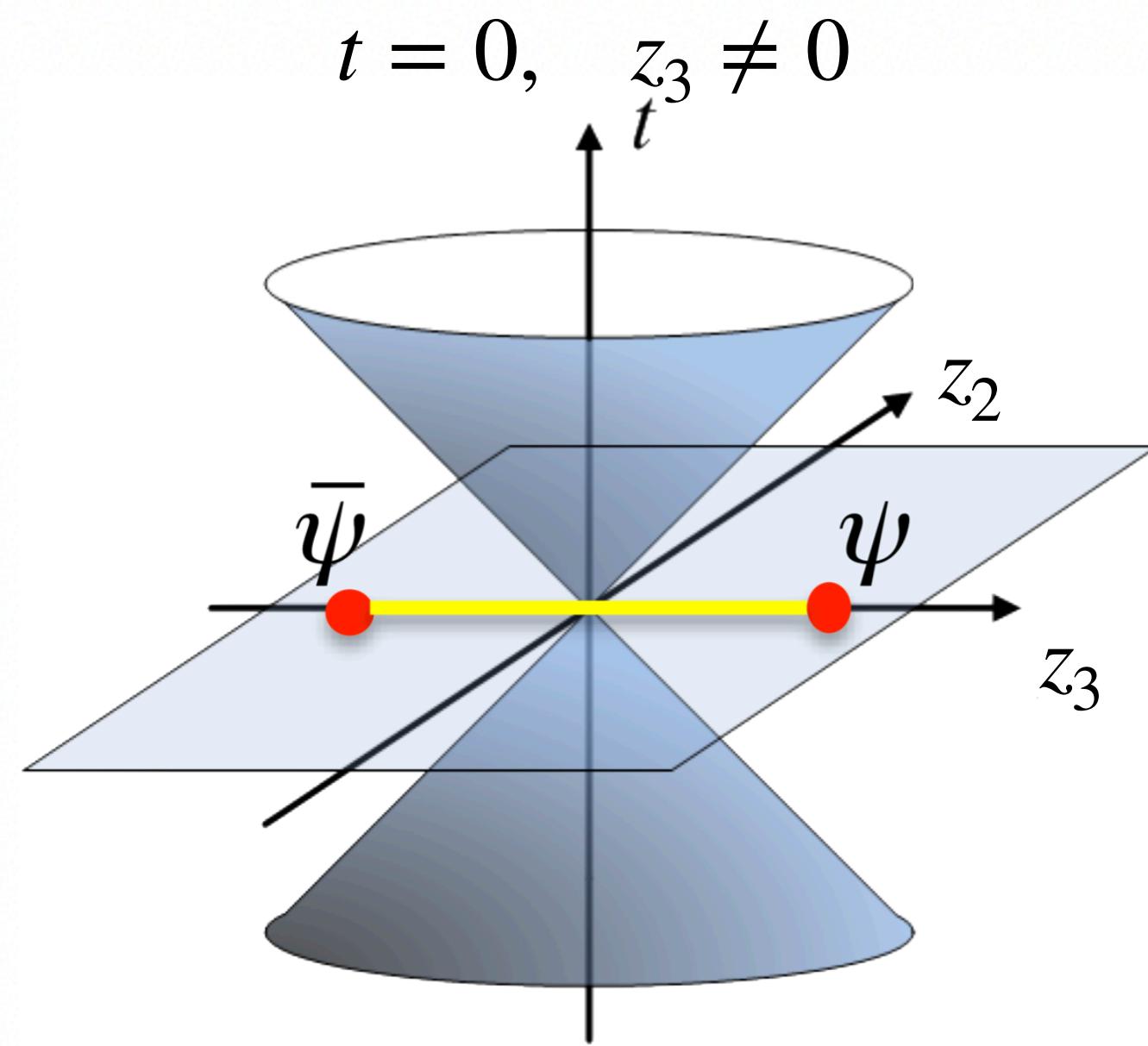
X. Ji, PRL 2013  
X. Ji, et al, RevModPhys 2021

- **Short distance factorization** of the **quasi-PDF matrix elements** in position space or the pseudo-PDF approach.

- A. Radyushkin, PRD 100 (2019)
- A. Radyushkin, Int.J.Mod.Phys.A 2020

● ...

# Short distance factorization



$$\mathcal{F}^\mu(z, P, \Delta)$$

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## SDF of the zero skewness GPD matrix elements:

- V. Braun et al., EPJC 55 (2008)
- A. V. Radyushkin et al., PRD 96 (2017)
- Y. Ma et al., PRL 120 (2018)
- T. Izubuchi et al., PRD 98 (2018)

$$\mathcal{F}^R(z, P, \Delta)$$

$$= \int_{-1}^1 d\alpha \mathcal{C}(\alpha, \mu^2 z^2) \int_{-1}^1 dy e^{-iy\alpha\lambda} F(x, \xi, \Delta, \mu) + \mathcal{O}(z^2 \Lambda_{QCD}^2)$$

$$= \sum_{n=0}^{\infty} \frac{(-izP)^n}{n!} C_n(z^2 \mu^2) \langle x^n \rangle(\mu) + \mathcal{O}(z^2 \Lambda_{QCD}^2)$$

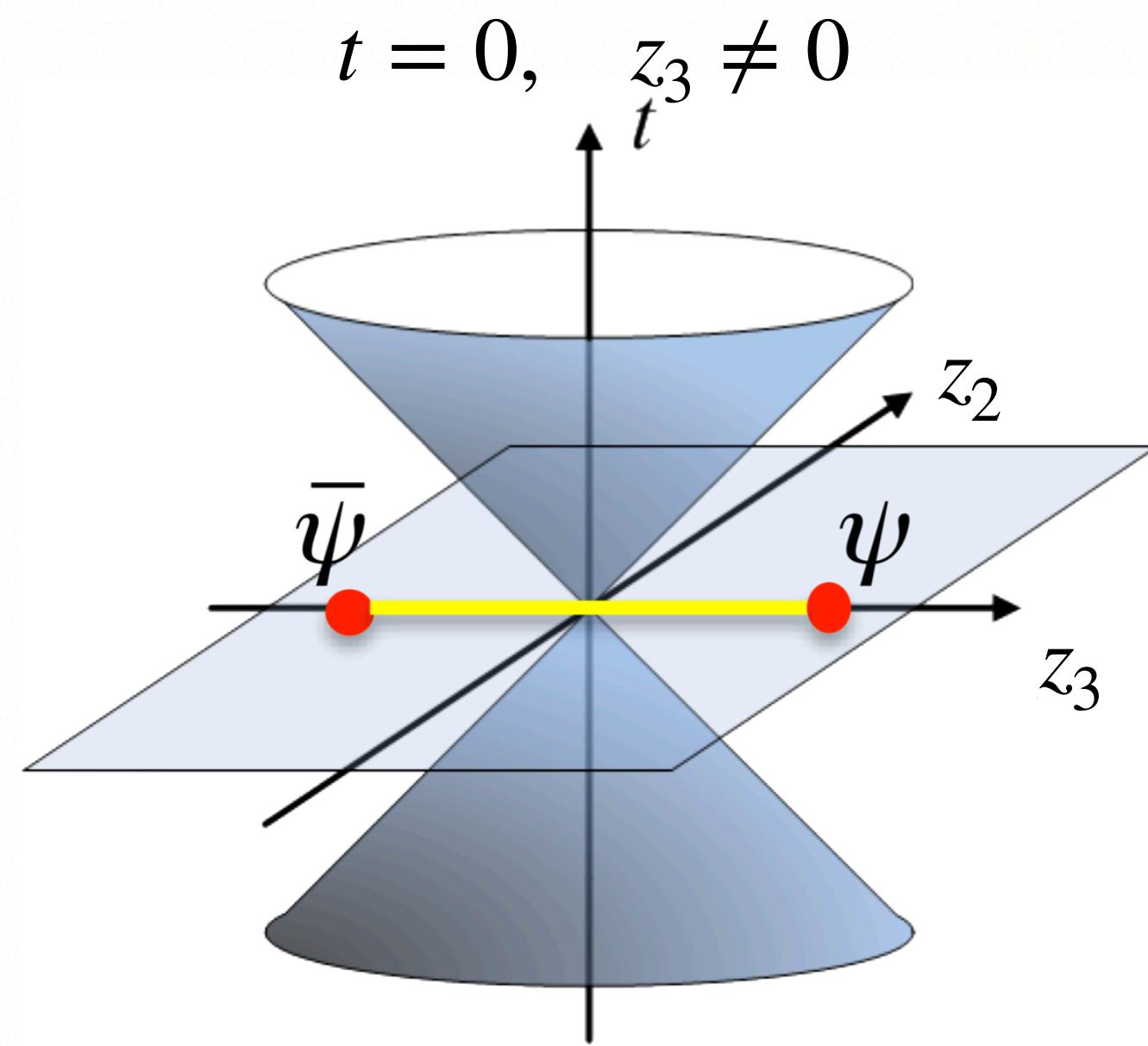
Perturbative matching

$$\lambda = zP$$

$$\int_{-1}^1 dx x^n H^q(x, \xi = 0, t) = A_{n+1,0}^q(t)$$

$$\int_{-1}^1 dx x^n E^q(x, \xi = 0, t) = B_{n+1,0}^q(t)$$

# Short distance factorization



$$\mathcal{F}^\mu(z, P, \Delta)$$

$$= \langle p_f | \bar{q}(-\frac{z}{2}) \gamma^\mu \mathcal{W}(-\frac{z}{2}, \frac{z}{2}) q(\frac{z}{2}) | p_i \rangle$$

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$$\mathcal{F}^R(z, P, \Delta)$$

$$\begin{aligned} &= \int_{-1}^1 d\alpha \mathcal{C}(\alpha, \mu^2 z^2) \int_{-1}^1 dy e^{-iy\alpha\lambda} F(x, \xi, \Delta, \mu) + \mathcal{O}(z^2 \Lambda_{QCD}^2) \\ &= \sum_{n=0}^{\infty} \frac{(-izP)^n}{n!} C_n(z^2 \mu^2) \langle x^n \rangle(\mu) + \mathcal{O}(z^2 \Lambda_{QCD}^2) \end{aligned}$$

Perturbative matching

$$\lambda = zP$$

- The perturbative matching is valid in **short range of  $z_3$** .
- The information that lattice data contains is limited by the range of **finite  $\lambda = zP$** .

# quasi-GPD matrix elements

The matrix elements can be parametrized in terms of linearly-independent Dirac structures:

S. Bhattacharya, et al., PRD 106 (2022)

$$F^\mu(z, P, \Delta) = \bar{u}(p_f, \lambda') \left[ \frac{P^\mu}{m} A_1 + mz^\mu A_2 + \frac{\Delta^\mu}{m} A_3 + im\sigma^{\mu z} A_4 + \frac{i\sigma^{\mu\Delta}}{m} A_5 + \frac{P^\mu i\sigma^{z\Delta}}{m} A_6 + mz^\mu i\sigma^{z\Delta} A_7 + \frac{\Delta^\mu i\sigma^{z\Delta}}{m} A_8 \right] u(p_i, \lambda) A_i(z \cdot P, z \cdot \Delta, \Delta^2, z^2)$$

## light-cone GPDs $H$ and $E$

$$F^+(z, P, \Delta) = \bar{u}(p_f, \lambda') \left[ \gamma^+ H(z, P, \Delta) + \frac{i\sigma^{+\mu} \Delta_\mu}{2m} E(z, P, \Delta) \right] u(p_i, \lambda)$$

$$H(z, P, \Delta) = A_1 + \frac{\Delta^+}{P^+} A_3$$

$$E(z, P, \Delta) = -A_1 - \frac{\Delta^+}{P^+} A_3 + 2A_5 + 2P^+ z^- A_6 + 2\Delta^+ z^- A_8$$

## Commonly used quasi-GPD matrix elements

$$\begin{aligned} \mathcal{F}^0(z, P, \Delta) &= \bar{u}(p_f, \lambda') \left[ \gamma^0 \mathcal{H}_0(z, P, \Delta) + \frac{i\sigma^{0\mu} \Delta_\mu}{2m} \mathcal{E}_0(z, P, \Delta) \right] u(p_i, \lambda) \\ \mathcal{H}_0^s(z, P^s, \Delta^s) &= A_1 + \frac{\Delta^{0,s}}{P^{0,s}} A_3 - \frac{m^2 \Delta^{0,s} z^3}{2P^{0,s} P^{3,s}} A_4 + \left[ \frac{(\Delta^{0,s})^2 z^3}{2P^{3,s}} - \frac{\Delta^{0,s} \Delta^{3,s} z^3 P^{0,s}}{2(P^{3,s})^2} - \frac{z^3 (\Delta_\perp^s)^2}{2P^{3,s}} \right] A_6 \\ &\quad + \left[ \frac{(\Delta^{0,s})^3 z^3}{2P^{0,s} P^{3,s}} - \frac{(\Delta^{0,s})^2 \Delta^{3,s} z^3}{2(P^{3,s})^2} - \frac{\Delta^{0,s} z^3 (\Delta_\perp^s)^2}{2P^{0,s} P^{3,s}} \right] A_8 \end{aligned}$$

...

See M. Constantinou's talk for more details.

- Lorentz invariant, frame independent

$$\bullet \frac{\Delta^+}{P^+} = \frac{\Delta \cdot z}{P \cdot z}, z^2 = 0$$

- Encouraging results were reported

- Frame dependent
- Computational expensive for multiple  $Q^2$

# quasi-GPD matrix elements

## light-cone GPDs $H$ and $E$

$$H(z, P, \Delta) = A_1 + \frac{\Delta^+}{P^+} A_3$$

$$E(z, P, \Delta) = -A_1 - \frac{\Delta^+}{P^+} A_3 + 2A_5 + 2P^+ z^- A_6 + 2\Delta^+ z^- A_8$$

- Lorentz invariant, frame independent
- $\frac{\Delta^+}{P^+} = \frac{\Delta \cdot z}{P \cdot z}$ ,  $A_i(z \cdot P, z \cdot \Delta, \Delta^2, z^2)$ ,  $z^2 = 0$

## Lorentz invariant quasi-GPD matrix elements

S. Bhattacharya, et al., PRD 106 (2022)

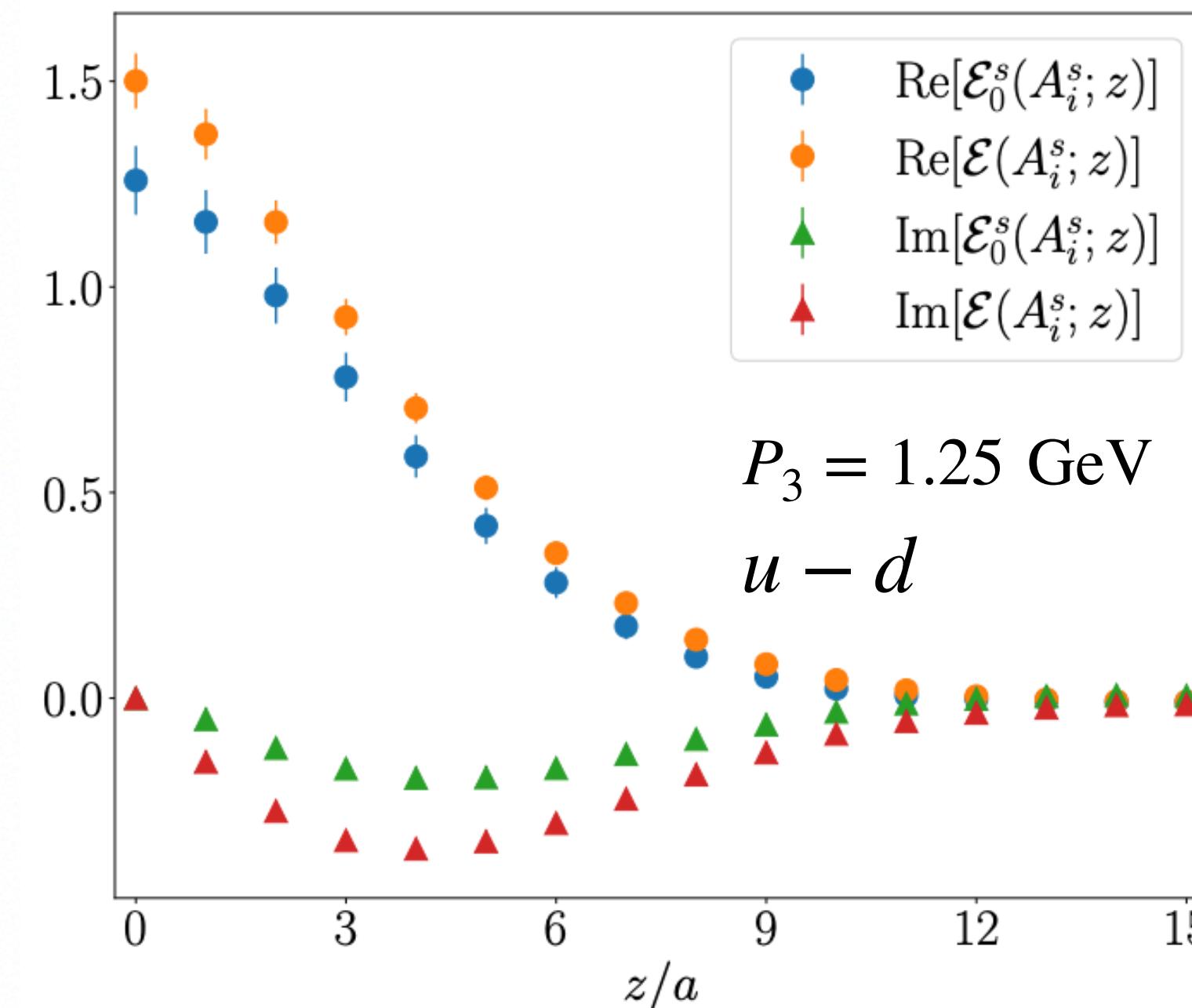
$$\mathcal{H}(z, P, \Delta) = A_1 + \frac{\Delta \cdot z}{P \cdot z} A_3$$

$$\mathcal{E}(z, P, \Delta) = -A_1 - \frac{\Delta \cdot z}{P \cdot z} A_3 + 2A_5 + 2P \cdot z A_6 + 2\Delta \cdot z A_8$$

- Lorentz invariant, frame independent
- $A_i(z \cdot P, z \cdot \Delta, \Delta^2, z^2)$ ,  $z^2 \neq 0$
- $A_i$  can be solve from matrix elements of  $\mathcal{F}^0, \mathcal{F}^1, \mathcal{F}^2$

# Bare matrix elements and renormalization

## Bare matrix elements of quasi-GPD $E$



- Nf=2+1+1 twisted mass (TM) fermions & clover improvement.
- $m_\pi = 260 \text{ MeV}$ ,  $a = 0.093 \text{ fm}$ ,  $32^3 \times 64$
- iso-vector (u-d) and iso-scalar (u+d), connected diagrams only.

The operator can be **multiplicatively renormalized**

- X. Ji, J. H. Zhang and Y. Zhao, PRL120.112001
- J. Green, K. Jansen and F. Steffens, PRL.121.022004

$$[\bar{q}(-\frac{z}{2})\gamma^\mu \mathcal{W}(-\frac{z}{2}, \frac{z}{2})q(\frac{z}{2})]_B \\ = e^{-\delta m(a)|z|} Z(a) [\bar{q}(-\frac{z}{2})\gamma^\mu \mathcal{W}(-\frac{z}{2}, \frac{z}{2})q(\frac{z}{2})]_R$$

$$\delta m = m_{-1}/a + m_0$$

### • Ratio scheme renormalization

- A. V. Radyushkin et al., PRD 96 (2017)
- BNL, PRD 102 (2020)

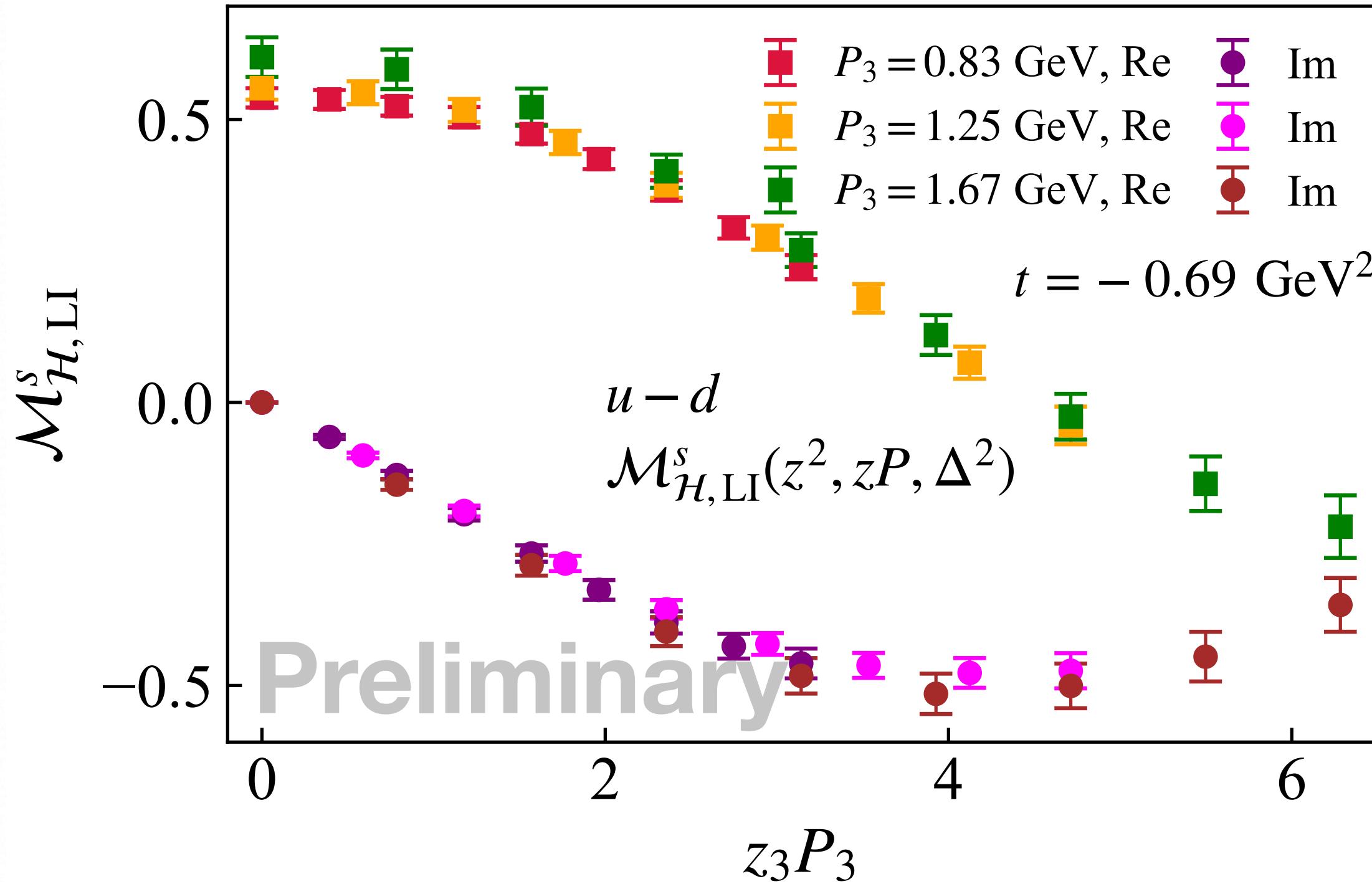
$$\mathcal{M}(z^2, zP, \Delta^2) = \frac{F^R(z, P, \Delta; \mu)}{F^R(z, P = 0, \Delta = 0; \mu)} = \frac{F^B(z, P, \Delta; a)}{F^B(z, P = 0, \Delta = 0; a)}$$

$$= \sum_{n=0}^{\infty} \frac{(-izP)^n}{n!} \frac{C_n(z^2\mu^2)}{C_n(z^2\mu^2)} \langle x^n \rangle(\Delta^2; \mu) + \mathcal{O}(z^2\Lambda_{\text{QCD}}^2)$$

Reduce to the standard Ioffe-time pseudo-distribution when  $\Delta = 0$

# Ratio scheme renormalization

## Ratio-scheme matrix elements for $H$

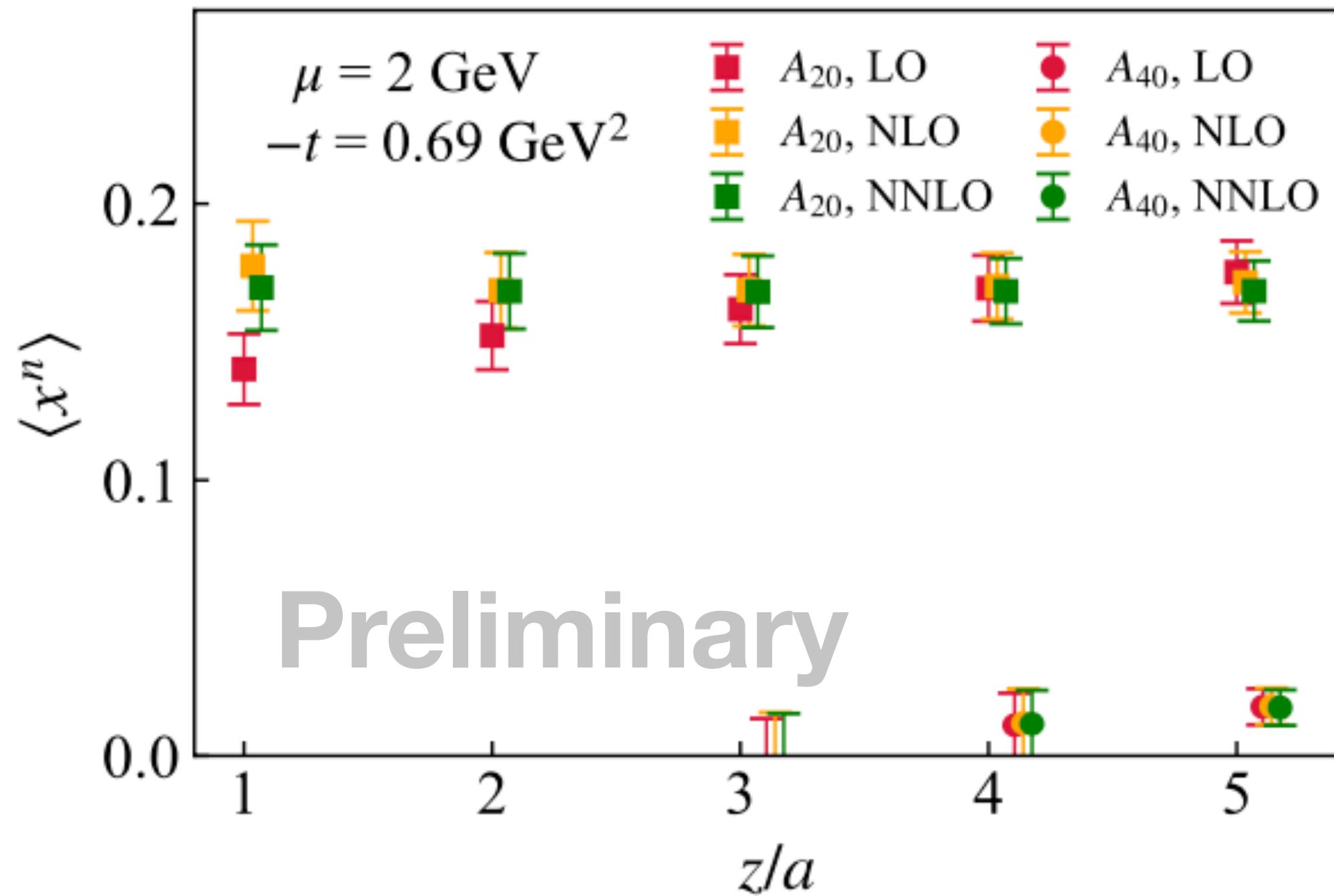


- At tree level ( $\alpha_s = 0, C_n(\mu^2 z^2) = 1$ ) approximation, simply a **polynomial function of  $zP$** .
- **Beyond LO**, the perturbative kernels  $C_n(z^2 \mu^2)/C_0(z^2 \mu^2)$  are supposed to describe the  $z$ -dependent evolution.
- Wilson-coefficients available up to **NNLO** for iso-vector case, while **NLO** for iso-scalar case.

$$\mathcal{M}(z^2, zP, \Delta^2) = \sum_{n=0}^{\infty} \frac{(-izP)^n}{n!} \frac{C_n(z^2 \mu^2)}{C_0(z^2 \mu^2)} \langle x^n \rangle(\mu) + \mathcal{O}(z^2 \Lambda_{\text{QCD}}^2)$$

# Mellin moments of GPDs

$$\mathcal{M}(z^2, zP, \Delta^2) = \sum_{n=0}^{\infty} \frac{(-izP)^n}{n!} \frac{C_n(z^2\mu^2)}{C_n(z^2\mu^2)} \langle x^n \rangle + \mathcal{O}(z^2\Lambda_{\text{QCD}}^2)$$

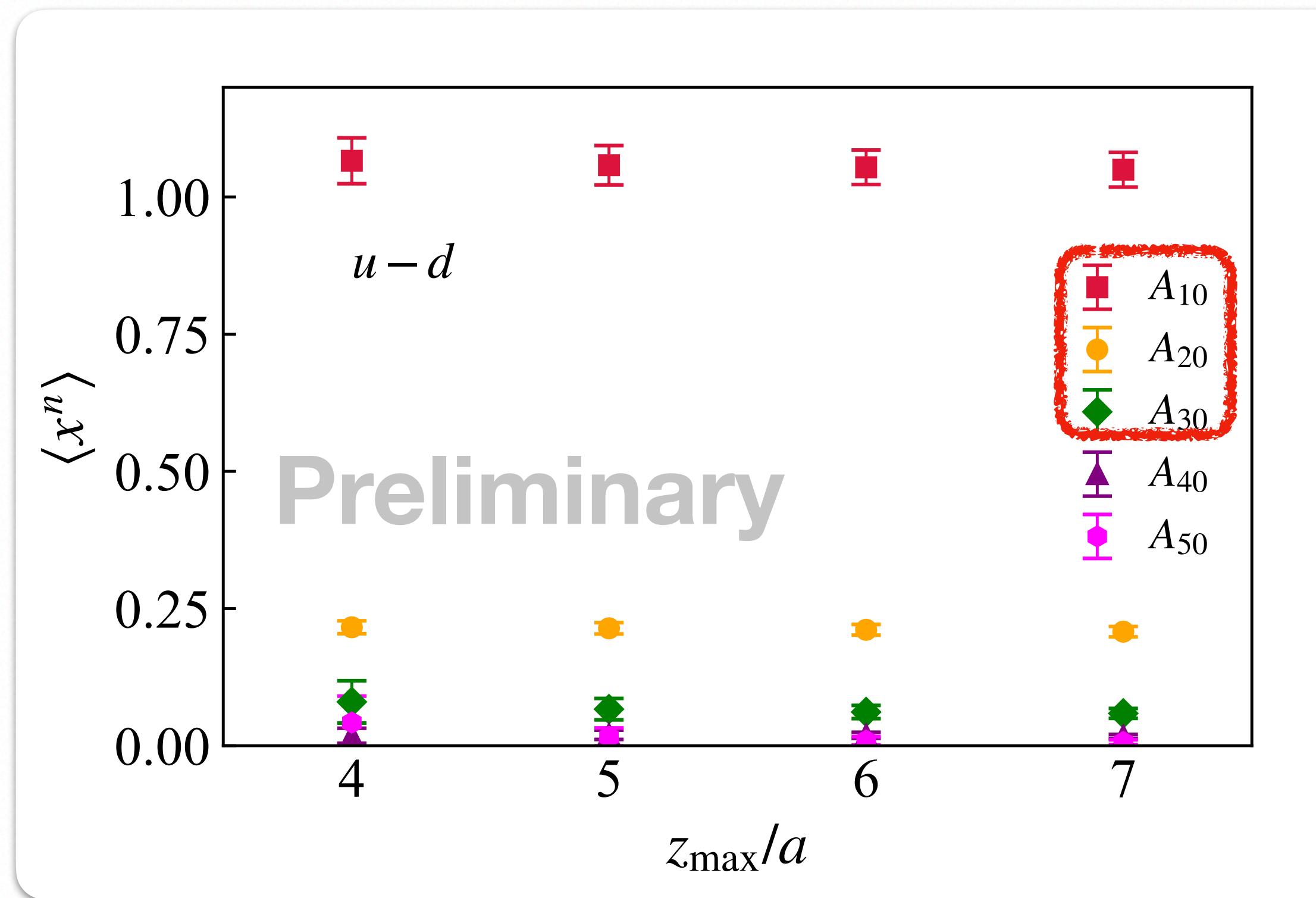


Odd moments  $\langle x \rangle$  and  $\langle x^3 \rangle$  (or  $A_{20}$  and  $A_{40}$ )  
extracted from imaginary part of matrix  
element at each  $z$  by fitting  $P$  dependence.

- The tree-level ( $\alpha_s = 0$ ) result show mild  $z$  dependence.
- Beyond LO, the  $z$  dependence is compensated by the Wilson coefficients and produce the  $z$ -independent plateau.
- NNLO produce similar results with NLO within current statistical errors.
- Signal for higher moments is weak: requiring higher momentum and statistics.

# Mellin moments of GPDs

$$\mathcal{M}(z^2, zP, \Delta^2) = \sum_{n=0}^{\infty} \frac{(-izP)^n}{n!} \frac{C_n(z^2\mu^2)}{C_n(z^2\mu^2)} \langle x^n \rangle + \mathcal{O}(z^2\Lambda_{\text{QCD}}^2)$$



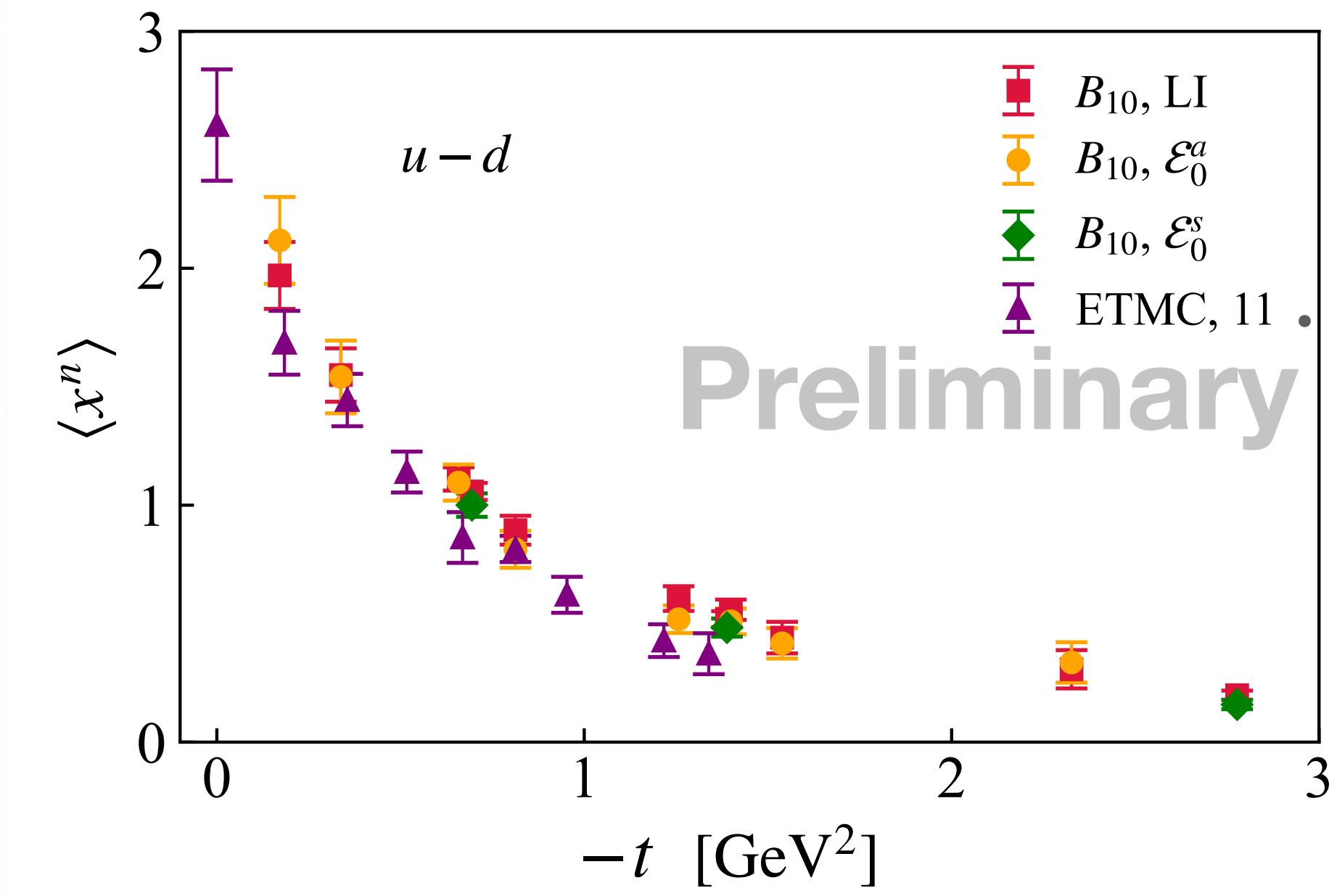
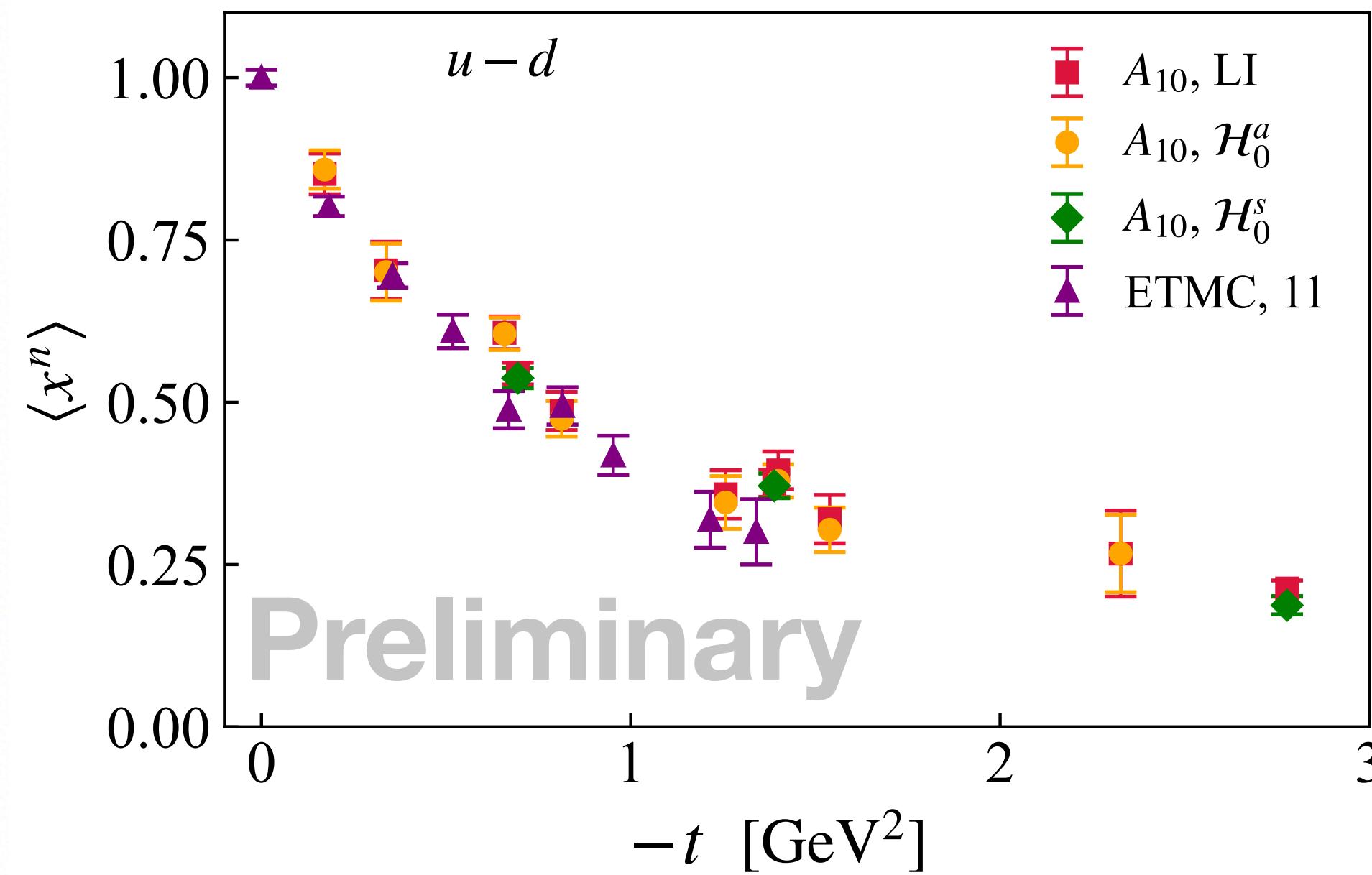
Combine  $z$  fit to stabilize the fit using

$$z \in [z_{\min}, z_{\max}]$$

Latt. artifact?      Higher twist?

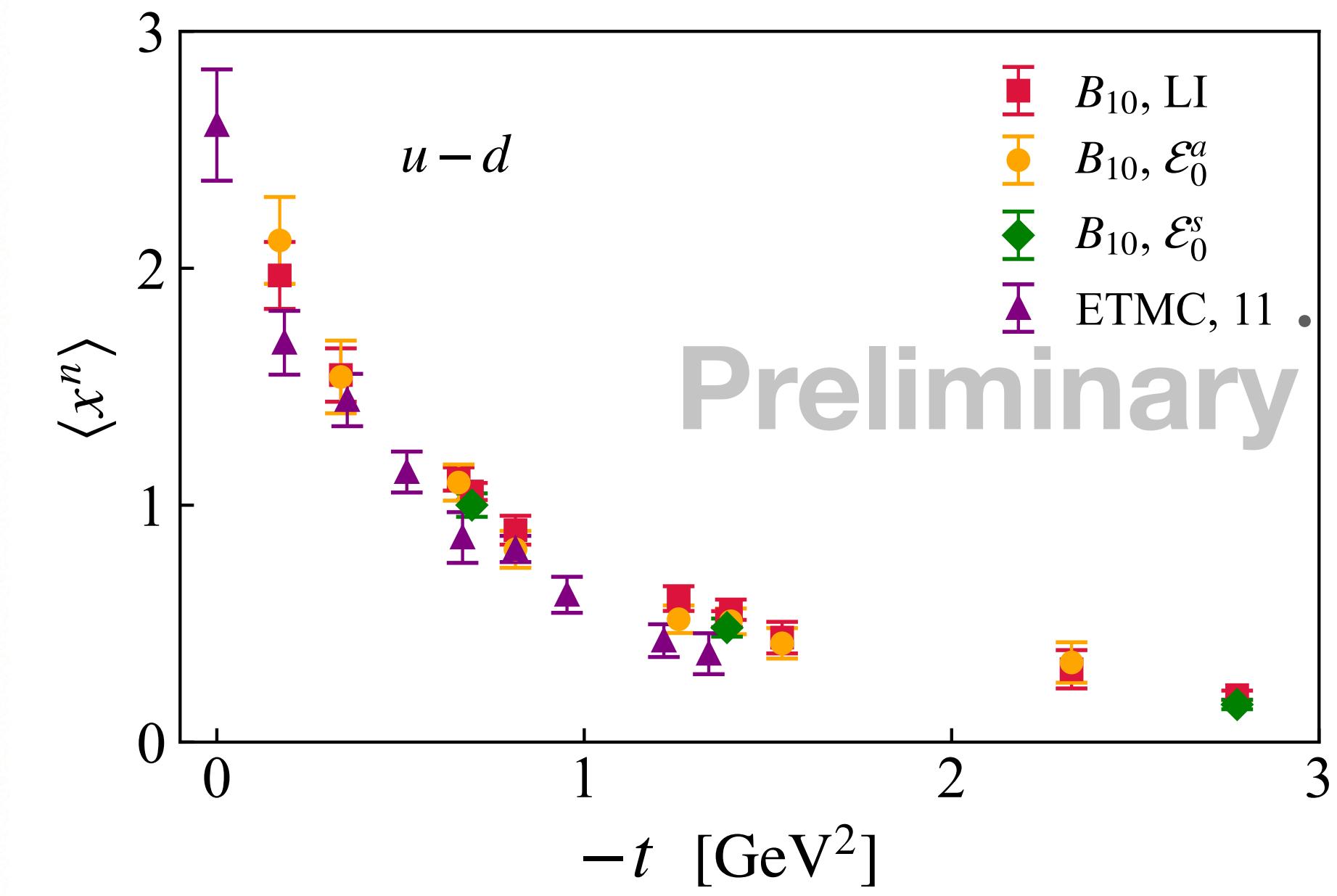
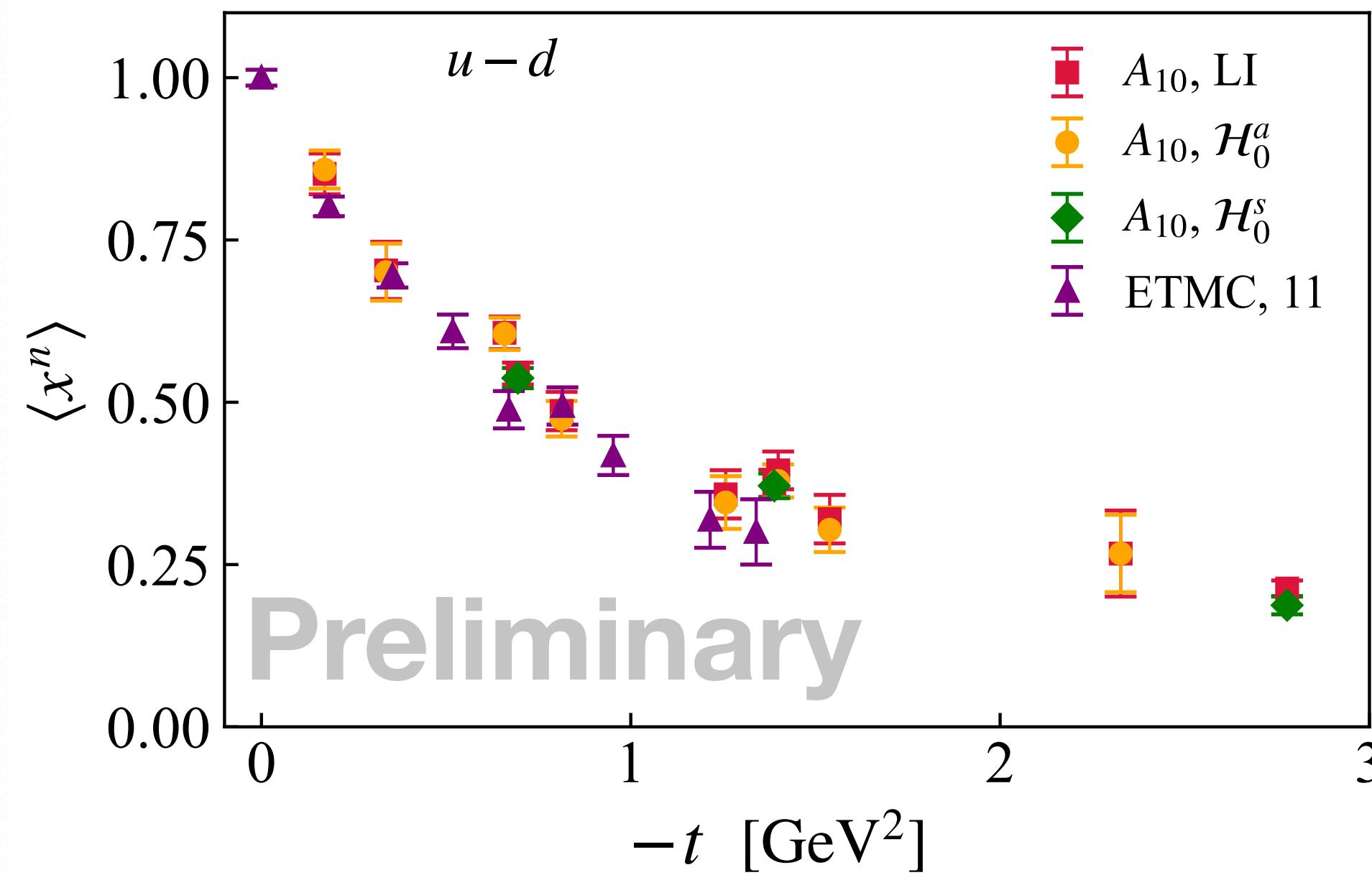
- We skipped  $z = a$  ( $z_{\min} = 2a$ ) to avoid the most serious discretization effect and vary  $z_{\max}$  to estimate the systematic errors.
- Reasonable signal up to  $\langle x^2 \rangle$  or  $A_{30}$ , higher momentum and statistics are needed to constrain higher moments.

# $t$ -dependence of moments



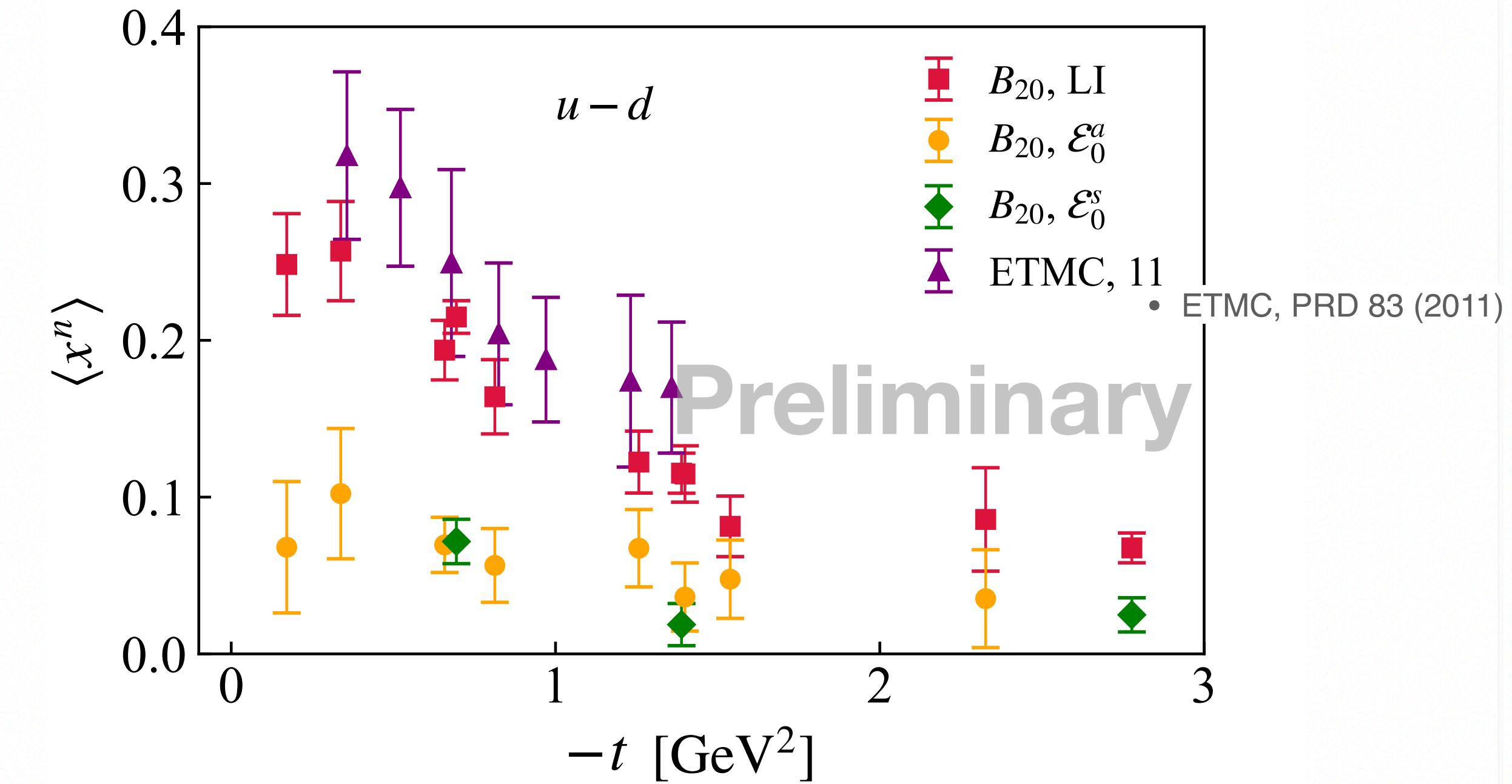
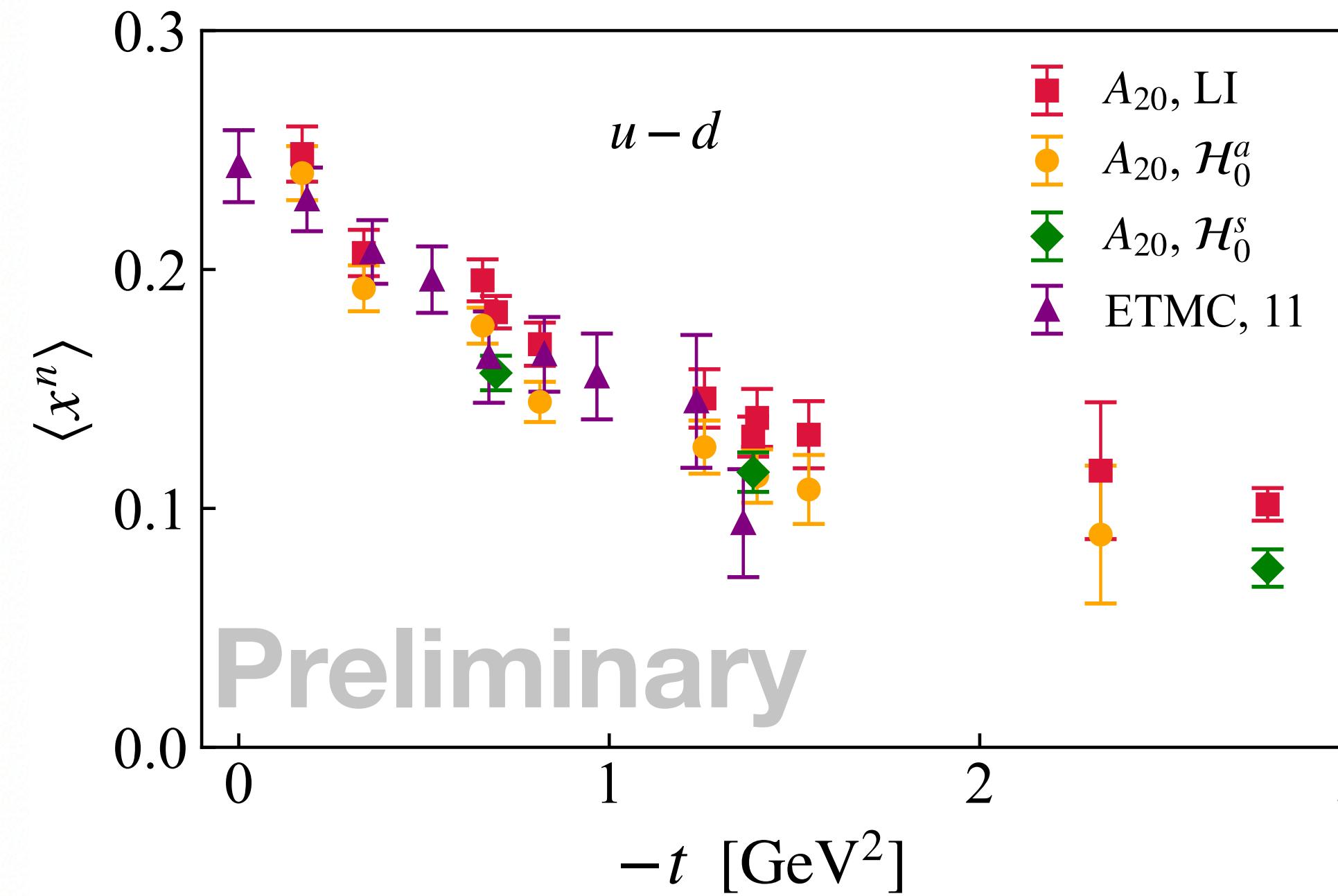
- LI: Lorentz invariant definition.
- $\mathcal{H}_0^a$  and  $\mathcal{E}_0^a$  are  $\gamma_0$  definition in asymmetric frame:  $p_f = P$ ,  $p_i = P - \Delta$ .
- $\mathcal{H}_0^s$  and  $\mathcal{E}_0^s$  are  $\gamma_0$  definition in symmetric frame:  $p_f = P + \Delta/2$ ,  $p_i = P - \Delta/2$ .

# $t$ -dependence of moments



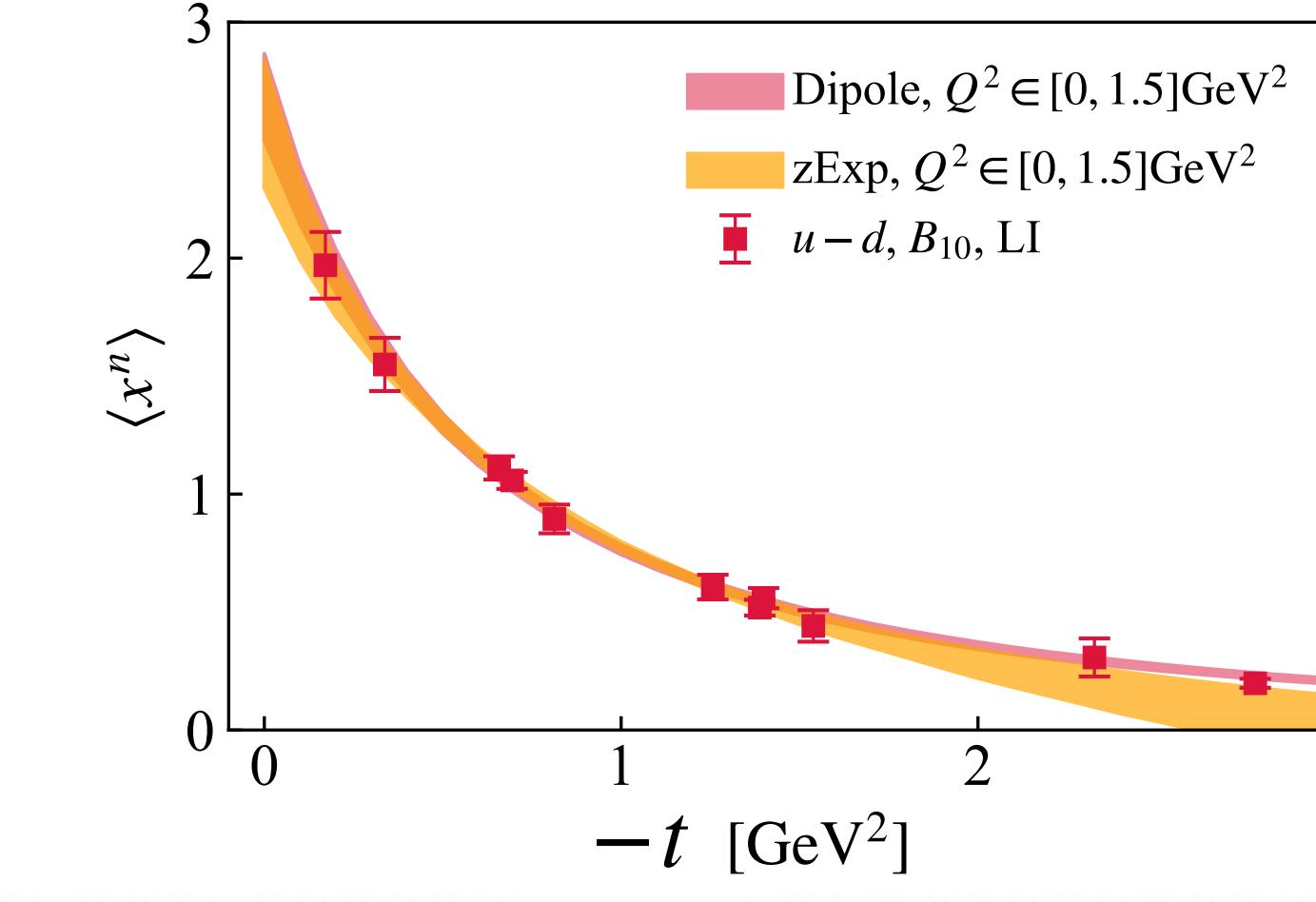
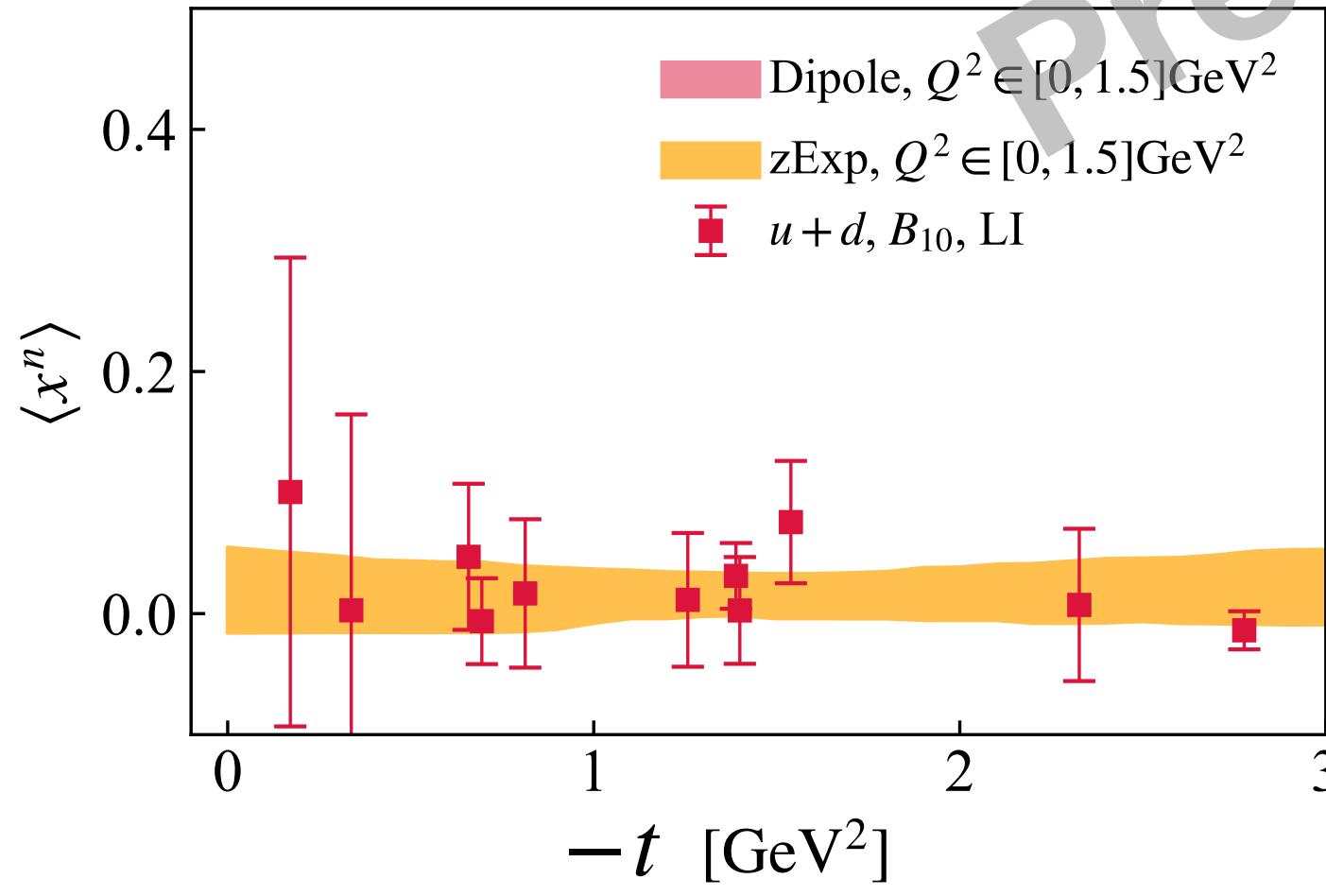
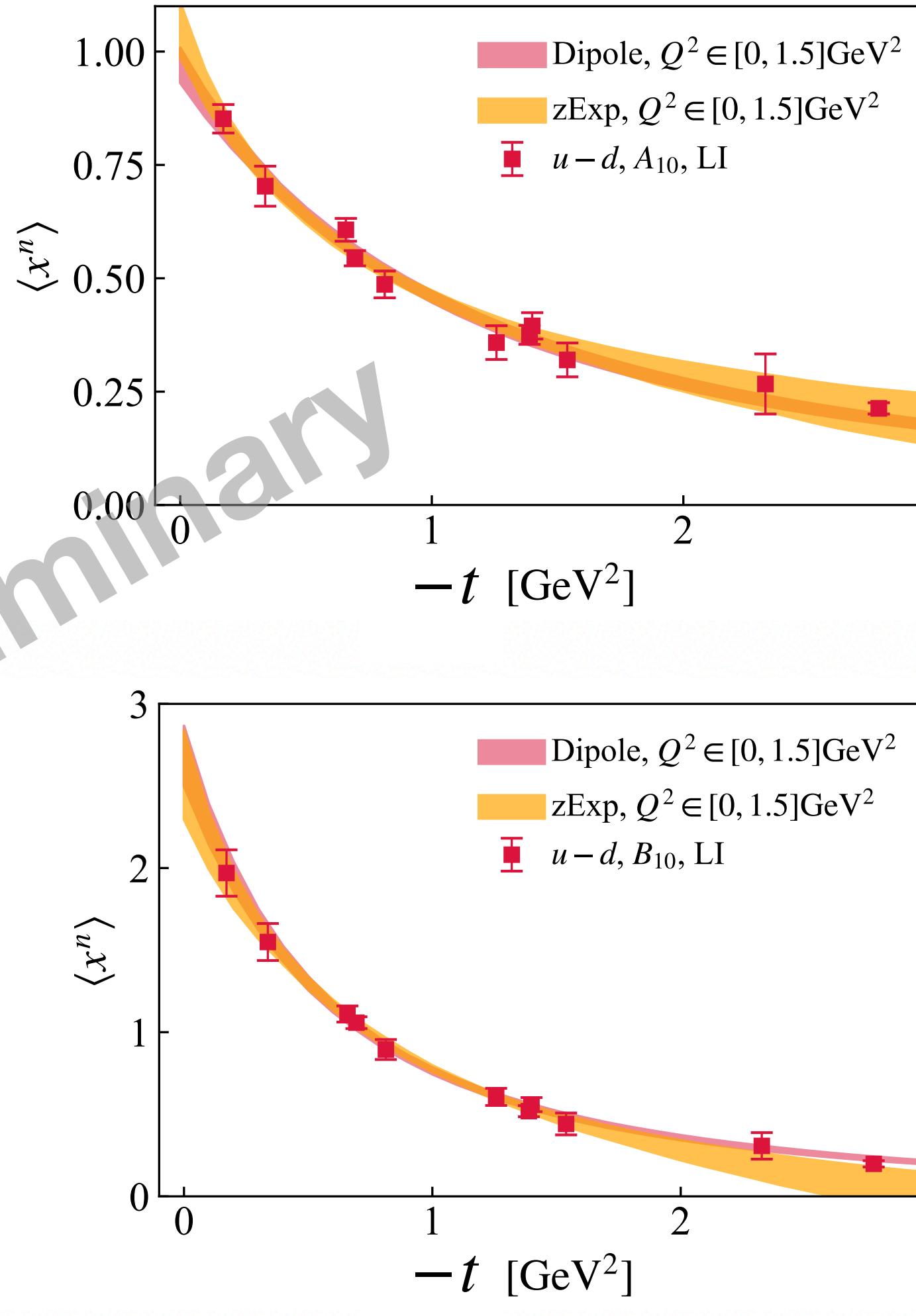
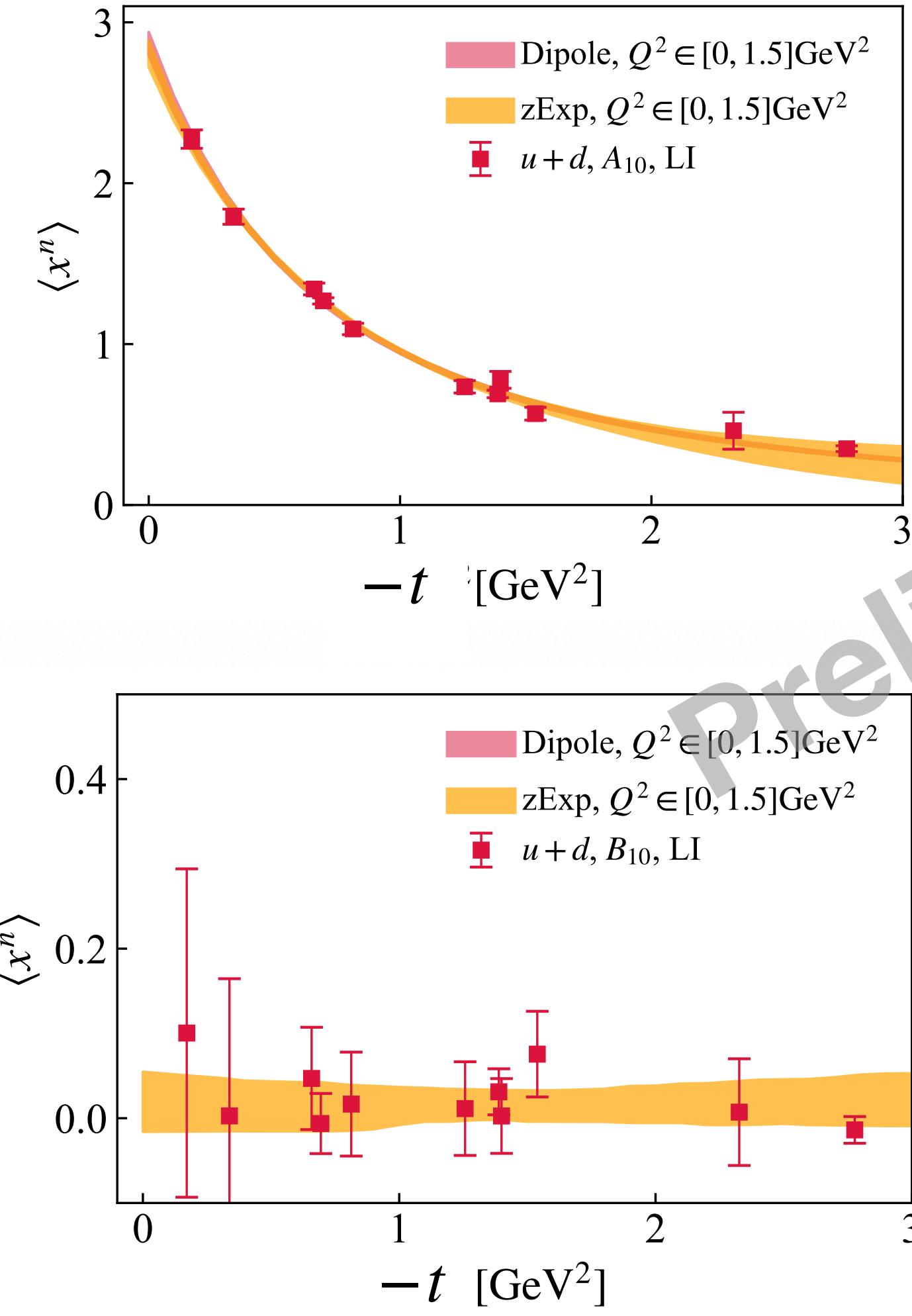
- The  $A_{10}$  and  $B_{10}$  are Dirac and Pauli form factors.
- No difference between different definition.
- Good agreement with literatures using similar lattice setup.

# $t$ -dependence of moments



- The  $A_{20}$  and  $B_{20}$  are gravitational form factors.
- Results from different definitions are inconsistent especially for  $B_{20}$ .
- The Lorentz invariant (LI) definition have better agreement with the traditional moments calculation.

# $t$ -dependence of moments



Di-pole fit:

$$\langle x^n \rangle(Q^2) = \frac{\langle x^n \rangle(0)}{(1 + \frac{Q^2}{M^2})^2}$$

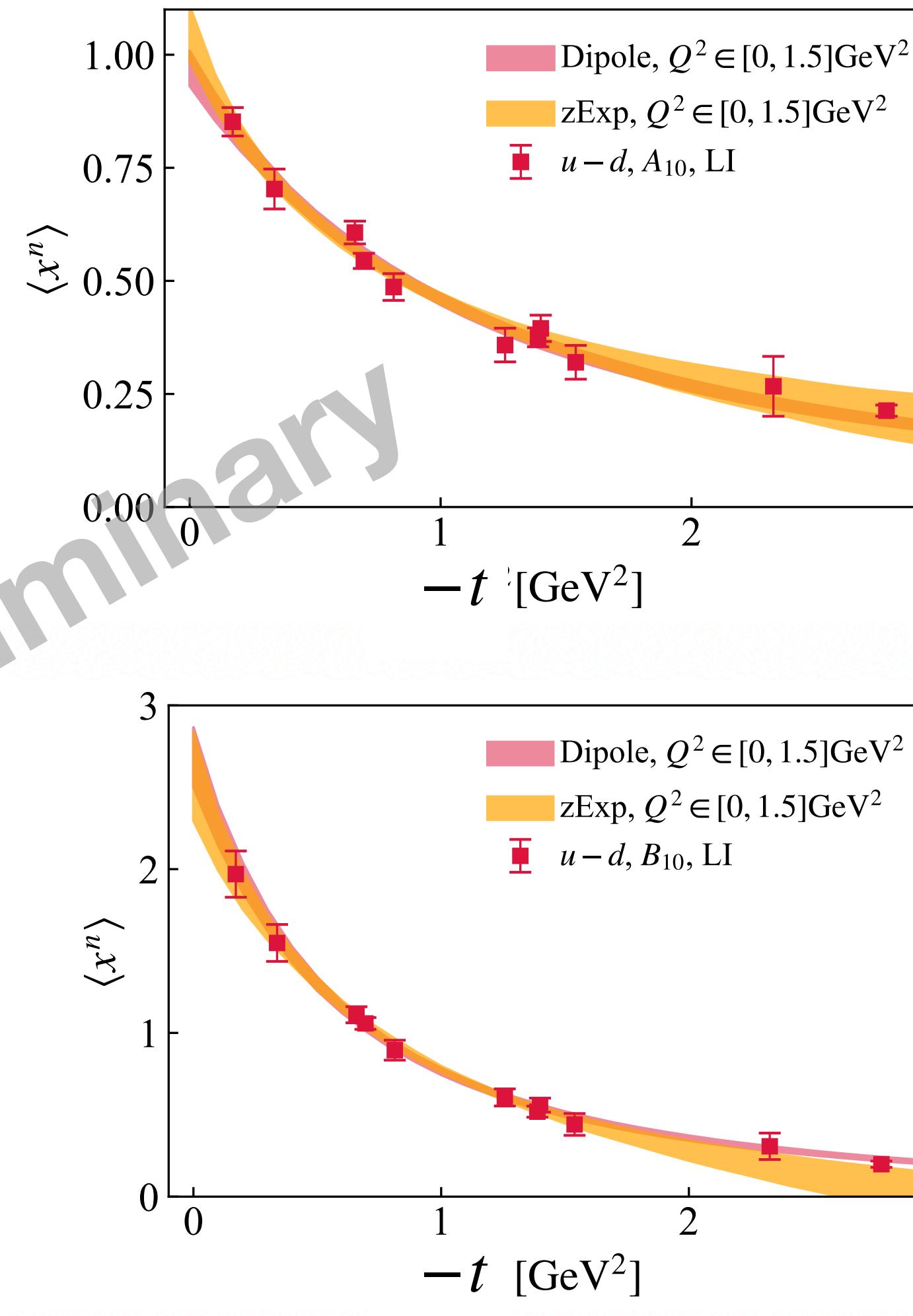
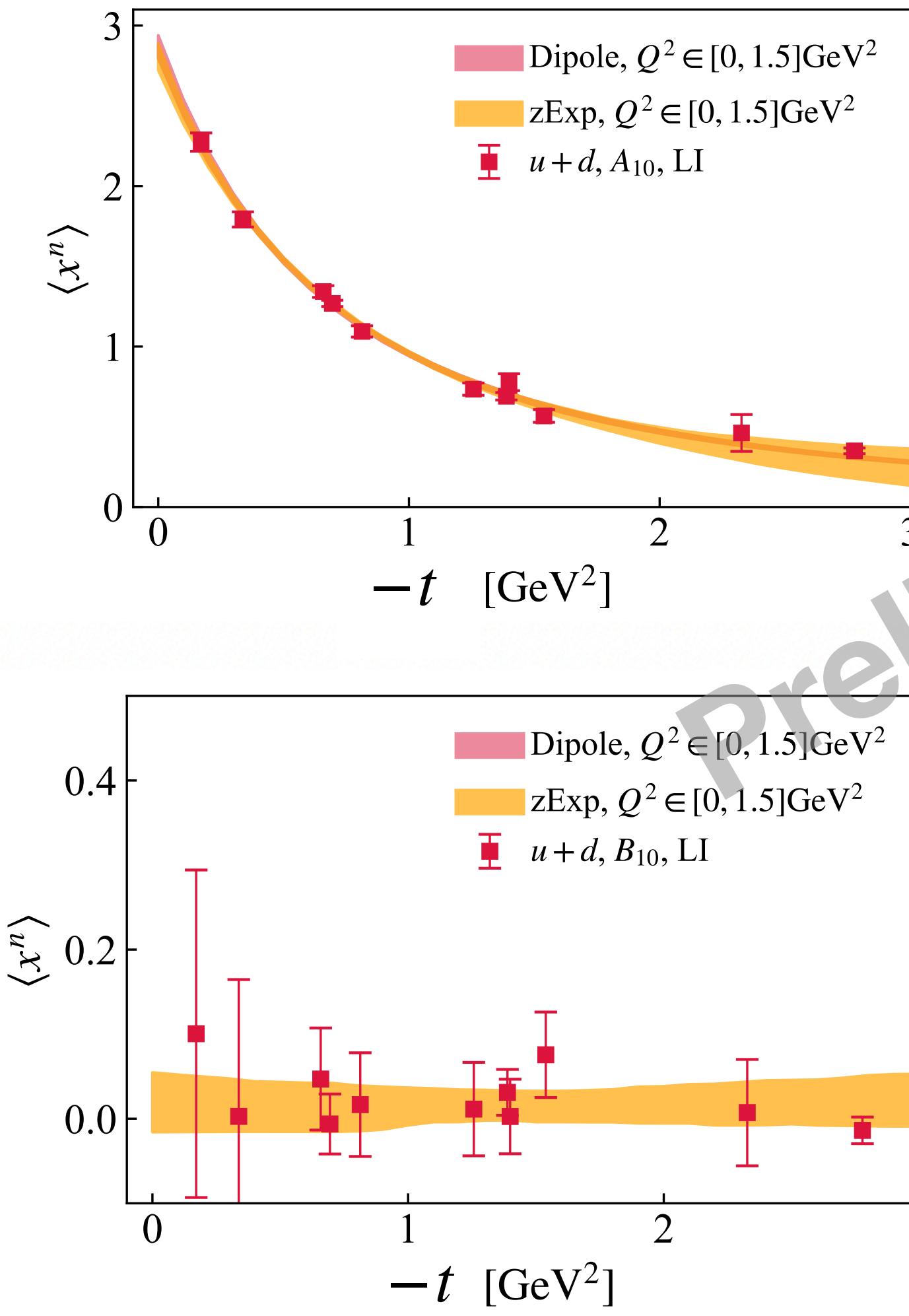
$z$ -expansion fit:

$$\langle x^n \rangle(Q^2) = \sum_{k=0}^{k_{\max}} a_k z(Q^2)^k$$

$$z(Q^2) = \frac{\sqrt{t_{\text{cut}} + Q^2} - \sqrt{t_{\text{cut}} - t_0}}{\sqrt{t_{\text{cut}} + Q^2} + \sqrt{t_{\text{cut}} - t_0}}$$

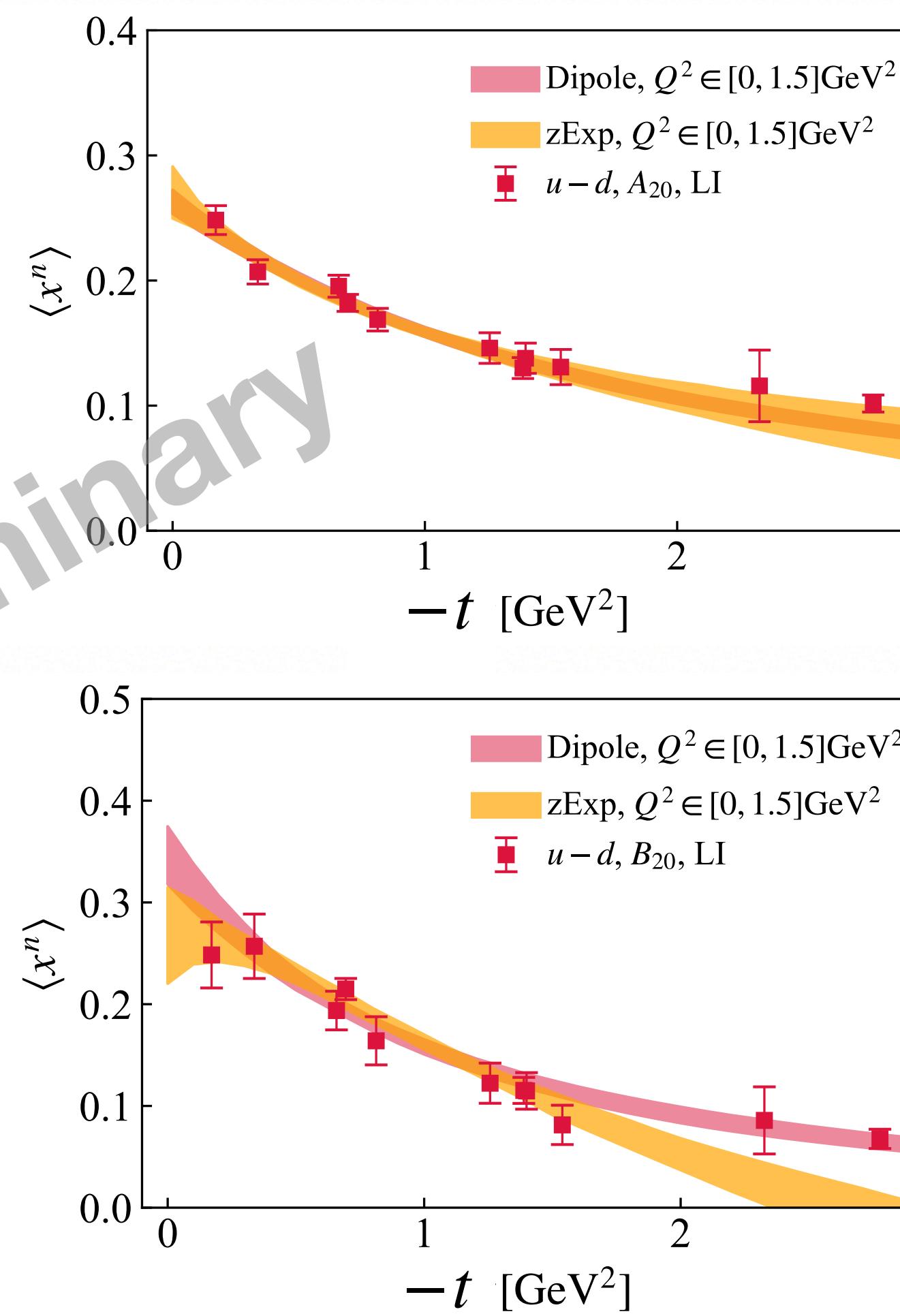
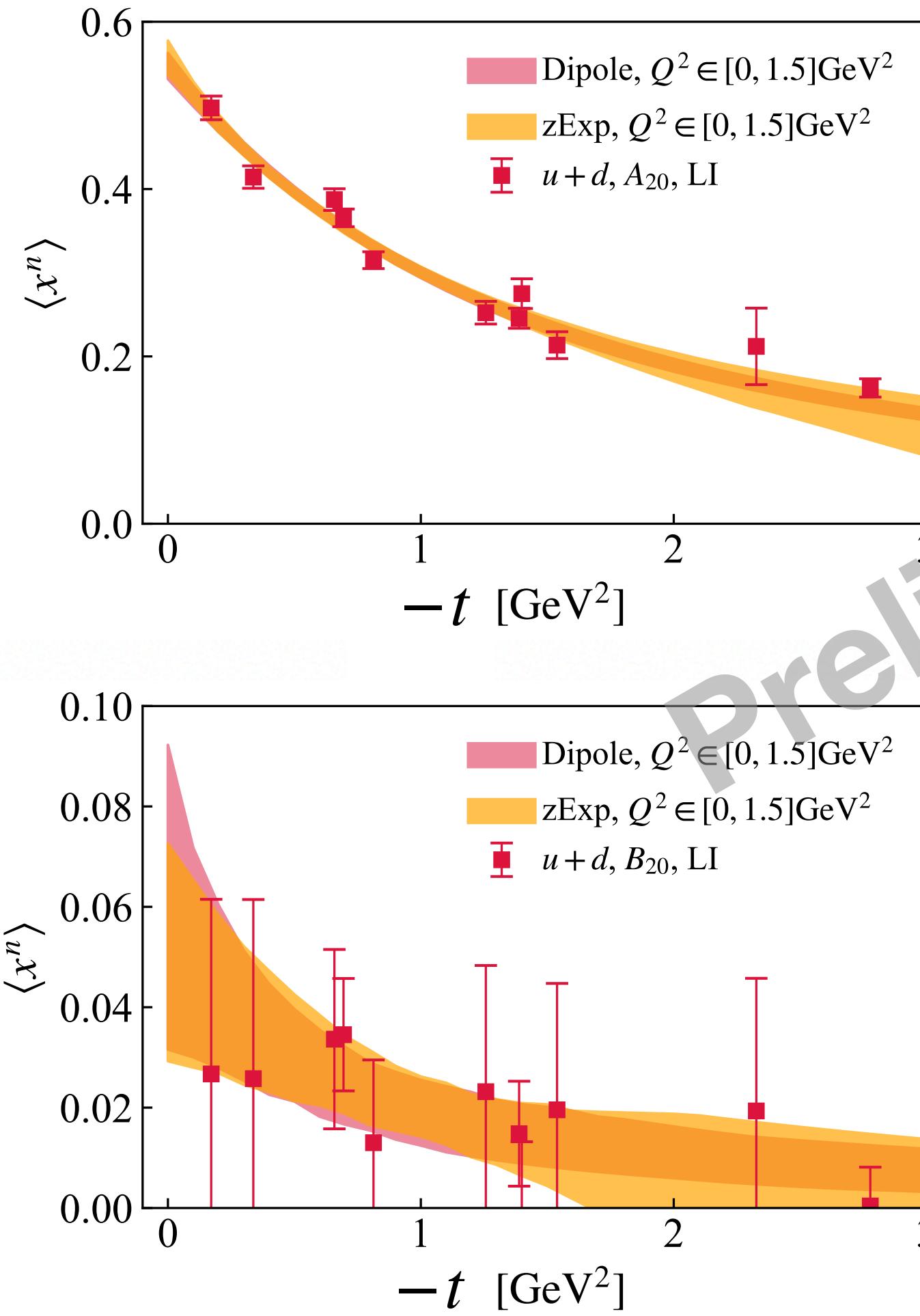
$$-t = Q^2$$

# $t$ -dependence of moments



$-t \rightarrow 0$	Di-pole	$z$ -expansion	ETMC'11
$A_{10}^{u-d}$	0.97(04)	1.05(06)	1
$A_{10}^{u+d}$	2.87(07)	2.80(08)	—
$B_{10}^{u-d}$	2.68(18)	2.56(27)	2.61(23)
$B_{10}^{u+d}$	0.38(38)	0.19(36)	—

# $t$ -dependence of moments



$-t \rightarrow 0$	Di-pole	$z$ -expansion	ETMC'11
$A_{20}^{u-d}$	0.263(10)	0.270(21)	0.264(13)
$A_{20}^{u+d}$	0.547(16)	0.557(21)	0.613(14)
$B_{20}^{u-d}$	0.346(28)	0.267(47)	0.301(47)
$B_{20}^{u+d}$	0.065(29)	0.050(22)	-0.046(43)

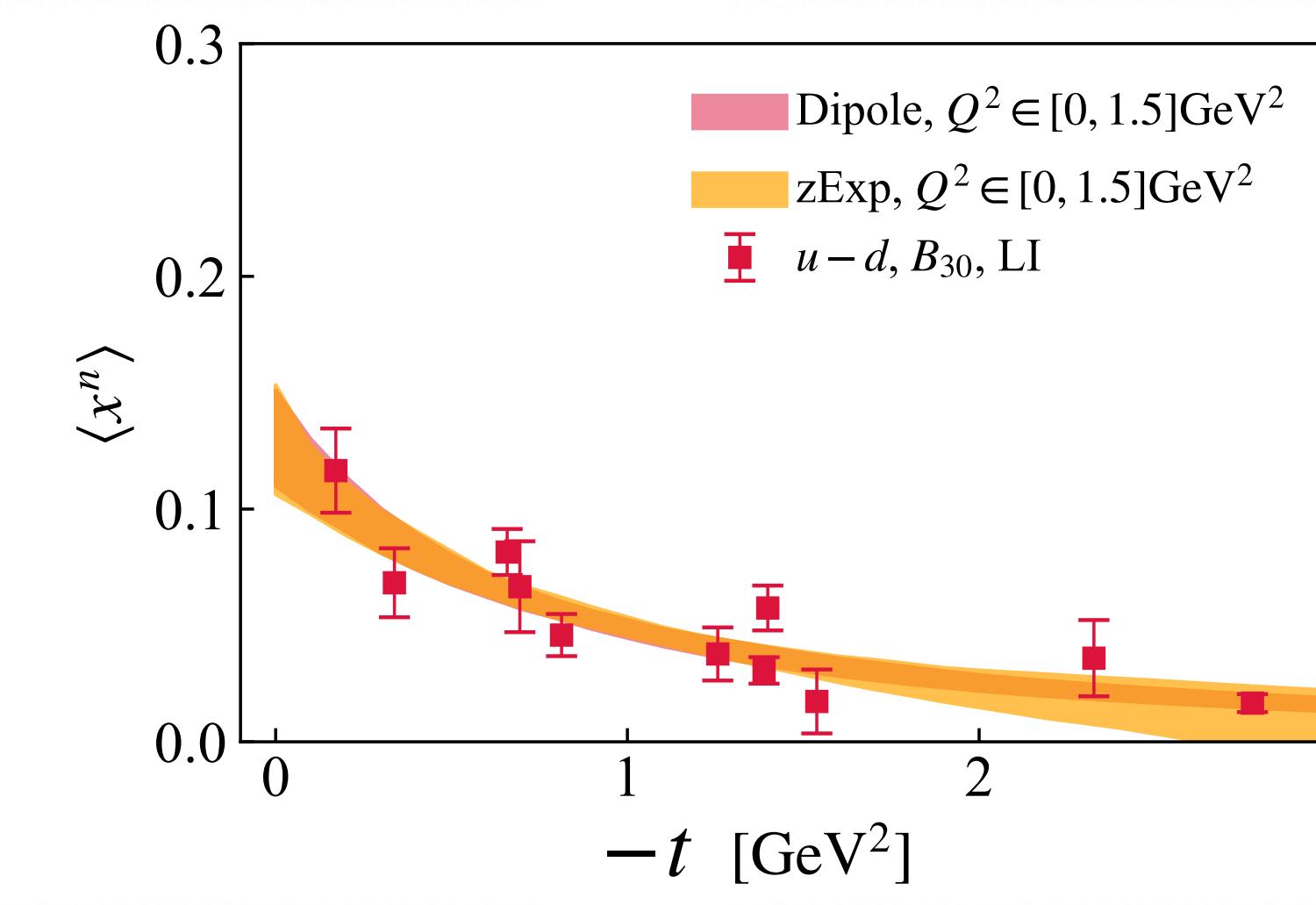
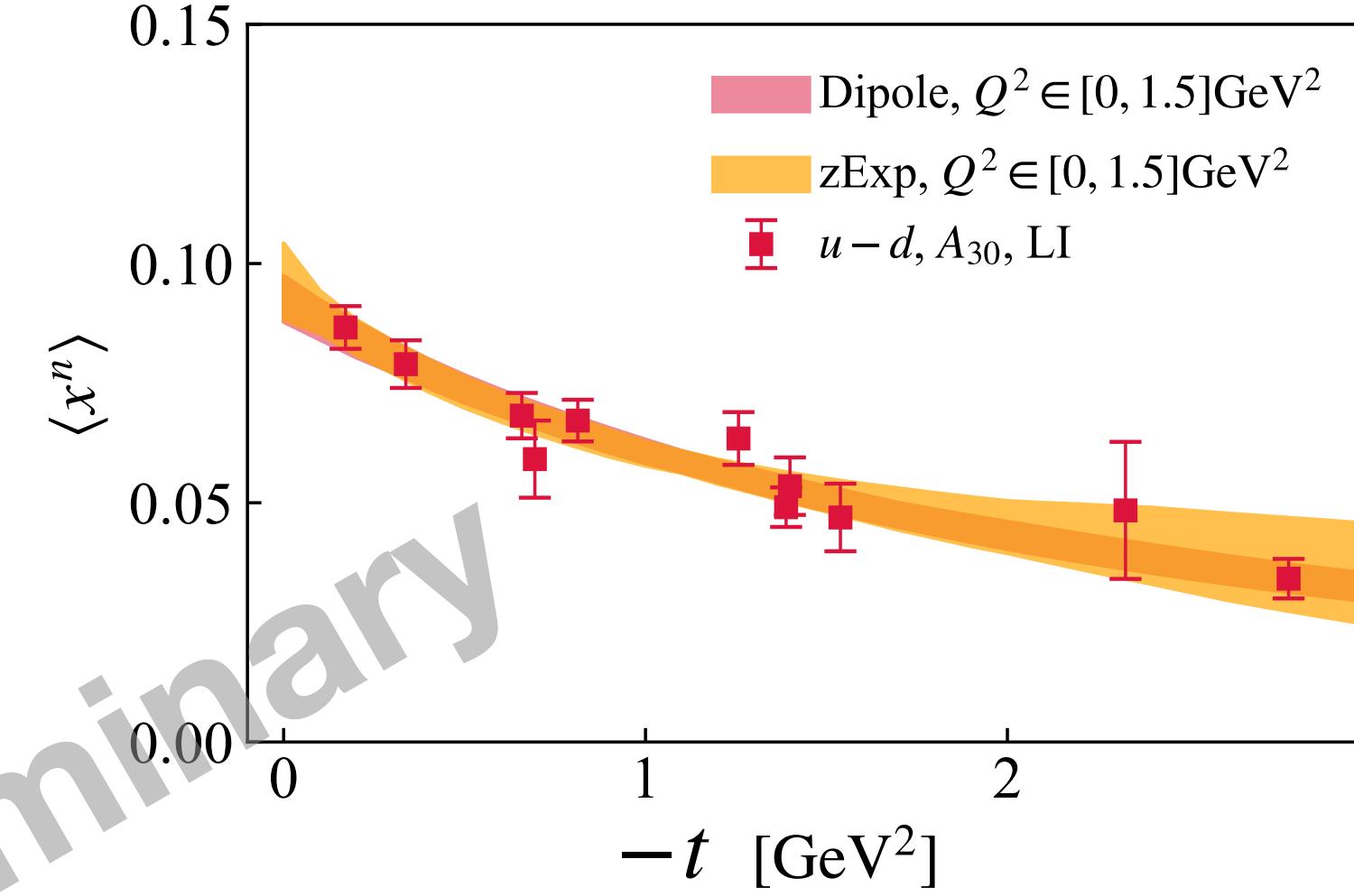
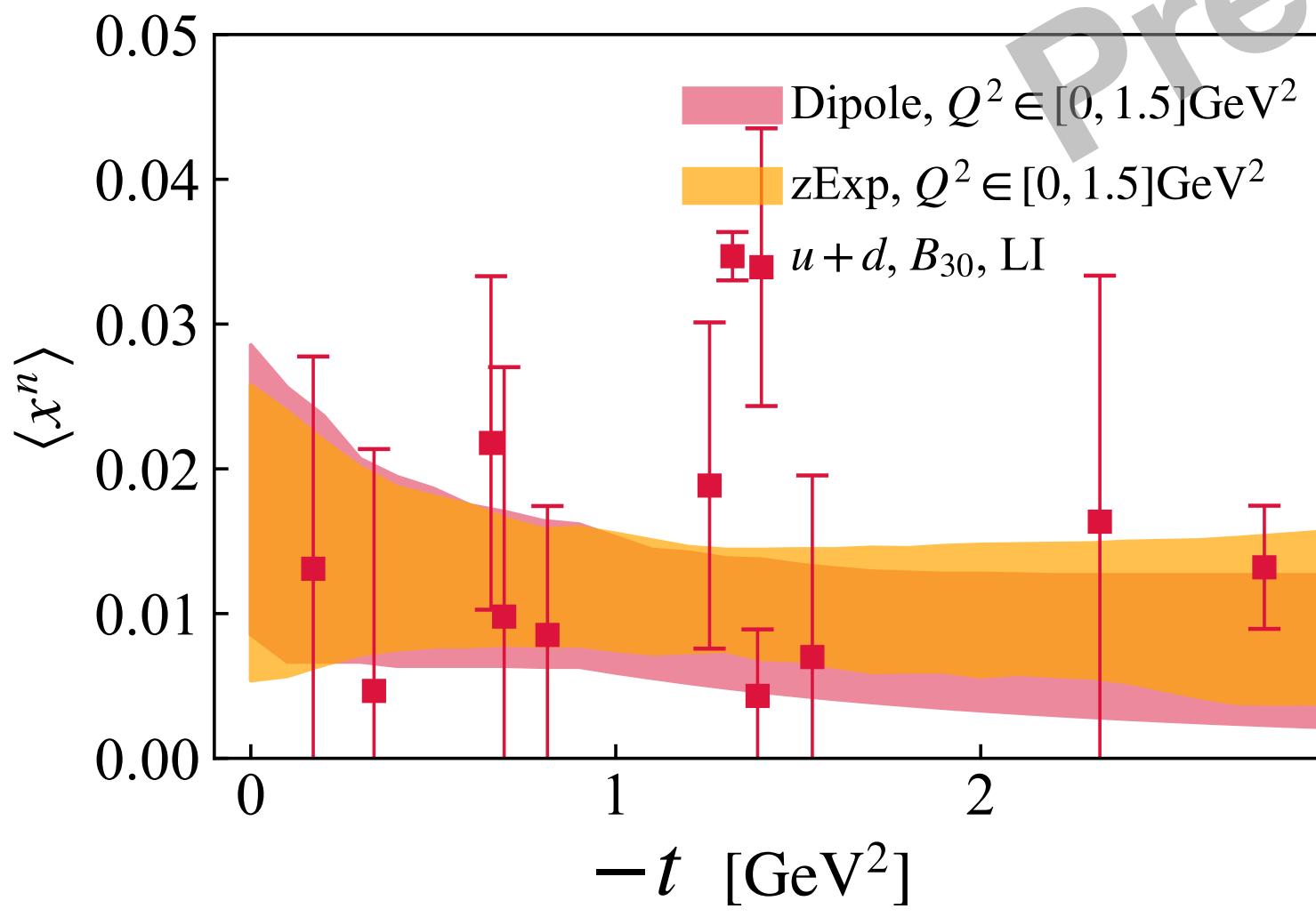
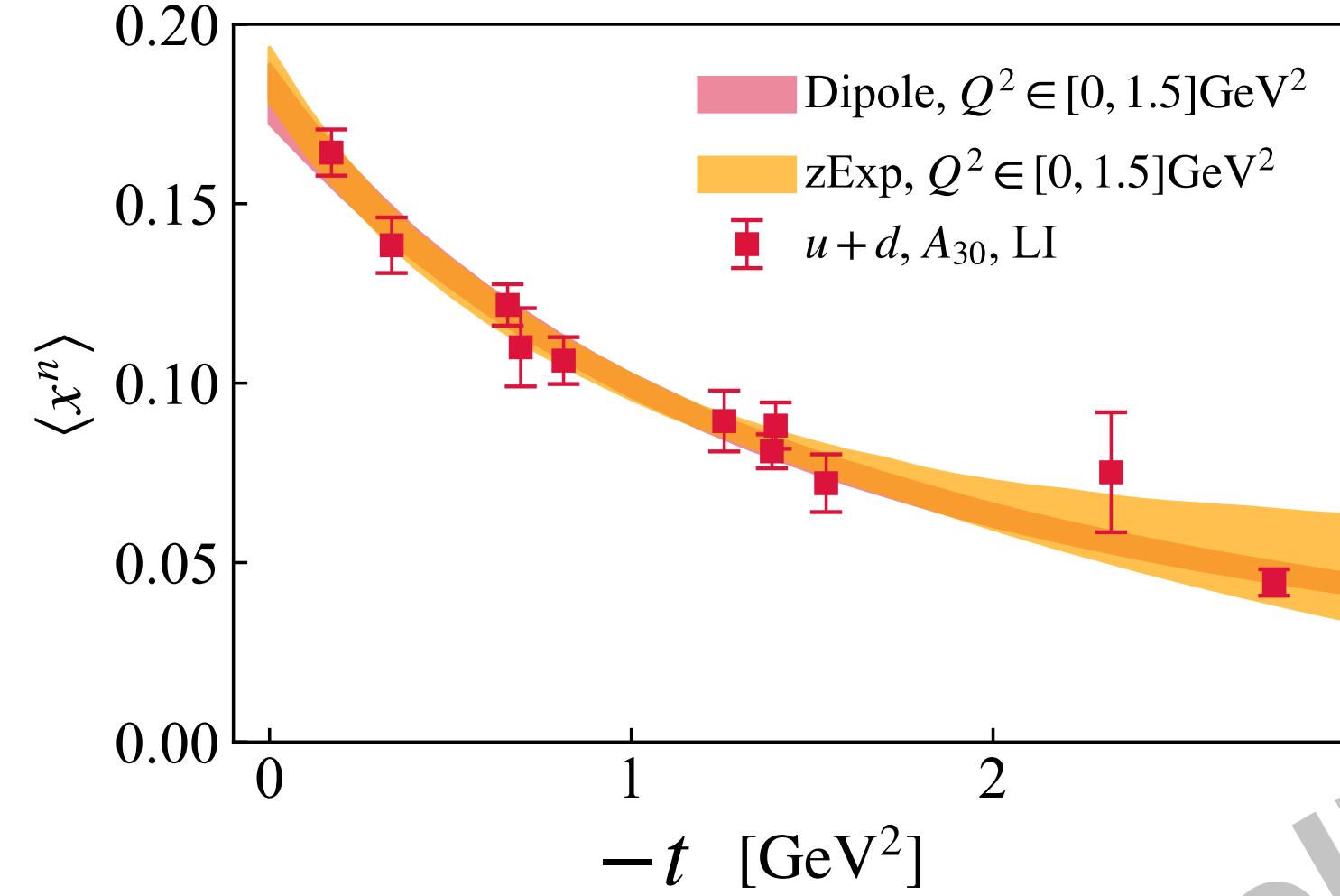
Quark total angular momentum:

$$J^q = \frac{1}{2}(A_{20}^q(0) + B_{20}^q(0))$$

$$J^{u-d} = 0.267(27)(39)$$

$$J^{u+d} = 0.301(14)(02)$$

# $t$ -dependence of moments



$-t \rightarrow 0$	Di-pole	$z$ -expansion
$A_{30}^{u-d}$	0.093(05)	0.096(08)
$A_{30}^{u+d}$	0.181(08)	0.186(07)
$B_{30}^{u-d}$	0.130(21)	0.130(23)
$B_{30}^{u+d}$	0.018(10)	0.015(10)

# Summary and outlook

- We carried out lattice calculation of the quasi-GPD matrix elements of proton using the Lorentz invariant definition.
- The matrix elements are renormalized in ratio scheme and the first few Mellin moments up to  $A_{30}$  and  $B_{30}$  were extracted using the leading-twist short distance factorization frame work.
- Higher moments can be constrained with higher momentum and statistics.
- The methods can be extended to non-zero skewness GPDs.
- Calculations with physical quark masses and smaller lattice spacings are needed to address the lattice artifacts.

Thanks for your attention