

Orbital Angular Momentum at Small- x

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Overview

- * Motivation: proton spin puzzle
- * Polarized observables/LCOT approach
- * OAM distributions in terms of polarized S-matrices
- * Polarized S-matrix (+moment) evolution
- * Numerical results:
 - * Small- x asymptotics of OAM distributions
 - * OAM/helicity PDF ratios

Motivation: proton spin puzzle

* Jaffe-Manohar sum rule: (Jaffe and Manohar, 1990)

$$\int_0^1 dx \left[\frac{1}{2} \Delta \Sigma(x, Q^2) + L_{q+\bar{q}}(x, Q^2) + \Delta G(x, Q^2) + L_G(x, Q^2) \right] = \frac{1}{2}$$

* Current estimates of proton spin

$$\frac{1}{2} \int_{0.001}^1 \Delta \Sigma(x, Q^2 = 10 \text{ GeV}^2) \approx [0.15, 0.20] \quad \int_{0.05}^1 \Delta G(x, Q^2 = 10 \text{ GeV}^2) \approx [0.13, 0.26]$$

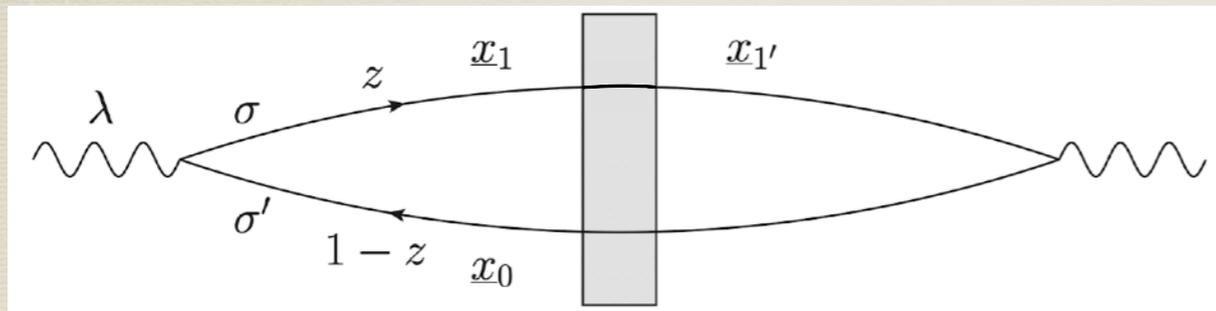
[arXiv:1501.01220](https://arxiv.org/abs/1501.01220)

* Missing spin needs to be constrained at small- x

* Focus on OAM distributions at small- x

Polarized observables

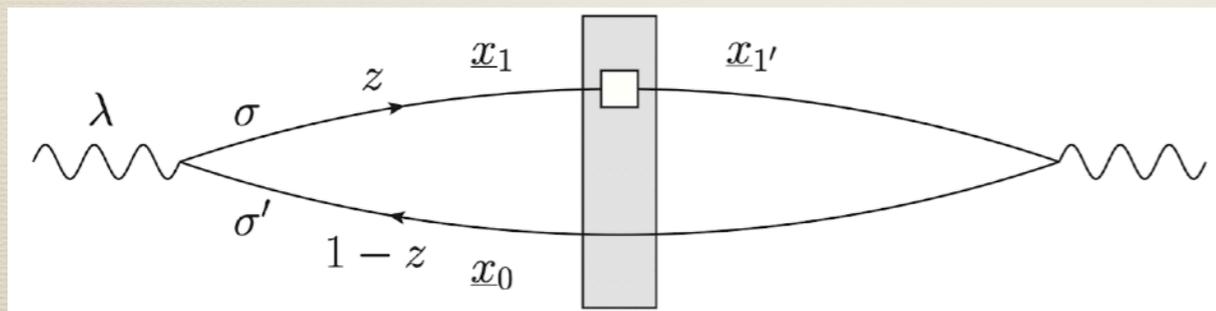
- * Unpolarized observables dominated by eikonal contributions



$$S \sim \text{tr} \left[V_{\underline{x}_1} V_{\underline{x}_0}^\dagger \right]$$

- * Polarized observables dominated by *sub*-eikonal contributions

[arXiv:2204.11898](https://arxiv.org/abs/2204.11898)



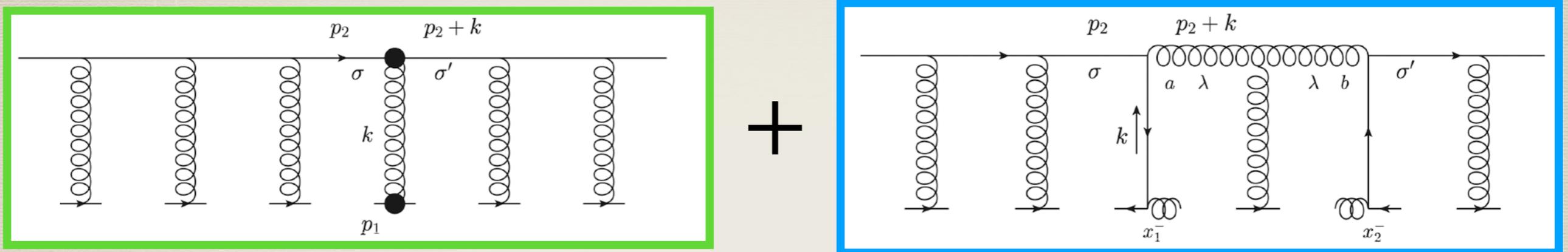
$$S \sim \text{tr} \left[V_{\underline{x}_1, \underline{x}_1'}^{\text{pol}} V_{\underline{x}_0}^\dagger \right]$$

$$V_{\underline{x}}[x_f^-, x_i^-] \equiv \mathcal{P} \exp \left[ig \int_{x_i^-}^{x_f^-} dx^- A^+(0^+, x^-, \underline{x}) \right]$$

$$V_{\underline{x}}[\infty, -\infty] \equiv V_{\underline{x}}$$

Polarized propagators

- Sub-eikonal propagators given by sub-eikonal vertex sandwiched between semi-infinite Wilson lines (LCOT approach) arXiv:1511.06737



$$V_{\underline{x}, \underline{y}; \sigma, \sigma'}^{\text{pol}} = \sigma \delta_{\sigma, \sigma'} \delta^2(\underline{x} - \underline{y}) \left(V_{\underline{x}}^{q[1]} + V_{\underline{x}}^{G[1]} \right) + \delta_{\sigma, \sigma'} \left(V_{\underline{x}, \underline{y}}^{G[2]} + V_{\underline{x}}^{q[2]} \delta^2(\underline{x} - \underline{y}) \right)$$

- Polarized dipole amplitudes

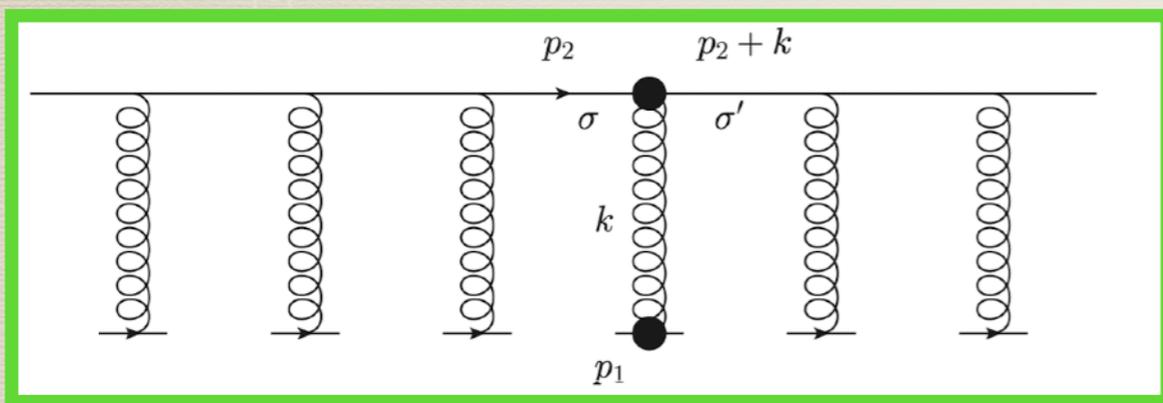
$$Q_{10}(zs) = \frac{s}{2N_c} \text{Re} \left\langle \text{tr} \left[V_{\underline{x}_0} V_{\underline{x}_1}^{\text{pol}[1]\dagger} + V_{\underline{x}_1}^{\text{pol}[1]} V_{\underline{x}_0} \right] \right\rangle \quad G_{10}^i(zs) = \frac{s}{2N_c} \text{Re} \left\langle \text{tr} \left[V_{\underline{x}_0}^\dagger V_{\underline{x}_1}^{iG[2]} + V_{\underline{x}_1}^{iG[2]\dagger} V_{\underline{x}_0} \right] \right\rangle$$

Simplified version

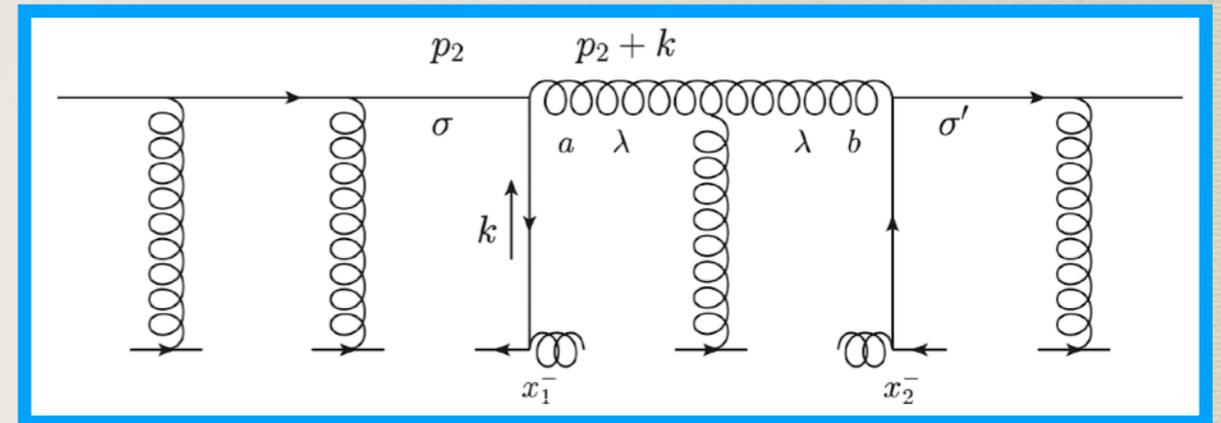
arXiv:2204.11898

Polarized propagators

- Sub-eikonal propagators given by sub-eikonal vertex sandwiched between semi-infinite Wilson lines (LCOT approach) arXiv:1511.06737



+



$$V_{\underline{x}, \underline{y}; \sigma, \sigma'}^{\text{pol}} = \sigma \delta_{\sigma, \sigma'} \delta^2(\underline{x} - \underline{y}) \left(V_{\underline{x}}^{q[1]} + V_{\underline{x}}^{G[1]} \right) + \delta_{\sigma, \sigma'} \left(V_{\underline{x}, \underline{y}}^{G[2]} + V_{\underline{x}}^{q[2]} \delta^2(\underline{x} - \underline{y}) \right)$$

- Polarized dipole amplitudes

NEW

$$Q_{10}(zs) = \frac{s}{2N_c} \text{Re} \left\langle \text{tr} \left[V_{\underline{x}_0} V_{\underline{x}_1}^{\text{pol}[1]\dagger} + V_{\underline{x}_1}^{\text{pol}[1]} V_{\underline{x}_0} \right] \right\rangle \quad G_{10}^i(zs) = \frac{s}{2N_c} \text{Re} \left\langle \text{tr} \left[V_{\underline{x}_0}^\dagger V_{\underline{x}_1}^{iG[2]} + V_{\underline{x}_1}^{iG[2]\dagger} V_{\underline{x}_0} \right] \right\rangle$$

Simplified version

arXiv:2204.11898

OAM operator

* Generic OAM operator

$$L_z(x, Q^2) = \frac{d}{dx} L_z(Q^2)$$

$$L_z(Q^2) = \int \frac{d^2 b_\perp db^- d^2 k_\perp dk^+}{(2\pi)^3} (\underline{b} \times \underline{k})_z W(k, b)$$

where Wigner functions are

$$W^{q, SIDIS}(k, b) = 2 \sum_X \int d^2 r dr^- e^{ik \cdot r} \left\langle \bar{\psi}_\alpha \left(b - \frac{1}{2} r \right) V_{\underline{b} - \frac{1}{2} \underline{r}} [b^- - \frac{1}{2} r^-, \infty] | X \rangle \left(\frac{1}{2} \gamma^+ \right)_{\alpha\beta} \langle X | V_{\underline{b} + \frac{1}{2} \underline{r}} [\infty, b^- + \frac{1}{2} r^-] \psi_\beta \left(b + \frac{1}{2} r \right) \right\rangle$$

$$W^{G dip}(k, b) = \frac{4}{x P^+} \int d\xi^- d^2 \xi_\perp e^{ix P^+ \xi^- - ik \cdot \xi} \left\langle \text{tr} \left[F^{+i} \left(b - \frac{1}{2} \xi \right) \mathcal{U}^{[+]} \left[b - \frac{1}{2} \xi, b + \frac{1}{2} \xi \right] F^{+i} \left(b + \frac{1}{2} \xi \right) \mathcal{U}^{[-]} \left[b + \frac{1}{2} \xi, b - \frac{1}{2} \xi \right] \right] \right\rangle$$

Future and past-pointing staples are (in $A^- = 0$ gauge)

$$\mathcal{U}^{[+]}[0, \xi] = V_{\underline{0}}[0^-, +\infty] V_{\underline{\xi}}[+\infty, \xi^-]$$

$$\mathcal{U}^{[-]}[\xi, 0] = V_{\underline{\xi}}[\xi^-, -\infty] V_{\underline{0}}[-\infty, 0^-]$$

[arXiv:1111.3547](https://arxiv.org/abs/1111.3547)

[arXiv:1901.07453](https://arxiv.org/abs/1901.07453)

[arXiv:1207.5221](https://arxiv.org/abs/1207.5221)

Polarized dipole amplitudes

- * Impact-integrated polarized dipole amplitudes appear in calculations:

$$Q(x_{10}^2, zS) = \int d^2 \left(\frac{\underline{x}_0 + \underline{x}_1}{2} \right) Q_{10}(zS)$$

$$\int d^2 \left(\frac{\underline{x}_0 + \underline{x}_1}{2} \right) G_{10}^i(zS) = x_{10}^i G_1(x_{10}^2, zS) + \epsilon^{ij} x_{10}^j G_2(x_{10}^2, zS)$$

- * For OAM, we also need the first x_1 -moments:

$$\int d^2 x_1 x_1^i Q_{10}(zS) = x_{10}^i I_3(x_{10}^2, zS)$$

Other structures possible but don't contribute

$$\int d^2 x_1 x_1^i G_{10}^j(zS) = \epsilon^{ij} I_4(x_{10}^2, zS) + \epsilon^{ik} x_{10}^k x_{10}^j I_5(x_{10}^2, zS) + \epsilon^{jk} x_{10}^k x_{10}^i I_6(x_{10}^2, zS)$$

OAM distributions at small- x

- * At small- x , the OAM distributions can be written in terms of the regular and moment polarized dipole amplitudes

$$L_{q+\bar{q}}(x, Q^2) = \frac{N_c N_f}{2\pi^3} \int_{\frac{\Lambda^2}{s}}^1 \frac{dz}{z} \int_{\frac{1}{zs}}^{\min\{\frac{1}{zQ^2}, \frac{1}{\Lambda^2}\}} \frac{dx_{10}^2}{x_{10}^2} [Q - 3G_2 - I_3 - 2I_4 + I_5 + 3I_6](x_{10}^2, zs)$$

$$L_G(x, Q^2) = -\frac{N_c}{\alpha_s \pi} \left\{ [4 + x_{10}^2 \nabla_{10}^2] [I_4(x_{10}^2, zs) + I_5(x_{10}^2, zs)] + \left[2 + 2x_{10}^2 \frac{\partial}{\partial x_{10}^2} \right] [I_5(x_{10}^2, zs) + I_6(x_{10}^2, zs)] \right\}_{zs=\frac{Q^2}{x}, x_{10}^2=\frac{1}{Q^2}}$$

- * Compare to helicity PDFs

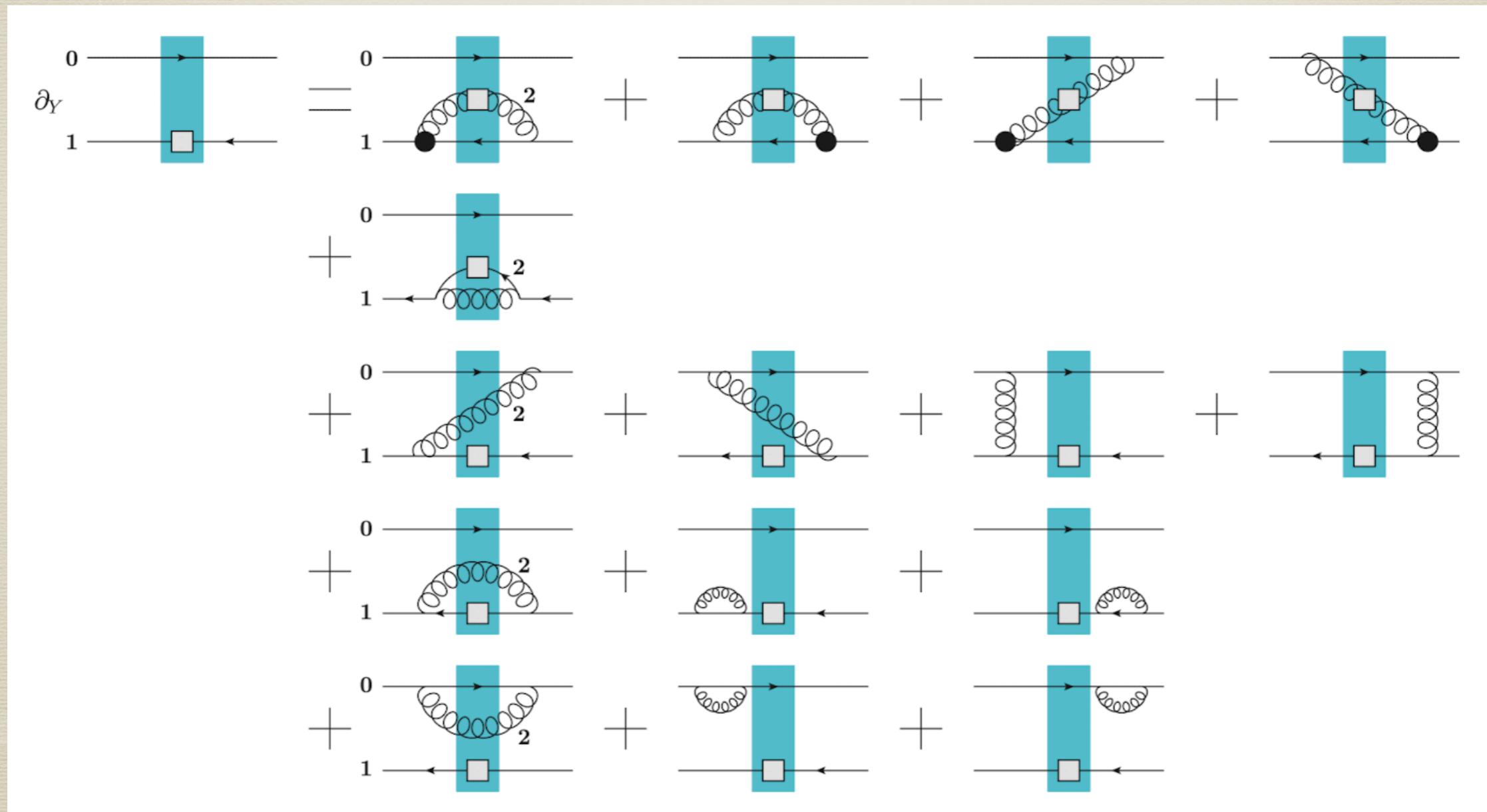
$$\Delta\Sigma(x, Q^2) = -\frac{N_c N_f}{2\pi^3} \int_{\frac{\Lambda^2}{s}}^1 \frac{dz}{z} \int_{\frac{1}{zs}}^{\min\{\frac{1}{zQ^2}, \frac{1}{\Lambda^2}\}} \frac{dx_{10}^2}{x_{10}^2} [Q + 2G_2](x_{10}^2, zs)$$

$$\Delta G(x, Q^2) = \frac{2N_c}{\alpha_s \pi} \left\{ \left[1 + x_{10}^2 \frac{\partial}{\partial x_{10}^2} \right] G_2 \left(x_{10}^2, zs = \frac{Q^2}{x} \right) \right\}_{x_{10}^2=\frac{1}{Q^2}}$$

[arXiv:2204.11898](https://arxiv.org/abs/2204.11898)

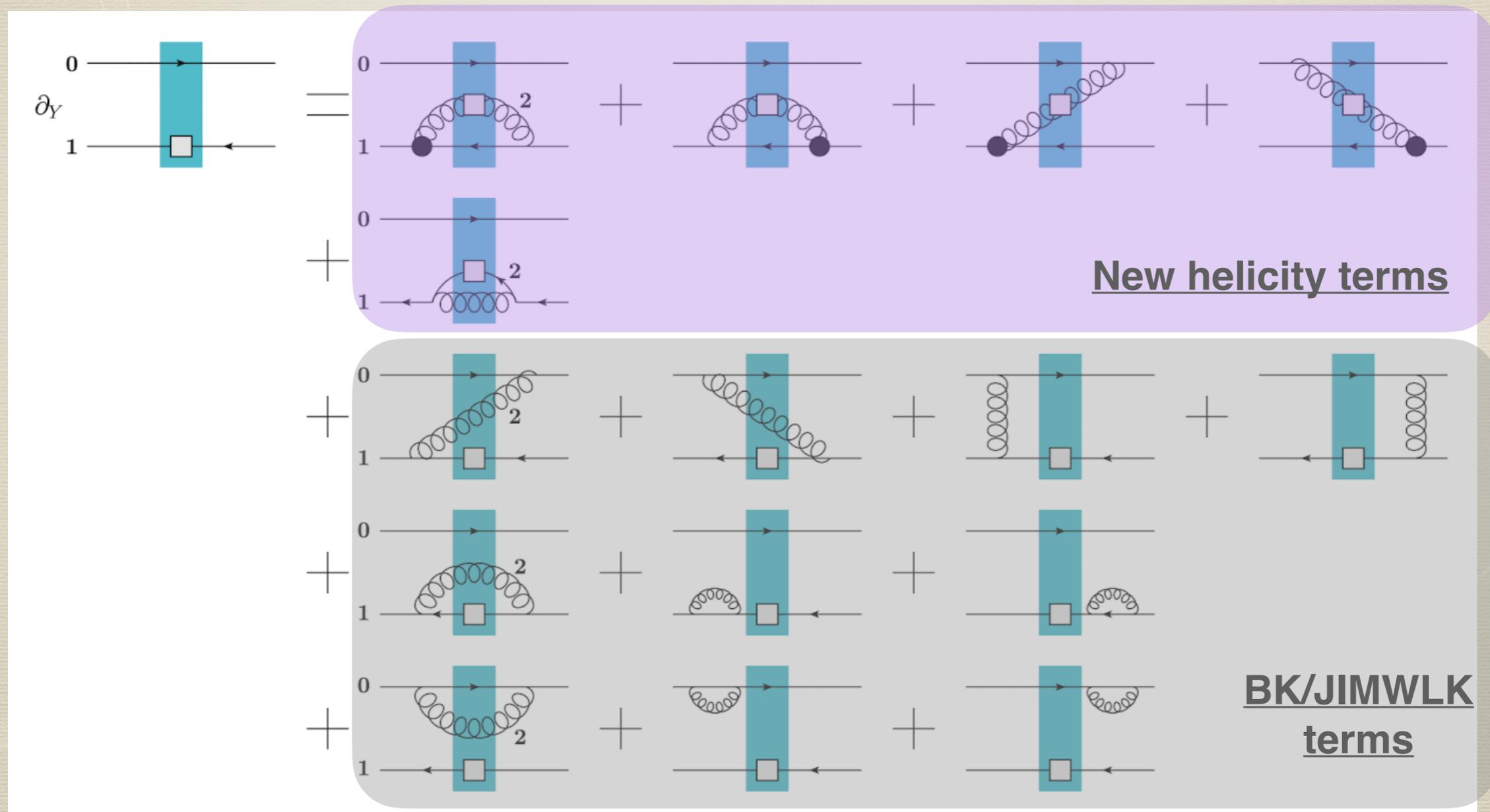
Polarized S-matrix evolution

- * Helicity evolution similar to BK/JIMWLK
(resummation parameter is $\alpha_s \ln^2 s$)



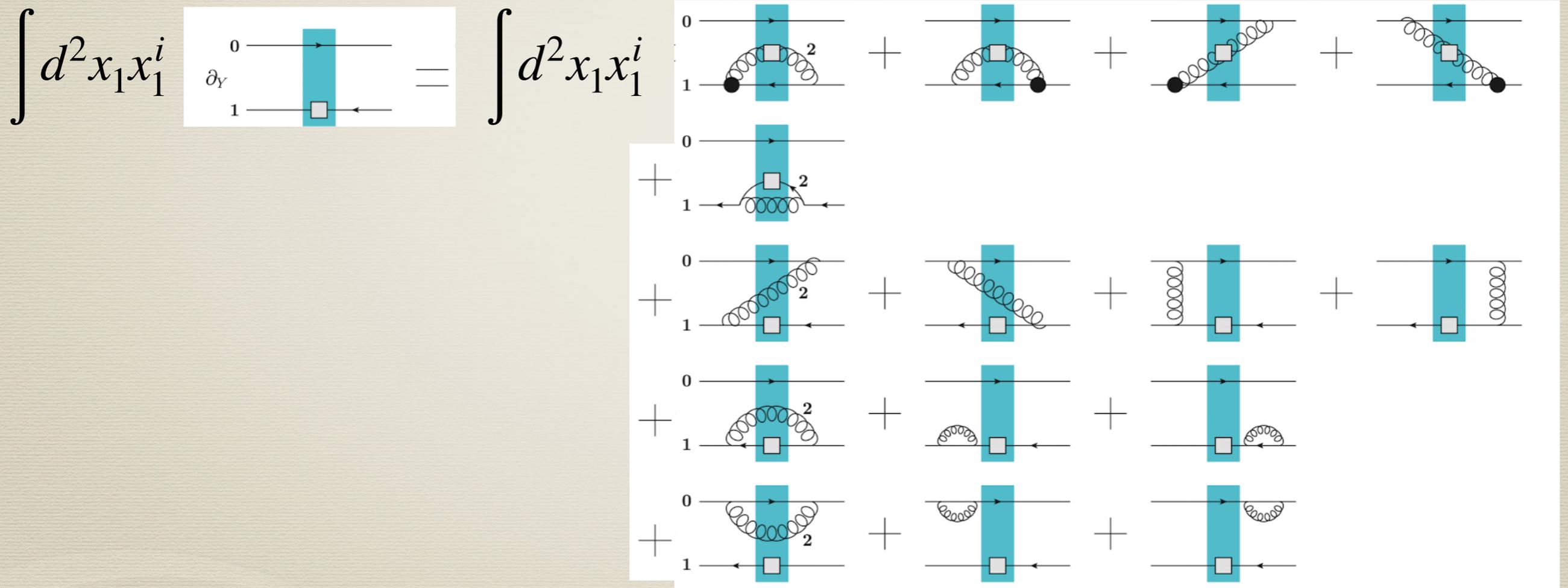
Polarized S-matrix evolution

- * Helicity evolution similar to BK/JIMWLK
(resummation parameter is $\alpha_s \ln^2 s$)



Moment amplitude evolution

* Moment evolution: integrate over x_1 and match tensor structures



Moment amplitude evolution

Large- N_c limit

$$\begin{aligned}
 \begin{pmatrix} I_3 \\ I_4 \\ I_5 \\ I_6 \end{pmatrix} (x_{10}^2, z s) &= \begin{pmatrix} I_3 \\ I_4 \\ I_5 \\ I_6 \end{pmatrix}_0 (x_{10}^2, z s) \\
 &+ \frac{\alpha_s N_c}{4\pi} \int_{\frac{1}{s x_{10}^2}}^z \frac{dz'}{z'} \int_{\frac{1}{z' s}}^{x_{10}^2} \frac{dx_{21}^2}{x_{21}^2} \begin{pmatrix} 2\Gamma_3 - 4\Gamma_4 + 2\Gamma_5 + 6\Gamma_6 - 2\Gamma_2 \\ 0 \\ 0 \\ 0 \end{pmatrix} (x_{10}^2, x_{21}^2, z' s) \\
 &+ \frac{\alpha_s N_c}{4\pi} \int_{\frac{\Lambda^2}{s}}^z \frac{dz'}{z'} \int_{\max[x_{10}^2, \frac{1}{z' s}]}^{\min[\frac{z}{z'} x_{10}^2, \frac{1}{\Lambda^2}]} \frac{dx_{21}^2}{x_{21}^2} \begin{pmatrix} 4 & -4 & 2 & 6 & -4 & -6 \\ 0 & 4 & 2 & -2 & 0 & 1 \\ -2 & 2 & -1 & -3 & 2 & 3 \\ 0 & 0 & 0 & 0 & 2 & 4 \end{pmatrix} \begin{pmatrix} I_3 \\ I_4 \\ I_5 \\ I_6 \\ G \\ G_2 \end{pmatrix} (x_{21}^2, z' s).
 \end{aligned}$$

Large N_c

$Q = G$

Moment amplitude evolution

Large- N_c limit

$$\begin{aligned}
 \begin{pmatrix} I_3 \\ I_4 \\ I_5 \\ I_6 \end{pmatrix} (x_{10}^2, z s) &= \begin{pmatrix} I_3 \\ I_4 \\ I_5 \\ I_6 \end{pmatrix}_0 (x_{10}^2, z s) \\
 &+ \frac{\alpha_s N_c}{4\pi} \int_{\frac{1}{s x_{10}^2}}^z \frac{dz'}{z'} \int_{\frac{1}{z' s}}^{x_{10}^2} \frac{dx_{21}^2}{x_{21}^2} \begin{pmatrix} 2\Gamma_3 - 4\Gamma_4 + 2\Gamma_5 + 6\Gamma_6 - 2\Gamma_2 \\ 0 \\ 0 \\ 0 \end{pmatrix} (x_{10}^2, x_{21}^2, z' s) \\
 &+ \frac{\alpha_s N_c}{4\pi} \int_{\frac{\Lambda^2}{s}}^z \frac{dz'}{z'} \int_{\max[x_{10}^2, \frac{1}{z' s}]}^{\min[\frac{z}{z'} x_{10}^2, \frac{1}{\Lambda^2}]} \frac{dx_{21}^2}{x_{21}^2} \begin{pmatrix} 4 & -4 & 2 & 6 & -4 & -6 \\ 0 & 4 & 2 & -2 & 0 & 1 \\ -2 & 2 & -1 & -3 & 2 & 3 \\ 0 & 0 & 0 & 0 & 2 & 4 \end{pmatrix} \begin{pmatrix} I_3 \\ I_4 \\ I_5 \\ I_6 \\ G \\ G_2 \end{pmatrix} (x_{21}^2, z' s).
 \end{aligned}$$

Mixing with “regular” amplitudes!

Numerical results: small-x asymptotics

- * OAM distributions have same asymptotics as helicity counterparts (large N_c)

[arXiv:2204.11898](https://arxiv.org/abs/2204.11898)

$$L_{q+\bar{q}}(x, Q^2) \sim L_G(x, Q^2) \sim \Delta\Sigma(x, Q^2) \sim \Delta G(x, Q^2) \sim \left(\frac{1}{x}\right)^{3.66\sqrt{\frac{\alpha_s N_c}{2\pi}}}$$

- * Previously found

$$L_{q+\bar{q}}(x, Q^2) \sim \Delta\Sigma(x, Q^2) \sim \left(\frac{1}{x}\right)^{2.31\sqrt{\frac{\alpha_s N_c}{2\pi}}}$$

$$L_G(x, Q^2) \sim \Delta G(x, Q^2) \sim \left(\frac{1}{x}\right)^{1.88\sqrt{\frac{\alpha_s N_c}{2\pi}}}$$

[arXiv:1901.07453](https://arxiv.org/abs/1901.07453)

Numerical results: OAM/hPDF ratios

- * Prediction for OAM/hPDF ratios (large Q^2)
(Boussarie, 2019)

[arXiv:1904.02693](https://arxiv.org/abs/1904.02693)

$$L_{q+\bar{q}}(x, Q^2) \approx -\frac{1}{1+\alpha} \Delta\Sigma(x, Q^2)$$

$$L_G(x, Q^2) \approx -\frac{2}{1+\alpha} \Delta G(x, Q^2)$$

$$\Delta\Sigma, \Delta G \sim \left(\frac{1}{x}\right)^\alpha$$

- * Previously found (Kovchegov, 2019)

$$L_{q+\bar{q}}(x, Q^2) = -\Delta\Sigma(x, Q^2)$$

$$L_G(x, Q^2) = (\# \ln Q^2) \Delta G(x, Q^2)$$

[arXiv:1901.07453](https://arxiv.org/abs/1901.07453)

Numerical results: OAM/hPDF ratios

- * Can numerically investigate OAM/hPDF ratios as a function of Q^2
- * BFKL-motivated ansätze for distribution functions

$$\ln \left| \Delta\Sigma \left(y = \ln \frac{1}{x}, Q^2 \right) \right| = \alpha y + \beta + \gamma \ln y$$

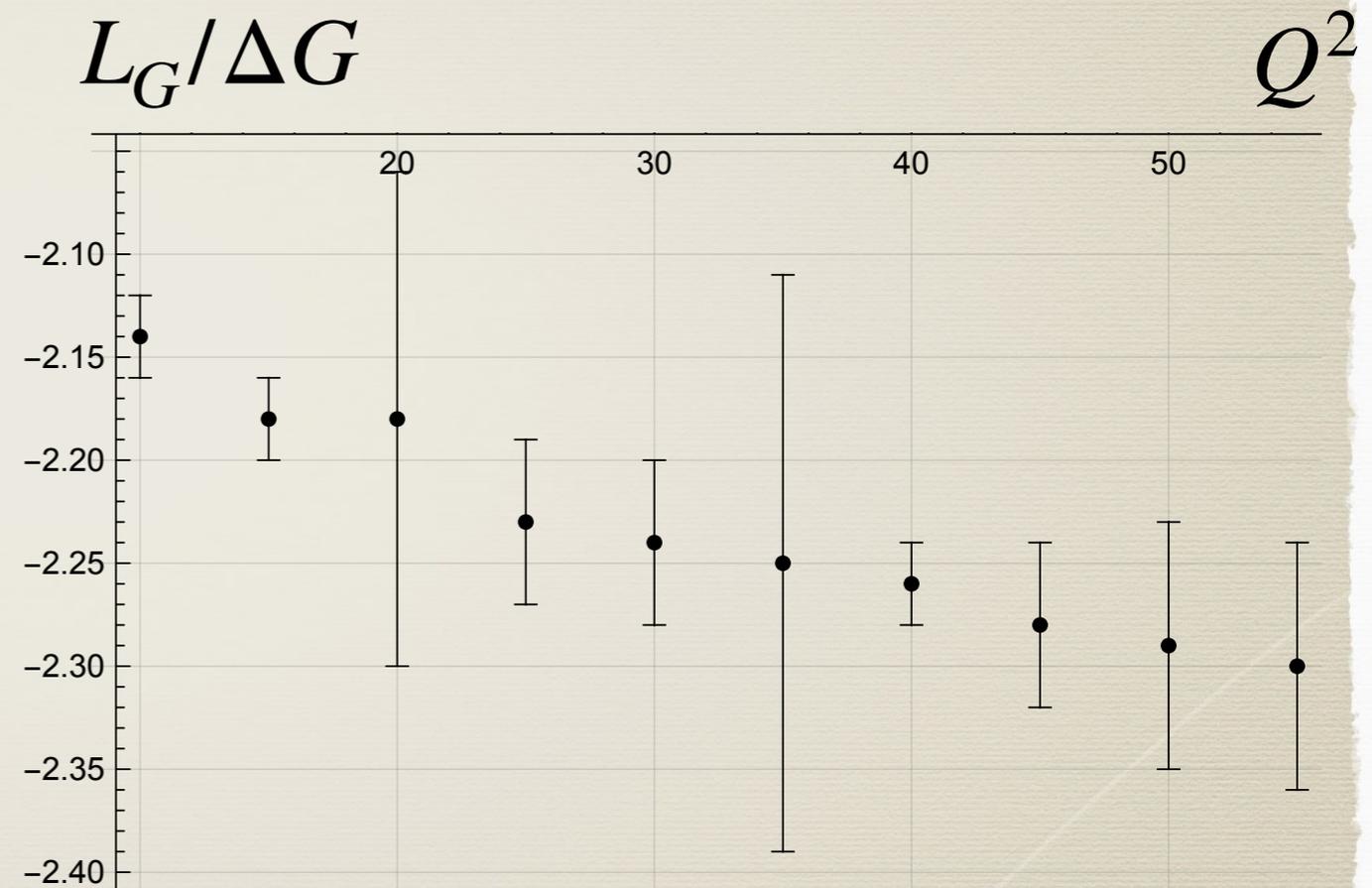
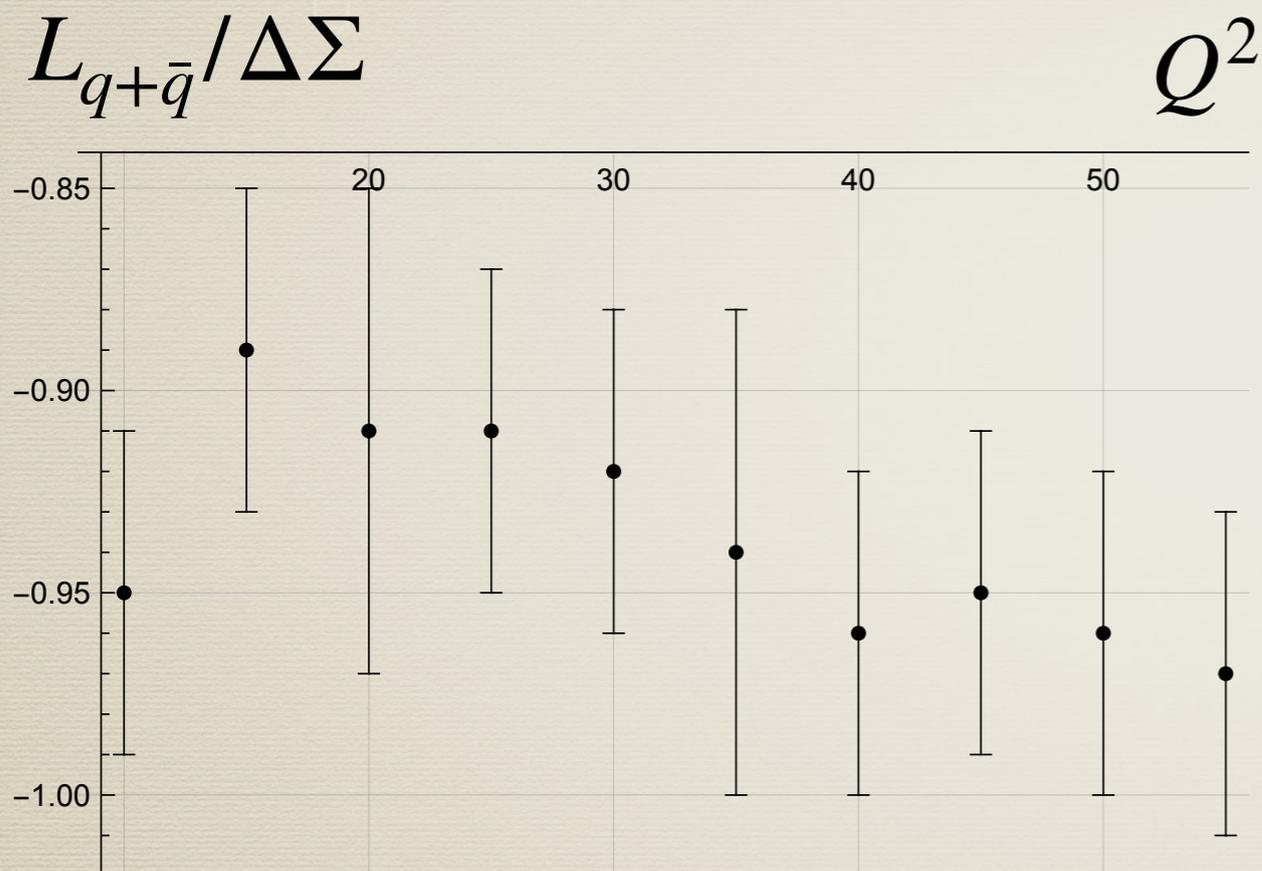
$$\ln \left| L_{q+\bar{q}} \left(y = \ln \frac{1}{x}, Q^2 \right) \right| = \bar{\alpha} y + \bar{\beta} + \bar{\gamma} \ln y$$

* We see

$$\alpha = \bar{\alpha}, \gamma = \bar{\gamma} \Rightarrow \lim_{x \rightarrow 0} \left| \frac{L_{q+\bar{q}}(x, Q^2)}{\Delta\Sigma(x, Q^2)} \right| = \exp(\bar{\beta} - \beta)$$

Numerical results: OAM/hPDF ratios

* Can numerically investigate OAM/hPDF ratios as a function of Q^2



* Predictions

$$L_{q+\bar{q}}(x, Q^2) \approx -\frac{1}{1+\alpha} \Delta\Sigma(x, Q^2)$$

$$L_G(x, Q^2) \approx -\frac{2}{1+\alpha} \Delta G(x, Q^2)$$

Summary

- * Revised helicity evolution technique works for OAM at small- x
- * Requires introduction of new *moment* polarized dipole amplitudes
 - * Derived novel large N_c evolution equations for moment amplitudes
- * Determined small- x asymptotics of OAM distributions

$$L_{q+\bar{q}}(x, Q^2) \sim L_G(x, Q^2) \sim \Delta\Sigma(x, Q^2) \sim \Delta G(x, Q^2) \sim \left(\frac{1}{x}\right)^{3.66\sqrt{\frac{\alpha_s N_c}{2\pi}}}$$

- * Numerically calculated OAM/hPDF ratio in the asymptotic limit

* Backup

Polarized Wilson lines

$$V_{\underline{x}, \underline{y}; \sigma, \sigma'} = \sigma \delta_{\sigma, \sigma'} V_{\underline{x}}^{\text{pol}[1]} \delta^2(\underline{x} - \underline{y}) + \delta_{\sigma, \sigma'} V_{\underline{x}, \underline{y}}^{\text{pol}[2]}$$

$$V_{\underline{x}}^{\text{pol}[1]} = V_{\underline{x}}^{\text{G}[1]} + V_{\underline{x}}^{\text{q}[1]}$$

$$V_{\underline{x}, \underline{y}}^{\text{pol}[2]} = \delta^2(\underline{x} - \underline{y}) V_{\underline{x}}^{\text{q}[2]} + V_{\underline{x}, \underline{y}}^{\text{G}[2]}$$

$$V_{\underline{x}}^{\text{G}[1]} = \frac{i g P^+}{s} \int_{-\infty}^{\infty} dx^- V_{\underline{x}}[\infty, x^-] F^{12}(x^-, \underline{x}) V_{\underline{x}}[x^-, -\infty],$$

$$V_{\underline{x}}^{\text{q}[1]} = \frac{g^2 P^+}{2s} \int_{-\infty}^{\infty} dx_1^- \int_{x_1^-}^{\infty} dx_2^- V_{\underline{x}}[\infty, x_2^-] t^b \psi_\beta(x_2^-, \underline{x}) U_{\underline{x}}^{ba}[x_2^-, x_1^-] [\gamma^+ \gamma^5]_{\alpha\beta} \bar{\psi}_\alpha(x_1^-, \underline{x}) t^a V_{\underline{x}}[x_1^-, -\infty],$$

$$V_{\underline{x}, \underline{y}}^{\text{G}[2]} = -\frac{i P^+}{s} \int_{-\infty}^{\infty} dz^- d^2 z V_{\underline{x}}[\infty, z^-] \delta^2(\underline{x} - \underline{z}) \check{D}^i(z^-, \underline{z}) D^i(z^-, \underline{z}) V_{\underline{y}}[z^-, -\infty] \delta^2(\underline{y} - \underline{z}),$$

$$V_{\underline{x}}^{\text{q}[2]} = -\frac{g^2 P^+}{2s} \int_{-\infty}^{\infty} dx_1^- \int_{x_1^-}^{\infty} dx_2^- V_{\underline{x}}[\infty, x_2^-] t^b \psi_\beta(x_2^-, \underline{x}) U_{\underline{x}}^{ba}[x_2^-, x_1^-] [\gamma^+]_{\alpha\beta} \bar{\psi}_\alpha(x_1^-, \underline{x}) t^a V_{\underline{x}}[x_1^-, -\infty].$$

$$V_{\underline{z}}^{i\text{G}[2]} \equiv \frac{P^+}{2s} \int_{-\infty}^{\infty} dz^- V_{\underline{z}}[\infty, z^-] \left[D^i(z^-, \underline{z}) - \check{D}^i(z^-, \underline{z}) \right] V_{\underline{z}}[z^-, -\infty].$$

CGC Averaging

* Standard *polarization-independent* CGC averaging

$$\langle \hat{\mathcal{O}}(b, r) \rangle = \frac{1}{2P^+} \int \frac{d^2\Delta d\Delta^+}{(2\pi)^3} e^{ib \cdot \Delta} \left\langle P + \frac{\Delta}{2} \left| \hat{\mathcal{O}}(0, r) \right| P - \frac{\Delta}{2} \right\rangle.$$

$$\langle \dots \rangle = \frac{1}{2} \sum_{S_L} \frac{1}{2P^+ V^-} \langle P, S_L | \dots | P, S_L \rangle$$

Moment amplitude asymptotics

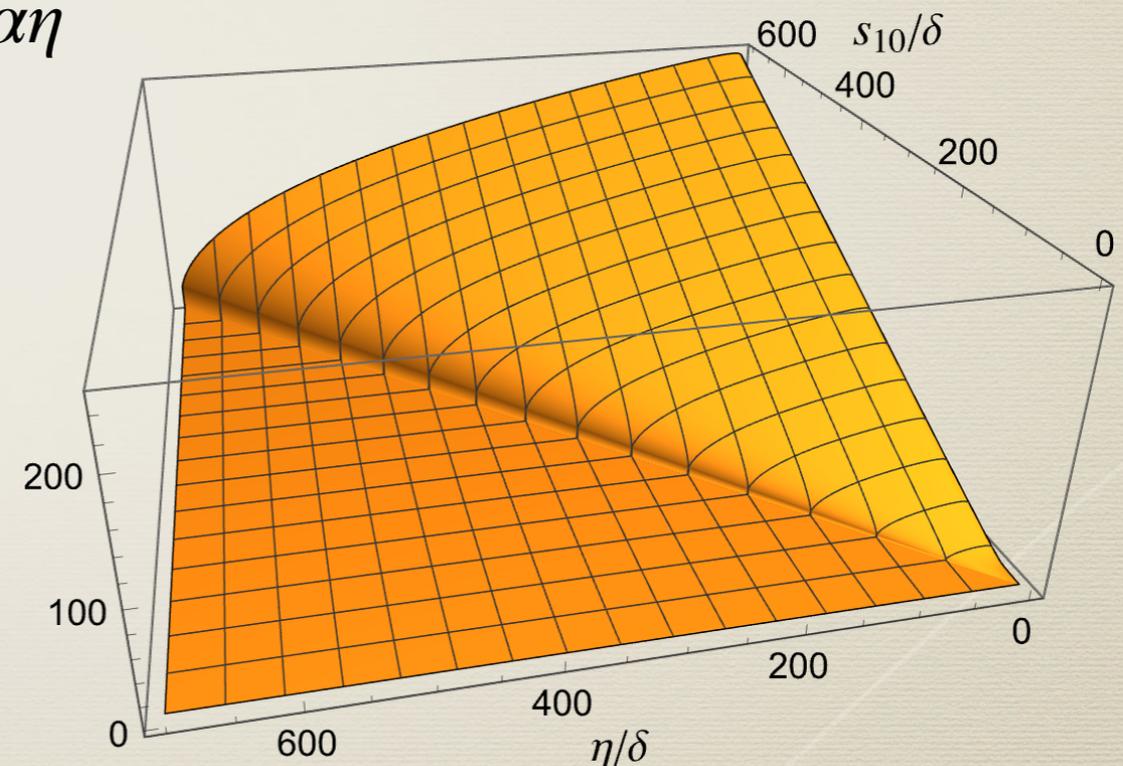
- * Define logarithmic coordinates $\eta = \sqrt{\frac{\alpha_s N_c}{2\pi}} \ln \frac{zs}{\Lambda^2}$ $s_{10} = \sqrt{\frac{\alpha_s N_c}{2\pi}} \ln \frac{1}{x_{10}^2 \Lambda^2}$
- * Moment amplitudes are exponential in η at large η

* Enables ansatz: $I(0, \eta) \propto e^{\alpha \eta}$

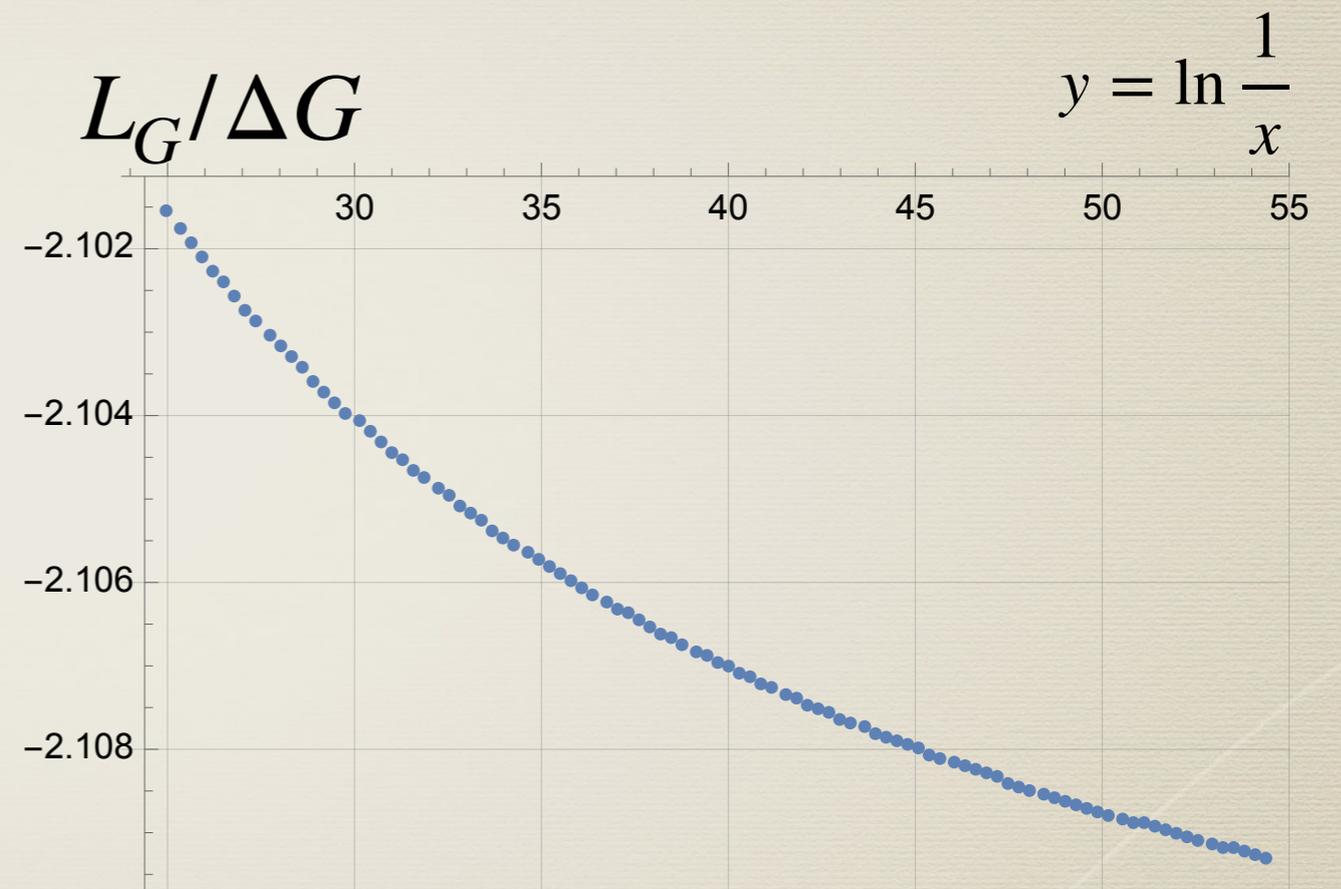
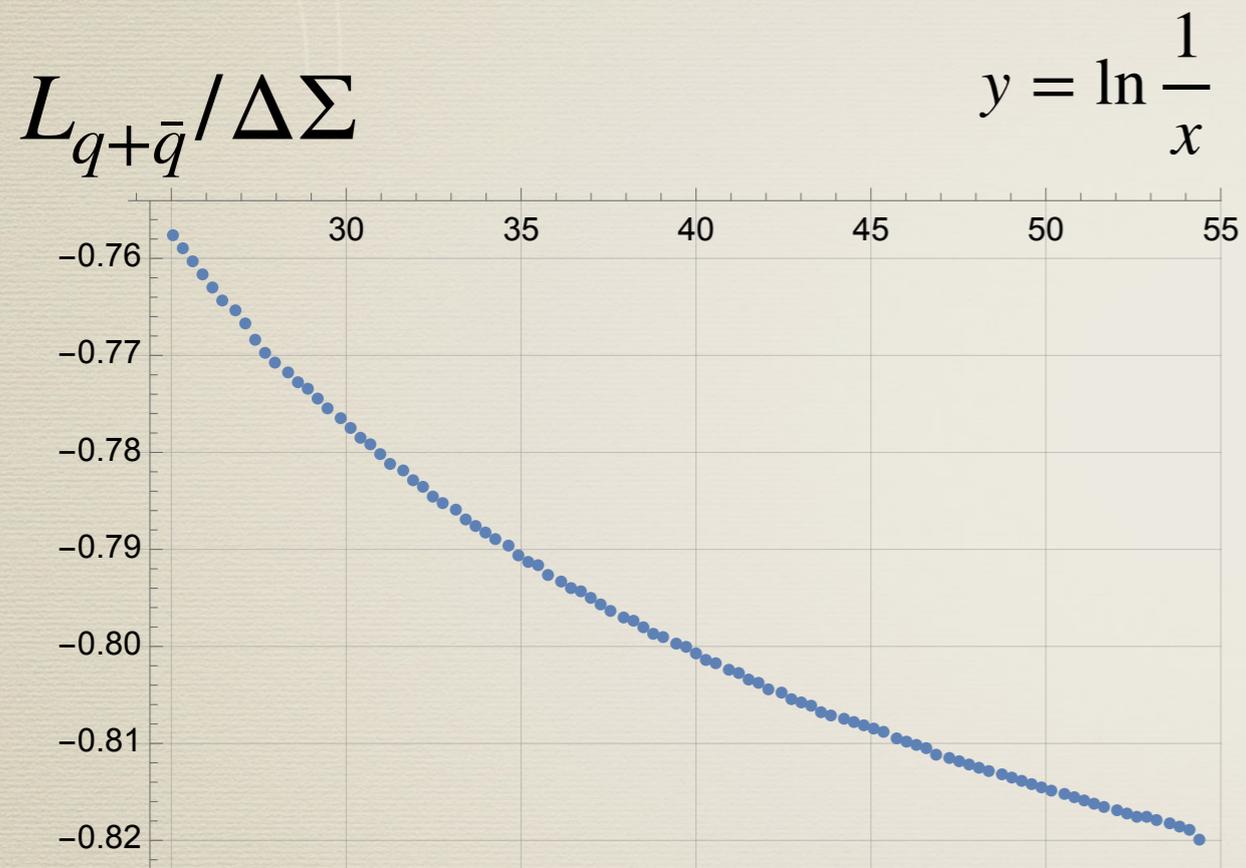
- * Can extract α numerically

find it agrees w/ helicity

$$\alpha_h = 3.66 \sqrt{\frac{\alpha_s N_c}{2\pi}}$$



OAM/hPDF ratios



$$\Delta\eta = 0.032$$