Analytic Solution for the Revised Helicity Evolution at Small x and Large N_C: New Resummed Gluon-Gluon Polarized Anomalous Dimension and Intercept

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Motivation

Proton spin sum rule: $S_q + L_q + S_G + L_G = \frac{1}{2}$ (Jaffe, Manohar '89) 10.1016/0550

$$S_q(Q^2) = \frac{1}{2}\int_0^1 \mathrm{d}x \Delta\Sigma(x, Q^2)$$

$$S_q(Q^2 = 10\,\mathrm{GeV}^2) \approx 0.15 \div 0.20$$

for

$$x \in [0.001, 1]$$

$$S_G(Q^2) = \int_0^1 \mathrm{d}x \Delta G(x, Q^2)$$

$$S_G(Q^2 = 10\,\mathrm{GeV}^2) \approx 0.13 \div 0.26$$

for

$$x \in [0.05, 1]$$

(see e.g. the review by Ji, Yuan, Zhao '20 2009.01291 or in particular DSSV: 1404.4293v1, 0904.3821v2)

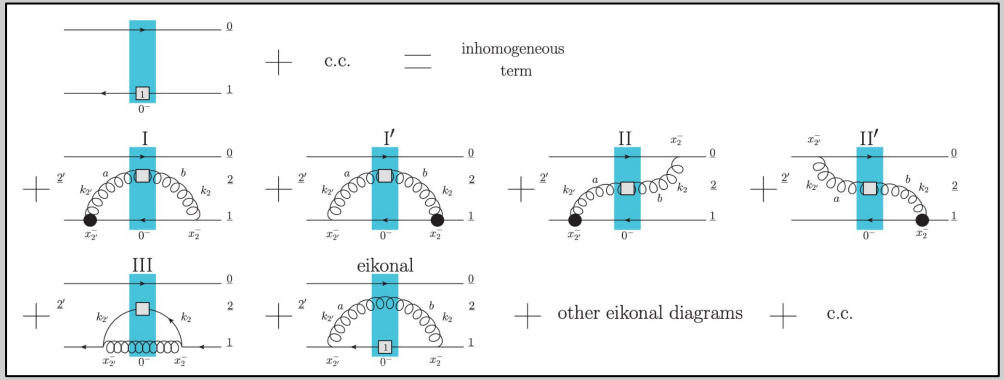
Still short of ½

How much spin at small-*x*?

Small-*x* Helicity Evolution

Cougoulic, Kovchegov, Tarasov, Tawabutr '22 2204.11898v3

Novel small-*x* helicity evolution equations



Already solved numerically (at large-N_c), giving numerical agreement with existing results

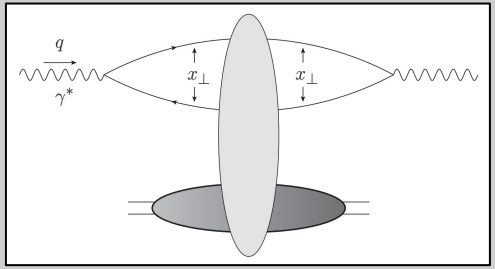
Bartels, Ermolaev, Ryskin '96 9603204v1

What about an analytic solution?

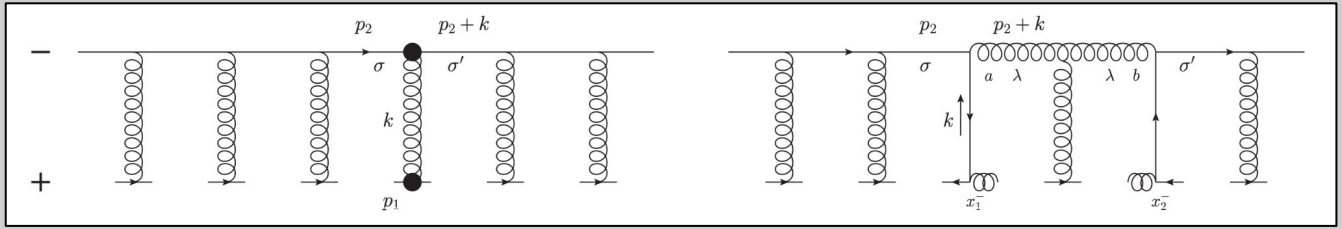
Cross check numerical results.
Can we learn anything new?

Quark and gluon helicity evolution at small-*x*

Dipole picture of DIS



Helicity evolution enters at the sub-eikonal level



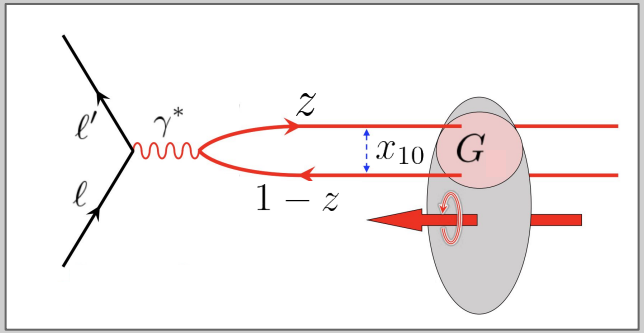
Turns out that lots of observables can be expressed in terms of the 'polarized dipole amplitudes' $G(x_{10}^2, zs),\ G_2(x_{10}^2, zs)$

$$g_1(x, Q^2) = -\sum_f \frac{N_c Z_f^2}{4\pi^3} \int_{\Lambda^2/s}^{1} \frac{\mathrm{d}z}{z} \int_{\frac{1}{zs}}^{\min\{\frac{1}{zQ^2}, \frac{1}{\Lambda^2}\}} \frac{\mathrm{d}x_{10}^2}{x_{10}^2} \left[G(x_{10}^2, zs) + 2G_2(x_{10}^2, zs)\right]$$

Polarized Dipole Amplitudes

$$G_{10}(zs) = \frac{1}{2N_c}\mathrm{Re}\left\langle\left\langle \mathrm{T\,tr}\left[V_{\underline{0}}V_{\underline{1}}^{G[1]\dagger}\right] + \mathrm{T\,tr}\left[V_{\underline{1}}^{G[1]}V_{\underline{0}}^{\dagger}\right]\right\rangle\right\rangle(zs)$$

$$G_{10}^{i}(zs) = \frac{1}{2N_c}\left\langle\left\langle \mathrm{tr}\left[V_{\underline{0}}^{\dagger}V_{\underline{1}}^{i\,G[2]} + \left(V_{\underline{1}}^{i\,G[2]}\right)^{\dagger}V_{\underline{0}}\right]\right\rangle\right\rangle(zs)$$



$$\int \mathrm{d}^2\left(\frac{x_0+x_1}{2}\right) G_{10}(zs) = \underline{G(x_{10}^2, zs)}$$

$$\int \mathrm{d}^2\left(\frac{x_1+x_0}{2}\right) G_{10}^{i}(zs) = (x_{10})_{\perp}^{i}\, G_1\left(x_{10}^2, zs\right) + \epsilon^{ij}\,(x_{10})_{\perp}^{j}\, \underline{G_2\left(x_{10}^2, zs\right)}$$

$V_{\underline{0}}$ is ordinary (unpolarized) fundamental Wilson line

$$V_{\underline{x}} = \mathcal{P}\exp\left[ig\int_{-\infty}^{\infty}\mathrm{d}x^{-}A^{+}\left(0^{+}, x^{-}, \underline{x}\right)\right]$$

$V_{\underline{1}}^{G[1]}$, $V_{\underline{1}}^{iG[2]}$ are polarized Wilson line operators

⇩

polarization-dependent interactions sandwiched between ordinary Wilson lines

Dipole Amplitudes Also Give helicity TMDs, PDFs

Gluon helicity TMD

$$g_{1L}^{G\,dip}(x,k_T^2) = \frac{N_c}{\alpha_s 2\pi^4}\int \mathrm{d}^2x_{10}\, e^{-i\underline{k}\cdot\underline{x}_{10}}\left[1 + x_{10}^2\frac{\partial}{\partial x_{10}^2}\right] G_2\left(x_{10}^2, zs = \frac{Q^2}{x}\right)$$

Flavor Singlet quark helicity TMD

$$g_{1L}^{S}(x,k_T^2) = \frac{8iN_cN_f}{(2\pi)^5}\int_{\Lambda^2/s}^{1}\frac{\mathrm{d}z}{z}\int \mathrm{d}^2x_{10}\, e^{i\underline{k}\cdot\underline{x}_{10}}\frac{\underline{x}_{10}}{x_{10}^2}\cdot\frac{\underline{k}}{\underline{k}^2}\left[G(x_{10}^2, zs) + 2G_2(x_{10}^2, zs)\right]$$

Gluon helicity PDF

$$\Delta G(x,Q^2) = \frac{2N_c}{\alpha_s\pi^2}\left[\left(1 + x_{10}^2\frac{\partial}{\partial x_{10}^2}\right) G_2\left(x_{10}^2, zs = \frac{Q^2}{x}\right)\right]\Bigg|_{x_{10}^2 = 1/Q^2}$$

Flavor Singlet quark helicity PDF

$$\Delta\Sigma(x,Q^2) = -\frac{N_cN_f}{2\pi^3}\int_{\Lambda^2/s}^{1}\frac{\mathrm{d}z}{z}\int_{\frac{1}{zs}}^{\min\{\frac{1}{zQ^2},\frac{1}{\Lambda^2}\}}\frac{\mathrm{d}x_{10}^2}{x_{10}^2}\left[G(x_{10}^2, zs) + 2G_2(x_{10}^2, zs)\right]$$

g_1 structure function

$$g_1(x,Q^2) = -\sum_f \frac{N_cZ_f^2}{4\pi^3}\int_{\Lambda^2/s}^{1}\frac{\mathrm{d}z}{z}\int_{\frac{1}{zs}}^{\min\{\frac{1}{zQ^2},\frac{1}{\Lambda^2}\}}\frac{\mathrm{d}x_{10}^2}{x_{10}^2}\left[G(x_{10}^2, zs) + 2G_2(x_{10}^2, zs)\right]$$

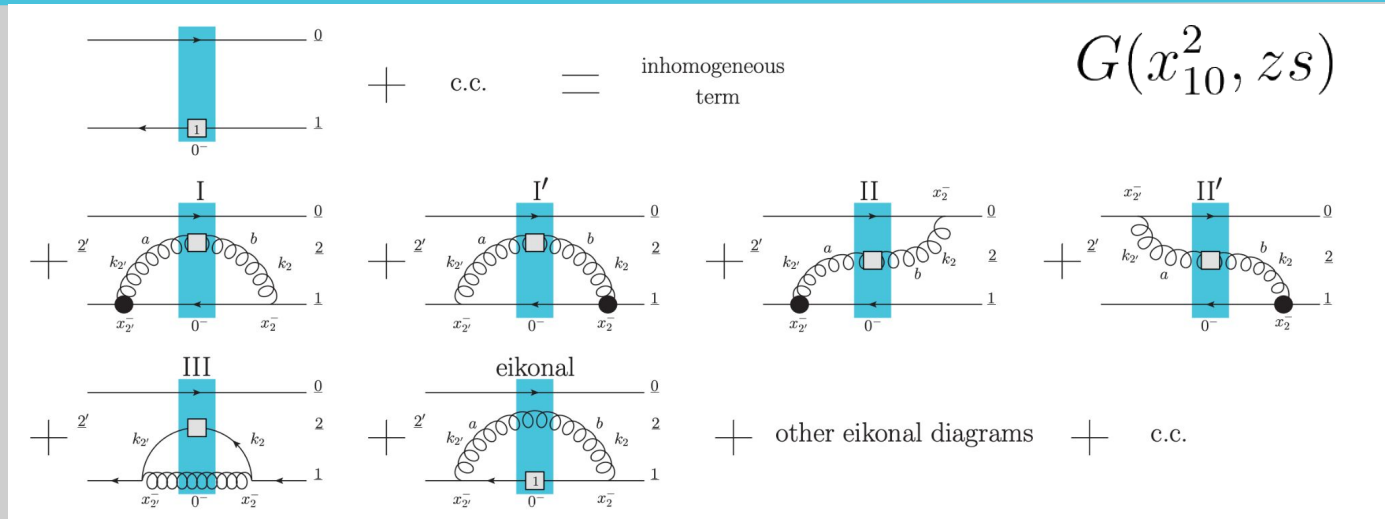
Small-*x* evolution of the dipole amplitudes

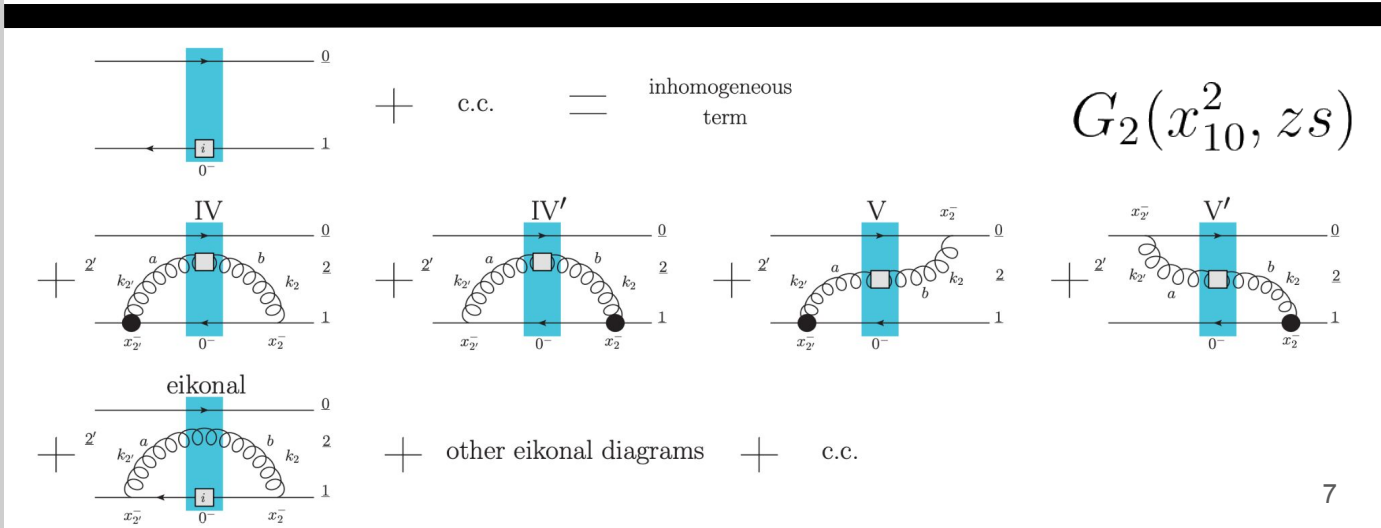
Cougoulic, Kovchegov, Tarasov, Tawabutr '22 2204.11898v3

Double-logarithmic approximation (resumming powers of $\alpha_s \ln^2(1/x)$)

Full evolution equations don't close (like Balitsky hierarchy)

Balitsky '95,'98 9509348v1,9812311v1





Equations do close in the large-N_c limit

Cougoulic, Kovchegov, Tarasov, Tawabutr '22
2204.11898v3

$$G(x_{10}^2, zs) = G^{(0)}(x_{10}^2, zs) + \frac{\alpha_s N_c}{2\pi} \int\limits_{\frac{1}{s x_{10}^2}}^{z} \frac{\mathrm{d}z'}{z'} \int\limits_{\frac{1}{z's}}^{x_{10}^2} \frac{\mathrm{d}x_{21}^2}{x_{21}^2} \left[\Gamma(x_{10}^2, x_{21}^2, z's) + 3G(x_{21}^2, z's) + 2G_2(x_{21}^2, z's) + 2\Gamma_2(x_{10}^2, x_{21}^2, z's) \right]$$

$$\Gamma(x_{10}^2, x_{21}^2, z's) = G^{(0)}(x_{10}^2, z's) + \frac{\alpha_s N_c}{2\pi} \int\limits_{\frac{1}{s x_{10}^2}}^{z'} \frac{\mathrm{d}z''}{z''} \int\limits_{\frac{1}{z''s}}^{\min\left[x_{10}^2, x_{21}^2 \frac{z'}{z''}\right]} \frac{\mathrm{d}x_{32}^2}{x_{32}^2} \left[\Gamma(x_{10}^2, x_{32}^2, z''s) + 3G(x_{32}^2, z''s) + 2G_2(x_{32}^2, z''s) + 2\Gamma_2(x_{10}^2, x_{32}^2, z''s) \right]$$

$$G_2(x_{10}^2, zs) = G_2^{(0)}(x_{10}^2, zs) + \frac{\alpha_s N_c}{\pi} \int\limits_{\frac{\Lambda^2}{s}}^{z} \frac{\mathrm{d}z'}{z'} \int\limits_{\max\left[x_{10}^2, \frac{1}{z's}\right]}^{\min\left[\frac{z}{z'} x_{10}^2, \frac{1}{\Lambda^2}\right]} \frac{\mathrm{d}x_{21}^2}{x_{21}^2} \left[G(x_{21}^2, z's) + 2G_2(x_{21}^2, z's) \right]$$

$$\Gamma_2(x_{10}^2, x_{21}^2, z's) = G_2^{(0)}(x_{10}^2, z's) + \frac{\alpha_s N_c}{\pi} \int\limits_{\frac{\Lambda^2}{s}}^{z' \frac{x_{21}^2}{x_{10}^2}} \frac{\mathrm{d}z''}{z''} \int\limits_{\max\left[x_{10}^2, \frac{1}{z''s}\right]}^{\min\left[\frac{z'}{z''} x_{21}^2, \frac{1}{\Lambda^2}\right]} \frac{\mathrm{d}x_{32}^2}{x_{32}^2} \left[G(x_{32}^2, z''s) + 2G_2(x_{32}^2, z''s) \right]$$

Γ and Γ_2 are auxiliary functions ('neighbor dipole amplitudes')

<u>Would like to solve these equations analytically</u>

Solution

Starting point - double Laplace transforms for G_2 and G

$$G_2(x_{10}^2, zs) = \frac{1}{\bar{\alpha}_s} \int \frac{\mathrm{d}\omega}{2\pi i} \int \frac{\mathrm{d}\gamma}{2\pi i} e^{\omega \ln\left(zsx_{10}^2\right) + \gamma \ln\left(\frac{1}{x_{10}^2 \Lambda^2}\right)} G_{2\omega\gamma}$$

$$G(x_{10}^2, zs) = \frac{1}{\bar{\alpha}_s} \int \frac{\mathrm{d}\omega}{2\pi i} \int \frac{\mathrm{d}\gamma}{2\pi i} e^{\omega \ln\left(zsx_{10}^2\right) + \gamma \ln\left(\frac{1}{x_{10}^2 \Lambda^2}\right)} G_{\omega\gamma}$$

$$\bar{\alpha}_s = \frac{\alpha_s N_c}{2\pi}$$

Along with corresponding transforms for the initial conditions of the evolution

Can then manipulate the large-N_c equations to find expressions for the other dipole amplitudes and constrain the coefficient $G_{2\omega\gamma}$

After some work, the results are…

Solution

$$G_2(x_{10}^2, zs) = \frac{1}{\bar{\alpha}_s} \int \frac{\mathrm{d}\omega}{2\pi i} \int \frac{\mathrm{d}\gamma}{2\pi i} e^{\omega \ln\left(zsx_{10}^2\right) + \gamma \ln\left(\frac{1}{x_{10}^2 \Lambda^2}\right)} G_{2\omega\gamma}$$

$$G(x_{10}^2, zs) = \frac{1}{\bar{\alpha}_s} \int \frac{\mathrm{d}\omega}{2\pi i} \int \frac{\mathrm{d}\gamma}{2\pi i} e^{\omega \ln\left(zsx_{10}^2\right) + \gamma \ln\left(\frac{1}{x_{10}^2 \Lambda^2}\right)} \left[\frac{1}{2} \frac{\omega\gamma}{\bar{\alpha}_s} \left(G_{2\omega\gamma} - G_{2\omega\gamma}^{(0)}\right) - 2G_{2\omega\gamma}\right]$$

$$G_{2\omega\gamma} = G_{2\omega\gamma}^{(0)} + \frac{\bar{\alpha}_s}{\omega\left(\gamma - \gamma_\omega^-\right)\left(\gamma - \gamma_\omega^+\right)} \left[2\left(\gamma - \delta_\omega^+\right)\left(G_{\delta_\omega^+\gamma}^{(0)} + 2G_{2\delta_\omega^+\gamma}^{(0)}\right) - 2\left(\gamma_\omega^+ - \delta_\omega^+\right)\left(G_{\delta_\omega^+\gamma_\omega^+}^{(0)} + 2G_{2\delta_\omega^+\gamma_\omega^+}^{(0)}\right) + 8\delta_\omega^-\left(G_{2\omega\gamma}^{(0)} - G_{2\omega\gamma_\omega^+}^{(0)}\right)\right]$$

$$\delta_\omega^\pm = \frac{\omega}{2}\left[1 \pm \sqrt{1 - \frac{4\bar{\alpha}_s}{\omega^2}}\right] \qquad \gamma_\omega^\pm = \frac{\omega}{2}\left[1 \pm \sqrt{1 - \frac{16\bar{\alpha}_s}{\omega^2}\sqrt{1 - \frac{4\bar{\alpha}_s}{\omega^2}}}\right]$$

Note $G_{2\omega\gamma}^{(0)}$, $G_{\omega\gamma}^{(0)}$ are the double-Laplace images of the initial conditions $G_2^{(0)}(x_{10}^2, zs)$, $G^{(0)}(x_{10}^2, zs)$

Using the Dipole Amplitudes

Can write down small-*x* large-N_c expressions for hTMDs and hPDFs (for arbitrary initial conditions)

$$g_{1L}^{G\,dip}(x,k_T^2) = \frac{4}{\alpha_s^2\pi^2}\frac{1}{k_T^2}\int\frac{\mathrm{d}\omega}{2\pi i}\int\frac{\mathrm{d}\gamma}{2\pi i}e^{\omega\ln\left(\frac{Q^2}{xk_T^2}\right)+\gamma\ln\left(\frac{k_T^2}{\Lambda^2}\right)}2^{2\omega-2\gamma}\frac{\Gamma\left(\omega-\gamma+1\right)}{\Gamma\left(\gamma-\omega\right)}G_{2\omega\gamma}$$

$$g_{1L}^{S}(x,k_T^2) = \frac{-N_f}{\alpha_s^2 N_c\pi^2}\frac{1}{k_T^2}\int\frac{\mathrm{d}\omega}{2\pi i}\int\frac{\mathrm{d}\gamma}{2\pi i}e^{\omega\ln\left(\frac{s}{k_T^2}\right)+\gamma\ln\left(\frac{k_T^2}{\Lambda^2}\right)}2^{2\omega-2\gamma}\frac{\Gamma\left(\omega-\gamma+1\right)}{\Gamma\left(\gamma-\omega+1\right)}\gamma\left(G_{2\omega\gamma}-G_{2\omega\gamma}^{(0)}\right)$$

Γ functions, not neighbor dipole amplitude

$$\Delta G(x,Q^2) = \frac{4}{\alpha_s^2\pi}\int\frac{\mathrm{d}\omega}{2\pi i}\int\frac{\mathrm{d}\gamma}{2\pi i}e^{\omega\ln\left(\frac{1}{x}\right)+\gamma\ln\left(\frac{Q^2}{\Lambda^2}\right)}G_{2\omega\gamma}$$

$$\Delta\Sigma(x,Q^2) = \frac{-N_f}{\alpha_s^2 N_c\pi}\int\frac{\mathrm{d}\omega}{2\pi i}\int\frac{\mathrm{d}\gamma}{2\pi i}\frac{1}{\omega-\gamma}\left(G_{2\omega\gamma}-G_{2\omega\gamma}^{(0)}\right)\left[\gamma e^{\omega\ln\left(\frac{s}{Q^2}\right)}+\omega\left(1+e^{\omega\ln\left(\frac{s}{Q^2}\right)}\left[e^{\gamma\ln\left(\frac{Q^2}{\Lambda^2}\right)}-1\right]-e^{\gamma\ln\left(\frac{s}{\Lambda^2}\right)}\right)\right]$$

$$g_1(x,Q^2) = -\sum_f\frac{Z_f^2}{2\pi\alpha_s^2 N_c}\int\frac{\mathrm{d}\omega}{2\pi i}\int\frac{\mathrm{d}\gamma}{2\pi i}\frac{1}{\omega-\gamma}\left(G_{2\omega\gamma}-G_{2\omega\gamma}^{(0)}\right)\left[\gamma e^{\omega\ln\left(\frac{s}{Q^2}\right)}+\omega\left(1+e^{\omega\ln\left(\frac{s}{Q^2}\right)}\left[e^{\gamma\ln\left(\frac{Q^2}{\Lambda^2}\right)}-1\right]-e^{\gamma\ln\left(\frac{s}{\Lambda^2}\right)}\right)\right]$$

Resummed Anomalous Dimension

Now fix the initial conditions of the evolution to be simply

$$G_2^{(0)}(x_{10}^2, zs) = 1$$

$$G^{(0)}(x_{10}^2, zs) = 0$$

Gluon helicity PDF takes the nice form

$$\Delta G(x, Q^2) = \frac{2N_c}{\alpha_s \pi^2} \int \frac{\mathrm{d}\omega}{2\pi i} e^{\omega \ln\left(\frac{1}{x}\right) + \gamma_\omega^- \ln\left(\frac{Q^2}{\Lambda^2}\right)} \frac{1}{\omega}$$

$$\bar{\alpha}_s = \frac{\alpha_s N_c}{2\pi}$$

Pure-glue polarized anomalous dimension

$$\Delta\gamma_{GG}(\omega) = \gamma_\omega^- = \frac{\omega}{2}\left[1 - \sqrt{1 - \frac{16\bar{\alpha}_s}{\omega^2}\sqrt{1 - \frac{4\bar{\alpha}_s}{\omega^2}}}\right] = \frac{4\bar{\alpha}_s}{\omega} + \frac{8\bar{\alpha}_s^2}{\omega^3} + \frac{56\bar{\alpha}_s^3}{\omega^5} + \frac{496\bar{\alpha}_s^4}{\omega^7} + \mathcal{O}\left(\alpha_s^5\right)$$

Agrees with fixed-order calculations up to $\mathcal{O}\left(\alpha_s^3\right)$

Mertig & van Neerven '18 9506451;
Moch, Vermaseren, & Vogt '14 1409.5131v1;
Blümlein, Marquard, Schneider, & Schönwald '21 2111.12401

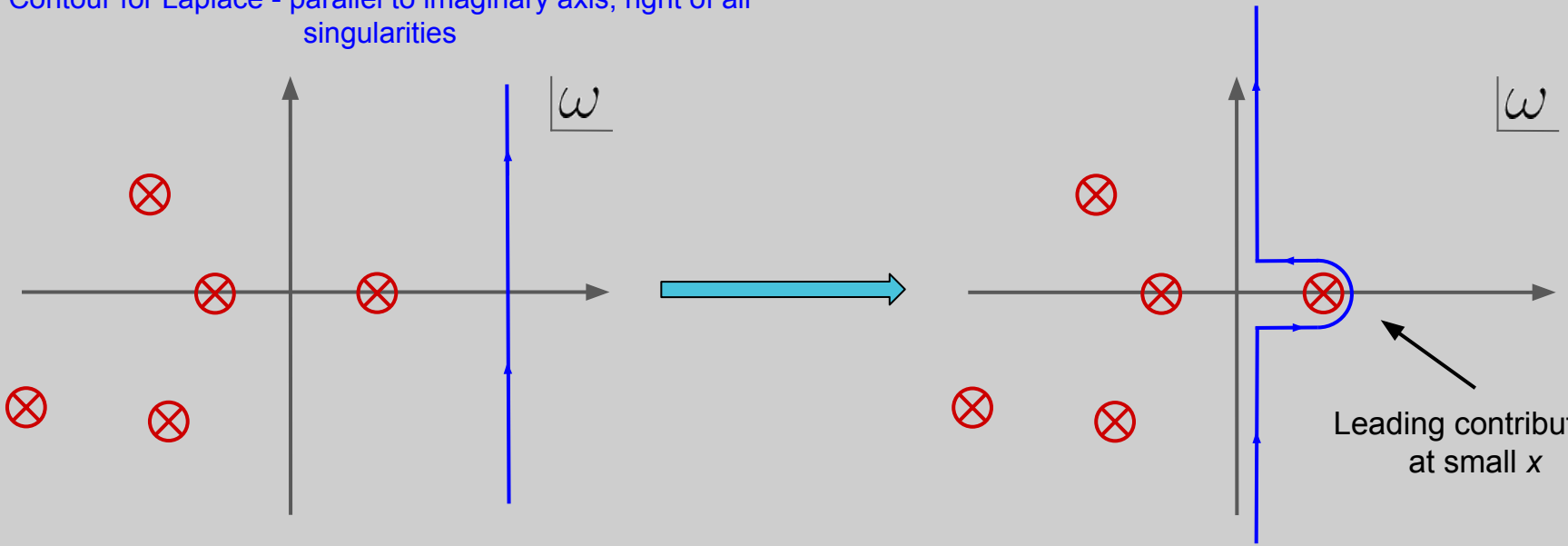
Small-*x* Asymptotics

Asymptotics governed by the 'intercept' α_h $\Rightarrow$ $\Delta\Sigma(x,Q^2) \sim \Delta G(x,Q^2) \sim g_1(x,Q^2) \sim \left(\frac{1}{x}\right)^{\alpha_h}$

corresponds to the rightmost singularity in the ω-plane

$$F(t) = \int \frac{\mathrm{d}\omega}{2\pi i} e^{\omega t} f_\omega$$

Contour for Laplace - parallel to imaginary axis, right of all singularities



Small-*x* Asymptotics

Rightmost singularity here comes from the polarized anomalous dimension γ_ω^-

See e.g. gluon helicity PDF

$$\Delta G(x,Q^2) = \frac{2N_c}{\alpha_s \pi^2} \int \frac{\mathrm{d}\omega}{2\pi i} e^{\omega \ln\left(\frac{1}{x}\right) + \gamma_\omega^- \ln\left(\frac{Q^2}{\Lambda^2}\right)} \frac{1}{\omega}$$

$$\gamma_\omega^- = \frac{\omega}{2}\left[1 - \underline{\sqrt{1 - \frac{16\bar{\alpha}_s}{\omega^2}\sqrt{1 - \frac{4\bar{\alpha}_s}{\omega^2}}}}\right]$$

Turns out to be a branch point from the large square root

$$\boxed{\alpha_h = \frac{4}{3^{1/3}}\sqrt{\mathrm{Re}\left[\left(-9 + i\sqrt{111}\right)^{1/3}\right]}\sqrt{\frac{\alpha_s N_c}{2\pi}} \approx 3.66074\sqrt{\frac{\alpha_s N_c}{2\pi}}}$$

Comparison to BER

Bartels, Ermolaev, and Ryskin (BER) IR evolution

Bartels, Ermolaev, Ryskin '96 9603204v1

$$\bar{\alpha}_s = \frac{\alpha_s N_c}{2\pi}$$

Polarized GG anomalous dimension

$$\Delta\gamma_{GG}^{BER}(\omega) = \frac{\omega}{2}\left[1 - \sqrt{1 - \frac{16\bar{\alpha}_s}{\omega^2}\frac{1 - \frac{3\bar{\alpha}_s}{\omega^2}}{1 - \frac{\bar{\alpha}_s}{\omega^2}}}\right] = \underline{\frac{4\bar{\alpha}_s}{\omega} + \frac{8\bar{\alpha}_s^2}{\omega^3} + \frac{56\bar{\alpha}_s^3}{\omega^5}} + \underline{\frac{504\bar{\alpha}_s^4}{\omega^7}} + \mathcal{O}\left(\alpha_s^5\right)$$

Compare to us

$$\Delta\gamma_{GG}(\omega) = \gamma_\omega^- = \frac{\omega}{2}\left[1 - \sqrt{1 - \frac{16\bar{\alpha}_s}{\omega^2}\sqrt{1 - \frac{4\bar{\alpha}_s}{\omega^2}}}\right] = \underline{\frac{4\bar{\alpha}_s}{\omega} + \frac{8\bar{\alpha}_s^2}{\omega^3} + \frac{56\bar{\alpha}_s^3}{\omega^5}} + \underline{\frac{496\bar{\alpha}_s^4}{\omega^7}} + \mathcal{O}\left(\alpha_s^5\right)$$

Comparison to BER

Bartels, Ermolaev, and Ryskin (BER) IR evolution

Bartels, Ermolaev, Ryskin '96 9603204v1

Small-*x* (pure-glue) intercept

$$\alpha_h^{BER} = \sqrt{\frac{17+\sqrt{97}}{2}}\sqrt{\frac{\alpha_s N_c}{2\pi}} \approx \underline{3.66394}\sqrt{\frac{\alpha_s N_c}{2\pi}}$$

Compare to us

$$\alpha_h = \frac{4}{3^{1/3}}\sqrt{\mathrm{Re}\left[\left(-9+i\sqrt{111}\right)^{1/3}\right]}\sqrt{\frac{\alpha_s N_c}{2\pi}} \approx \underline{3.66074}\sqrt{\frac{\alpha_s N_c}{2\pi}}$$

Why the (*very small*) disagreement with BER?

Could be an issue with two softest gluons as part of a non-ladder diagram, not included in IREE (?)

Kovchegov, Pitonyak, & Sievert '16 1610.06197v1

Takeaways

- Analytic solution at small-*x* and large-N_c for the dipole amplitudes
  - → Analytic expressions in the same regime for gluon and flavor-singlet quark helicity TMDs and PDFs, along with g_1
- We find small-*x* asymptotics

$$\Delta\Sigma(x,Q^2) \sim \Delta G(x,Q^2) \sim g_1(x,Q^2) \sim \left(\frac{1}{x}\right)^{\alpha_h} \qquad \alpha_h \approx \underline{3.66074}\sqrt{\frac{\alpha_s N_c}{2\pi}}$$

  - A *very* small discrepancy compared to the prediction of BER: $\alpha_h^{BER} \approx \underline{3.66394}\sqrt{\frac{\alpha_s N_c}{2\pi}}$

- We find a resummed small-*x* anomalous dimension

$$\Delta\gamma_{GG}(\omega) = \frac{\omega}{2}\left[1-\sqrt{1-\frac{16\bar{\alpha}_s}{\omega^2}\sqrt{1-\frac{4\bar{\alpha}_s}{\omega^2}}}\right] = \underline{\frac{4\bar{\alpha}_s}{\omega}+\frac{8\bar{\alpha}_s^2}{\omega^3}+\frac{56\bar{\alpha}_s^3}{\omega^5}} + \underline{\frac{496\bar{\alpha}_s^4}{\omega^7}} + \mathcal{O}\left(\alpha_s^5\right)$$

  - Comparison with BER again yields a very small discrepancy at $\mathcal{O}\left(\alpha_s^4\right)$

$$\Delta\gamma_{GG}^{BER}(\omega) = \frac{\omega}{2}\left[1-\sqrt{1-\frac{16\bar{\alpha}_s}{\omega^2}\frac{1-\frac{3\bar{\alpha}_s}{\omega^2}}{1-\frac{\bar{\alpha}_s}{\omega^2}}}\right] = \underline{\frac{4\bar{\alpha}_s}{\omega}+\frac{8\bar{\alpha}_s^2}{\omega^3}+\frac{56\bar{\alpha}_s^3}{\omega^5}} + \underline{\frac{504\bar{\alpha}_s^4}{\omega^7}} + \mathcal{O}\left(\alpha_s^5\right)$$

Large - N_c & N_f limit in the future?

Extra Slides

Polarized (Fundamental) Wilson Line Operators

$$V^{\mathrm{pol}[1]}_{\underline{x}} = V^{\mathrm{G}[1]}_{\underline{x}} + V^{\mathrm{q}[1]}_{\underline{x}}, \quad V^{\mathrm{pol}[2]}_{\underline{x},\underline{y}} = V^{\mathrm{G}[2]}_{\underline{x},\underline{y}} + V^{\mathrm{q}[2]}_{\underline{x}}\, \delta^2(\underline{x}-\underline{y}),$$

$$V^{\mathrm{G}[1]}_{\underline{x}} = \frac{i\,g\,P^+}{s} \int_{-\infty}^{\infty} dx^- V_{\underline{x}}[\infty, x^-]\, F^{12}(x^-, \underline{x})\; V_{\underline{x}}[x^-, -\infty],$$

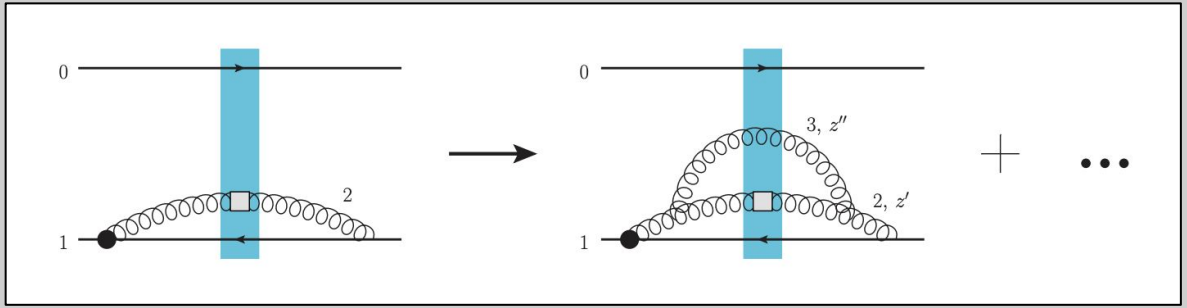
$$V^{\mathrm{q}[1]}_{\underline{x}} = \frac{g^2 P^+}{2\,s} \int_{-\infty}^{\infty} dx_1^- \int_{x_1^-}^{\infty} dx_2^- V_{\underline{x}}[\infty, x_2^-]\, t^b\, \psi_\beta(x_2^-, \underline{x})\, U^{ba}_{\underline{x}}[x_2^-, x_1^-]\; \left[\gamma^+\gamma^5\right]_{\alpha\beta}\, \bar{\psi}_\alpha(x_1^-, \underline{x})\, t^a\, V_{\underline{x}}[x_1^-, -\infty],$$

$$V^{\mathrm{G}[2]}_{\underline{x},\underline{y}} = -\frac{i\,P^+}{s} \int_{-\infty}^{\infty} dz^- d^2z\; V_{\underline{x}}[\infty, z^-]\, \delta^2(\underline{x}-\underline{z})\, \overleftarrow{D}^i(z^-, \underline{z})\, D^i(z^-, \underline{z})\, V_{\underline{y}}[z^-, -\infty]\, \delta^2(\underline{y}-\underline{z}),$$

$$V^{\mathrm{q}[2]}_{\underline{x}} = -\frac{g^2 P^+}{2\,s} \int_{-\infty}^{\infty} dx_1^- \int_{x_1^-}^{\infty} dx_2^- V_{\underline{x}}[\infty, x_2^-]\, t^b\, \psi_\beta(x_2^-, \underline{x})\, U^{ba}_{\underline{x}}[x_2^-, x_1^-]\; \left[\gamma^+\right]_{\alpha\beta}\, \bar{\psi}_\alpha(x_1^-, \underline{x})\, t^a\, V_{\underline{x}}[x_1^-, -\infty].$$

$$V^{i\,\mathrm{G}[2]}_{\underline{z}} \equiv \frac{P^+}{2s} \int_{-\infty}^{\infty} dz^-\, V_{\underline{z}}[\infty, z^-]\, \left[D^i(z^-, \underline{z}) - \overleftarrow{D}^i(z^-, \underline{z})\right]\, V_{\underline{z}}[z^-, -\infty].$$

Neighbor Dipole Amplitudes



One step in evolution of neighbor dipole amplitude

(1) DLA lifetime ordering → $x_{21}^2 z' \gg x_{32}^2 z''$

(2) But also have IR cutoff for dipole 02 → $x_{32} \ll x_{20}$

When $x_{20}^2 > x_{21}^2 \frac{z'}{z''}$ → (1) is more constraining than (2)

So for everything to be ordered properly, subsequent evolution in dipole 02 (here evolving to give dipole 32) 'knows' about dipole 21

$$G(x_{10}^2, zs)$$

$$\frac{1}{2N_c}\left\langle\!\!\left\langle \mathrm{tr}\left[V_{\underline{0}}\, V_{\underline{1}}^{\mathrm{pol}[1]\,\dagger}\right] + \mathrm{c.c.}\right\rangle\!\!\right\rangle(zs) = \frac{1}{2N_c}\left\langle\!\!\left\langle \mathrm{tr}\left[V_{\underline{0}}\, V_{\underline{1}}^{\mathrm{pol}[1]\,\dagger}\right] + \mathrm{c.c.}\right\rangle\!\!\right\rangle_0(zs)$$

$$+\frac{\alpha_s N_c}{2\pi^2}\int\limits_{\frac{\Lambda^2}{s}}^{z}\frac{dz'}{z'}\int d^2x_2\Bigg\{\left[\frac{1}{x_{21}^2}-\frac{\underline{x}_{21}}{x_{21}^2}\cdot\frac{\underline{x}_{20}}{x_{20}^2}\right]\frac{1}{N_c^2}\left\langle\!\!\left\langle \mathrm{tr}\left[t^b V_{\underline{0}} t^a V_{\underline{1}}^\dagger\right]\left(U_{\underline{2}}^{\mathrm{pol}[1]}\right)^{ba} + \mathrm{c.c.}\right\rangle\!\!\right\rangle(z's)$$

$$+\left[2\frac{\epsilon^{ij}\,x_{21}^j}{x_{21}^4}-\frac{\epsilon^{ij}\,(x_{20}^j+x_{21}^j)}{x_{20}^2\,x_{21}^2}-\frac{2\,\underline{x}_{20}\times\underline{x}_{21}}{x_{20}^2\,x_{21}^2}\left(\frac{x_{21}^i}{x_{21}^2}-\frac{x_{20}^i}{x_{20}^2}\right)\right]\frac{1}{N_c^2}\left\langle\!\!\left\langle \mathrm{tr}\left[t^b V_{\underline{0}} t^a V_{\underline{1}}^\dagger\right]\left(U_{\underline{2}}^{i\,\mathrm{G}[2]}\right)^{ba} + \mathrm{c.c.}\right\rangle\!\!\right\rangle(z's)\Bigg\}$$

$$+\frac{\alpha_s N_c}{4\pi^2}\int\limits_{\frac{\Lambda^2}{s}}^{z}\frac{dz'}{z'}\int\frac{d^2x_2}{x_{21}^2}\left\{\frac{1}{N_c^2}\left\langle\!\!\left\langle \mathrm{tr}\left[t^b\, V_{\underline{0}}\, t^a\, V_{\underline{2}}^{\mathrm{pol}[1]\,\dagger}\right]\, U_{\underline{1}}^{ba}\right\rangle\!\!\right\rangle(z's) + 2\,\frac{\epsilon^{ij}\,\underline{x}_{21}^j}{x_{21}^2}\,\frac{1}{N_c^2}\left\langle\!\!\left\langle \mathrm{tr}\left[t^b\, V_{\underline{0}}\, t^a\, V_{\underline{2}}^{i\,\mathrm{G}[2]\,\dagger}\right]\, U_{\underline{1}}^{ba}\right\rangle\!\!\right\rangle(z's) + \mathrm{c.c.}\right\}$$

$$+\frac{\alpha_s N_c}{2\pi^2}\int\limits_{\frac{\Lambda^2}{s}}^{z}\frac{dz'}{z'}\int d^2x_2\,\frac{x_{10}^2}{x_{21}^2\,x_{20}^2}\left\{\frac{1}{N_c^2}\left\langle\!\!\left\langle \mathrm{tr}\left[t^b\, V_{\underline{0}}\, t^a\, V_{\underline{1}}^{\mathrm{pol}[1]\,\dagger}\right]\, U_{\underline{2}}^{ba}\right\rangle\!\!\right\rangle(z's) - \frac{C_F}{N_c^2}\left\langle\!\!\left\langle \mathrm{tr}\left[V_{\underline{0}}\, V_{\underline{1}}^{\mathrm{pol}[1]\,\dagger}\right]\right\rangle\!\!\right\rangle(z's) + \mathrm{c.c.}\right\}.$$

$$G_2(x_{10}^2, zs)$$

$$\frac{1}{2N_c}\left\langle\!\!\left\langle \mathrm{tr}\left[V_{\underline{0}}\, V_{\underline{1}}^{i\,\mathrm{G}[2]\,\dagger}\right] + \mathrm{c.c.}\right\rangle\!\!\right\rangle(zs) = \frac{1}{2N_c}\left\langle\!\!\left\langle \mathrm{tr}\left[V_{\underline{0}}\, V_{\underline{1}}^{i\,\mathrm{G}[2]\,\dagger}\right] + \mathrm{c.c.}\right\rangle\!\!\right\rangle_0(zs)$$

$$+\frac{\alpha_s N_c}{4\pi^2}\int\limits_{\frac{\Lambda^2}{s}}^{z}\frac{dz'}{z'}\int d^2x_2\Bigg\{\left[\frac{\epsilon^{ij}x_{21}^j}{x_{21}^2}-\frac{\epsilon^{ij}x_{20}^j}{x_{20}^2}+2x_{21}^i\frac{\underline{x}_{21}\times\underline{x}_{20}}{x_{21}^2\,x_{20}^2}\right]\frac{1}{N_c^2}\left\langle\!\!\left\langle \mathrm{tr}\left[t^b V_{\underline{0}} t^a V_{\underline{1}}^\dagger\right]\left(U_{\underline{2}}^{\mathrm{pol}[1]}\right)^{ba} + \mathrm{c.c.}\right\rangle\!\!\right\rangle(z's)$$

$$+\left[\delta^{ij}\left(\frac{3}{x_{21}^2}-2\,\frac{\underline{x}_{20}\cdot\underline{x}_{21}}{x_{20}^2\,x_{21}^2}-\frac{1}{x_{20}^2}\right)-2\frac{x_{21}^i\,x_{20}^j}{x_{21}^2\,x_{20}^2}\left(2\frac{\underline{x}_{20}\cdot\underline{x}_{21}}{x_{20}^2}+1\right)+2\frac{x_{21}^i\,x_{21}^j}{x_{21}^2\,x_{20}^2}\left(2\frac{\underline{x}_{20}\cdot\underline{x}_{21}}{x_{21}^2}+1\right)+2\frac{x_{20}^i\,x_{20}^j}{x_{20}^4}-2\frac{x_{21}^i\,x_{21}^j}{x_{21}^4}\right]$$

$$\times\frac{1}{N_c^2}\left\langle\!\!\left\langle \mathrm{tr}\left[t^b V_{\underline{0}} t^a V_{\underline{1}}^\dagger\right]\left(U_{\underline{2}}^{j\,\mathrm{G}[2]}\right)^{ba} + \mathrm{c.c.}\right\rangle\!\!\right\rangle(z's)\Bigg\}$$

$$+\frac{\alpha_s N_c}{2\pi^2}\int\limits_{\frac{\Lambda^2}{s}}^{z}\frac{dz'}{z'}\int d^2x_2\,\frac{x_{10}^2}{x_{21}^2\,x_{20}^2}\left\{\frac{1}{N_c^2}\left\langle\!\!\left\langle \mathrm{tr}\left[t^b\, V_{\underline{0}}\, t^a\, V_{\underline{1}}^{i\,\mathrm{G}[2]\,\dagger}\right]\,\left(U_{\underline{2}}\right)^{ba}\right\rangle\!\!\right\rangle(z's) - \frac{C_F}{N_c^2}\left\langle\!\!\left\langle \mathrm{tr}\left[V_{\underline{0}}\, V_{\underline{1}}^{i\,\mathrm{G}[2]\,\dagger}\right]\right\rangle\!\!\right\rangle(z's) + \mathrm{c.c.}\right\}.$$

Full equations for the fundamental dipole amplitudes (don't close)

Useful properties of the large-N_c equations

Scaling between $\mathbf{G_2}$ and $\mathbf{\Gamma}_2$

$$\Gamma_2(s_{10}, s_{21}, \eta') - G_2^{(0)}(s_{10}, \eta') = G_2(s_{10}, \eta = \eta' + s_{10} - s_{21}) - G_2^{(0)}(s_{10}, \eta = \eta' + s_{10} - s_{21})$$

Boundary conditions for neighbors

$$\Gamma_2(s_{10}, s_{21} = s_{10}, \eta) = G_2(s_{10}, \eta)$$

$$\Gamma(s_{10}, s_{21} = s_{10}, \eta) = G(s_{10}, \eta)$$

PDE for $\mathbf{\Gamma}$

$$\frac{\partial^2 \Gamma(s_{10}, s_{21}, \eta')}{\partial s_{21}^2} + \frac{\partial^2 \Gamma(s_{10}, s_{21}, \eta')}{\partial s_{21} \partial \eta'} + \Gamma(s_{10}, s_{21}, \eta') = -3G(s_{21}, \eta') - 2G_2(s_{21}, \eta') - 2\Gamma_2(s_{10}, s_{21}, \eta')$$

Note the rescaled variables

$$\eta = \sqrt{\bar{\alpha}_s} \ln \frac{zs}{\Lambda^2} \qquad \eta' = \sqrt{\bar{\alpha}_s} \ln \frac{z's}{\Lambda^2}$$

$$s_{10} = \sqrt{\bar{\alpha}_s} \ln \frac{1}{x_{10}^2 \Lambda^2} \qquad s_{21} = \sqrt{\bar{\alpha}_s} \ln \frac{1}{x_{21}^2 \Lambda^2}$$

with $\bar{\alpha}_s = \dfrac{\alpha_s N_c}{2\pi}$

Full Solution

$$G_2(x_{10}^2, zs) = \frac{1}{\bar{\alpha}_s} \int \frac{\mathrm{d}\omega}{2\pi i} \int \frac{\mathrm{d}\gamma}{2\pi i} e^{\omega \ln\left(zsx_{10}^2\right) + \gamma \ln\left(\frac{1}{x_{10}^2 \Lambda^2}\right)} G_{2\omega\gamma}$$

$$\Gamma_2(x_{10}^2, x_{21}^2, z's) = \frac{1}{\bar{\alpha}_s} \int \frac{\mathrm{d}\omega}{2\pi i} \int \frac{\mathrm{d}\gamma}{2\pi i} \left[e^{\omega \ln\left(z'sx_{21}^2\right) + \gamma \ln\left(\frac{1}{x_{10}^2 \Lambda^2}\right)} \left(G_{2\omega\gamma} - G_{2\omega\gamma}^{(0)} \right) + e^{\omega \ln\left(z'sx_{10}^2\right) + \gamma \ln\left(\frac{1}{x_{10}^2 \Lambda^2}\right)} G_{2\omega\gamma}^{(0)} \right]$$

$$G(x_{10}^2, zs) = \frac{1}{\bar{\alpha}_s} \int \frac{\mathrm{d}\omega}{2\pi i} \int \frac{\mathrm{d}\gamma}{2\pi i} e^{\omega \ln\left(zsx_{10}^2\right) + \gamma \ln\left(\frac{1}{x_{10}^2 \Lambda^2}\right)} \left[\frac{1}{2} \frac{\omega\gamma}{\bar{\alpha}_s} \left(G_{2\omega\gamma} - G_{2\omega\gamma}^{(0)} \right) - 2G_{2\omega\gamma} \right]$$

$$\begin{aligned}
\Gamma(x_{10}^2, x_{21}^2, z's) = {} & \frac{1}{\sqrt{\bar{\alpha}_s}} \int \frac{\mathrm{d}\omega}{2\pi i} e^{\omega \ln\left(z'sx_{21}^2\right)} \left[\Gamma_\omega^+(x_{10}^2) e^{\delta_\omega^+ \ln\left(\frac{1}{x_{21}^2 \Lambda^2}\right)} + \Gamma_\omega^-(x_{10}^2) e^{\delta_\omega^- \ln\left(\frac{1}{x_{21}^2 \Lambda^2}\right)} \right] \\
& + \frac{1}{\bar{\alpha}_s} \int \frac{\mathrm{d}\omega}{2\pi i} \int \frac{\mathrm{d}\gamma}{2\pi i} e^{\omega \ln\left(z'sx_{21}^2\right) + \gamma \ln\left(\frac{1}{x_{21}^2 \Lambda^2}\right)} \left[\frac{(-\frac{3}{2}\omega\gamma + 4\bar{\alpha}_s) G_{2\omega\gamma} + \frac{3}{2}\omega\gamma G_{2\omega\gamma}^{(0)}}{\gamma^2 - \omega\gamma + \bar{\alpha}_s} \right] \\
& - \frac{1}{\bar{\alpha}_s} \int \frac{\mathrm{d}\omega}{2\pi i} \int \frac{\mathrm{d}\gamma}{2\pi i} \left[2e^{\omega \ln\left(z'sx_{21}^2\right) + \gamma \ln\left(\frac{1}{x_{10}^2 \Lambda^2}\right)} \left(G_{2\omega\gamma} - G_{2\omega\gamma}^{(0)} \right) + 2e^{\omega \ln\left(z'sx_{10}^2\right) + \gamma \ln\left(\frac{1}{x_{10}^2 \Lambda^2}\right)} G_{2\omega\gamma}^{(0)} \right]
\end{aligned}$$

Full Solution

$$G_{2\omega\gamma} = G^{(0)}_{2\omega\gamma} + \frac{\bar{\alpha}_s}{\omega\left(\gamma-\gamma^-_\omega\right)\left(\gamma-\gamma^+_\omega\right)}\left[2\left(\gamma-\delta^+_\omega\right)\left(G^{(0)}_{\delta^+_\omega\gamma}+2G^{(0)}_{2\delta^+_\omega\gamma}\right)-2\left(\gamma^+_\omega-\delta^+_\omega\right)\left(G^{(0)}_{\delta^+_\omega\gamma^+_\omega}+2G^{(0)}_{2\delta^+_\omega\gamma^+_\omega}\right)+8\delta^-_\omega\left(G^{(0)}_{2\omega\gamma}-G^{(0)}_{2\omega\gamma^+_\omega}\right)\right]$$

$$G^{(0)}(x^2_{10}, zs) = \frac{1}{\bar{\alpha}_s}\int\frac{\mathrm{d}\omega}{2\pi i}\int\frac{\mathrm{d}\gamma}{2\pi i}e^{\omega\ln\left(zsx^2_{10}\right)+\gamma\ln\left(\frac{1}{x^2_{10}\Lambda^2}\right)}G^{(0)}_{\omega\gamma}$$

$$G^{(0)}_2(x^2_{10}, zs) = \frac{1}{\bar{\alpha}_s}\int\frac{\mathrm{d}\omega}{2\pi i}\int\frac{\mathrm{d}\gamma}{2\pi i}e^{\omega\ln\left(zsx^2_{10}\right)+\gamma\ln\left(\frac{1}{x^2_{10}\Lambda^2}\right)}G^{(0)}_{2\omega\gamma}$$

$$\Gamma^+_\omega(x^2_{10}) = \frac{1}{(\bar{\alpha}_s)^{3/2}}\frac{e^{-\delta^+_\omega\ln\left(\frac{1}{x^2_{10}\Lambda^2}\right)}}{\delta^+_\omega-\delta^-_\omega}\int\frac{\mathrm{d}\gamma}{2\pi i}e^{\gamma\ln\left(\frac{1}{x^2_{10}\Lambda^2}\right)}\frac{\frac{1}{2}\omega\delta^+_\omega}{\gamma-\delta^+_\omega}\left[G_{2\omega\gamma}\left(\gamma^2-\omega\gamma+4\bar{\alpha}_s-8\bar{\alpha}_s\frac{\delta^-_\omega}{\omega}\right)-G^{(0)}_{2\omega\gamma}\left(\gamma^2-\omega\gamma+4\bar{\alpha}_s\right)\right]$$

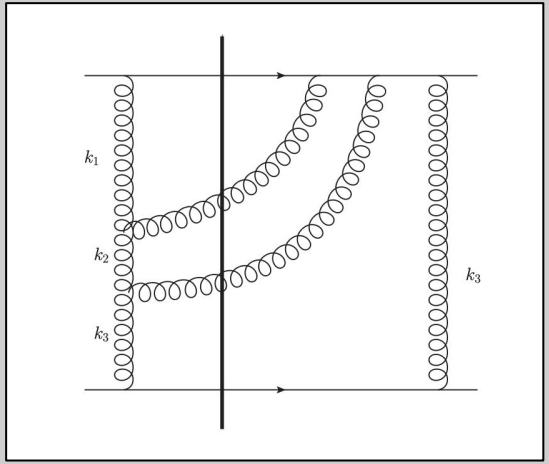
$$\Gamma^-_\omega(x^2_{10}) = \frac{1}{(\bar{\alpha}_s)^{3/2}}\frac{e^{-\delta^-_\omega\ln\left(\frac{1}{x^2_{10}\Lambda^2}\right)}}{\delta^-_\omega-\delta^+_\omega}\int\frac{\mathrm{d}\gamma}{2\pi i}e^{\gamma\ln\left(\frac{1}{x^2_{10}\Lambda^2}\right)}\frac{\frac{1}{2}\omega\delta^-_\omega}{\gamma-\delta^-_\omega}\left[G_{2\omega\gamma}\left(\gamma^2-\omega\gamma+4\bar{\alpha}_s-8\bar{\alpha}_s\frac{\delta^+_\omega}{\omega}\right)-G^{(0)}_{2\omega\gamma}\left(\gamma^2-\omega\gamma+4\bar{\alpha}_s\right)\right]$$

$$\delta^\pm_\omega = \frac{\omega}{2}\left[1\pm\sqrt{1-\frac{4\bar{\alpha}_s}{\omega^2}}\right] \qquad \gamma^\pm_\omega = \frac{\omega}{2}\left[1\pm\sqrt{1-\frac{16\bar{\alpha}_s}{\omega^2}\sqrt{1-\frac{4\bar{\alpha}_s}{\omega^2}}}\right]$$

Disagreement with BER

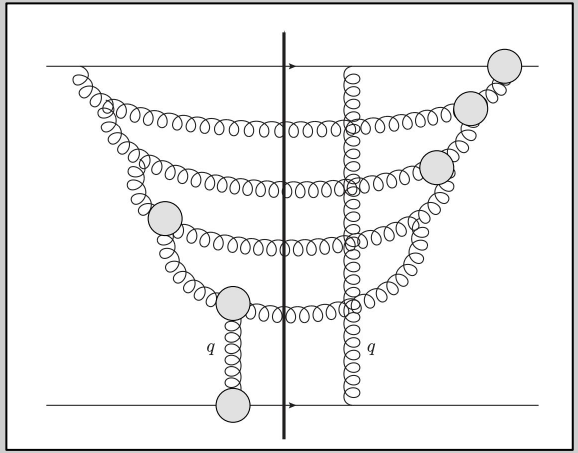
One softest gluon → Bremsstrahlung theorem

Two softest gluons $\xrightarrow{BER?}$ must be a ladder



$k_3 \ll k_2 \ll k_1$

Two brehmsstrahlung gluon correction to ladder?



Higher order diagram in a similar spirit

How do these enter IREE?