Renormalization of twist-two operators in covariant gauge to three loops in QCD

Tong-Zhi Yang

with Thomas Gehrmann and Andreas von Manteuffel
Based on 2302.00022

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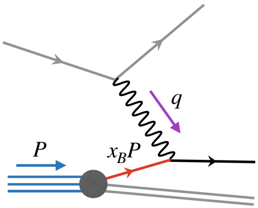
Parton denstities and splitting functions



- Bjorken variable

$$x_B = \frac{-q^2}{2P \cdot q}$$

- Factorization

$$\sigma \sim \sum_a f_{a|N}(x_B) \otimes \hat{\sigma}_a(x_B)$$

- Quark parton density in axial gauge

$$f_{q|N}(x_B) = \int \frac{dt}{2\pi} e^{-i\, x_B\, t\Delta\cdot p} \langle N(P)|\bar{\psi}(t\Delta)\frac{\not{\Delta}}{2}\psi(0)|N(P)\rangle \, , \; \Delta^2 = 0$$

- Splitting functions (SFs) govern the DGLAP evolutions of PDFs

$$\frac{df_{i|N}}{d\ln\mu} = 2\sum_k P_{ik} \otimes f_{k|N}$$

Why 4-loop SFs for the evolutions of N3LO PDFs?

- Expand PDFs and SFs with $a_s = \alpha_s/(4\pi)$

$$f_{i|N} = f_{i|N}^{(0)} + f_{i|N}^{(1)} a_s + \cdots + f_{i|N}^{(3)} a_s^3 + \cdots$$
$$P_{ij} = P_{ij}^{(0)} a_s + \cdots + P_{ij}^{(3)} a_s^4 + \cdots$$

- Evolution of a_s

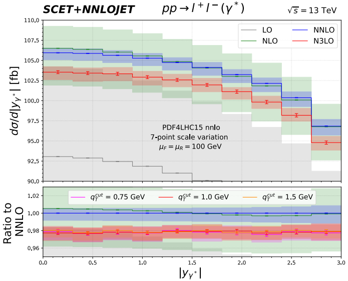
$$\frac{da_s}{d\ln\mu} = -2(a_s^2\beta_0 + a_s^3\beta_1 + \cdots)$$

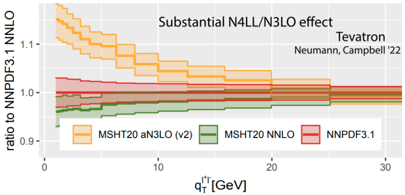
- A consistent evolution of N3LO PDFs requires 4-loop SFs

$$f_{i|N}^{(3)} \frac{da_s^3}{d\ln\mu} = f_{i|N}^{(3)}(-6a_s^4\beta_0 + \cdots) = \sum_k P_{ik}^{(3)} a_s^4 \otimes f_{k|N}^{(0)} + \cdots$$

Motivations for four-loop splitting functions

- Several $\hat{\sigma}$ are available at N3LO, but N3LO PDFs are missing
- The fields in fitting N3LO PDFs are active
  - MSHT20 aN3LO talks by Thomas Cridge,Lucian Harland-Lang
  - NNPDF in progress towards aN3LO talk by Giacomo Magni
  - CT are planning talk by Pavel Nadolsky





- Scale uncertainty at N3LO using NNLO PDF remains at 1% level talk by Thomas Gehrmann
- Seems that aN3LO PDF introduces large corrections to σ talk by Tobias Neumman

Splitting functions & Anomalous dimensions

- Mellin transformation

$$f_q(n) = -\int_0^1 dz\, z^{n-1} f_q(z)\,, \quad \gamma_{ij}(n) = -\int_0^1 dz\, z^{n-1} P_{ij}(z)$$

- DGLAP evolution in n-space

$$\frac{d}{d\ln\mu} f_q(n,\mu^2) = -2\sum_j \gamma_{qj}(n)\, f_j(n,\mu^2)$$

- PDFs in n-space are hadronic operator matrix elements (OMEs)

$$f_q(n) \sim \langle N(P)|\, \bar{\psi}\not{\Delta}(\Delta\cdot D)^{n-1}\psi\, |N(P)\rangle$$

Twist-two operators

According to the flavor group,

- Non-singlet: a single operator

$$O_{q,k} = \frac{i^{n-1}}{2}\left[\bar{\psi}_i \not{\Delta} (\Delta \cdot D)_{ij}^{n-1} \frac{\lambda_k}{2} \psi_j\right], k = 3, 8, \cdots n_f^2 - 1$$

  $\lambda_k/2$ is the diagonal generator of the flavor group

- Singlet: two operators

$$O_q = \frac{i^{n-1}}{2}\left[\bar{\psi}_i \not{\Delta} (\Delta \cdot D)_{ij}^{n-1} \psi_j\right],$$
$$O_g = -\frac{i^{n-2}}{2}\left[\Delta_{\mu_1} G_{a,\mu}^{\mu_1} (\Delta \cdot D)_{ab}^{n-2} \Delta_{\mu_n} G_b^{\mu_n \mu}\right]$$

  $G_a^{\mu\nu}$ is the gluon field strength tensor.

Renormalization of twist-two operators

- The non-singlet operator $O_{q,k}$ is multiplicatively renormalized,

$$O_{q,k}^{\mathsf{R}} = Z^{\mathsf{ns}} O_{q,k}^{\mathsf{B}}$$

- The two singlet operators mix under renormalization,

$$\begin{pmatrix} O_q \\ O_g \end{pmatrix}^{\mathsf{R,\ naive}} = \begin{pmatrix} Z_{qq} & Z_{qg} \\ Z_{gq} & Z_{gg} \end{pmatrix} \begin{pmatrix} O_q \\ O_g \end{pmatrix}^{\mathsf{B}}$$

- Evolution equation for the renormalization constants

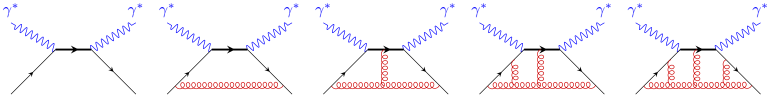
$$\frac{dZ_{ij}}{d\ln\mu} = -2 \sum_{k=q,\,g} \gamma_{ik}(n) Z_{kj}$$

- Extract anomalous dimensions from the renormalization constants

$$Z_{ij} = \delta_{ij} + \sum_{l=1}^{\infty} a_s^l \frac{1}{l\,\epsilon} \gamma_{ij}^{(k-1)} + \cdots$$

DIS method vs OME method

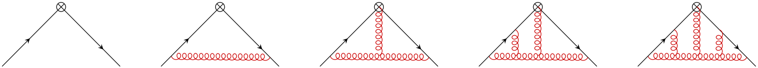
- Forward DIS (gauge invariant)



2 diagrams | 10 diagrams | 143 diagrams | 2922 diagrams | 73634 diagrams

Shrinking the heavy lines into effective vertices

- Partonic off-shell OME (fewer diagrams, easier integrals)



1 diagram | 3 diagrams | 37 diagrams | 684 diagrams | 15901 diagrams

- Off-shell OMEs are not gauge invariant (due to off-shell external gluons), physical operators mix with unknown gauge-variant (GV) operators
- Main goal: find all GV operators or their Feynman rules

Once all GV operators are known, the off-shell OME method
can be used to determine SFs efficiently

A bit of history about the calculations of SFs

- The first one-loop results are from off-shell OMEs
  - Non-singlet and singlet [D.J. Gross, F. Wilczek, 1973, 1974]
- Two-loop results
  - Non-singlet from off-shell OMEs[E.G. Floratos et al. 1977]
  - Singlet: inconsistences from off-shell OMEs (Flaws due to omitting GV operators)[E.G. Floratos et al. 1978] and from DIS (Correct) [Furmanski and Petronzio, 1980]
- The first three-loop results are from DIS
  - Non-singlet and Singlet[Moch, Vermaseren and Vogt, 2004,2004]
- Partial four-loop results
  - Non-singlet with $n \leq 16$ from off-shell OMEs[S. Moch et al. 2017]
  - Singlet with $n \leq 8$ from DIS[S. Moch et al. 2021]
  - Pure singlet with $n \leq 20$ from off-shell OMEs[G. Falcioni et al. 2023]

Only a few low-n results at four-loop are available

The off-shell OMEs method is much more efficient than DIS method

Significant efforts in deriving GV operators

- [D.J. Gross, F. Wilczek, 1974] pointed out possible mixing with GV operators
- [J.A. Dixon and J.C. Taylor, 1974] constructed order g_s GV operators, not clear how to generalize to higher order
- [Joglekar and Lee, 1975] gave a general theorem about the renormalization of gauge invariant operators No explicit results were given
- [J. C. Collins and R. J. Scalise, 1994] studied the renormalization of energy-momentum tensor in detail and pointed out subtleties of theorem by Joglekar and Lee
- [G. Falcioni and F. Herzog, 2022] constructed the GV operators for a fixed n based on a generalized BRST symmetry. Promising, however more and more number of operators are needed for higher n.
- This talk: A new framework which enables the derivation of all-n GV operator (counterterm) Feynman rules to all loop orders

A new framework of deriving gauge-variant operators

- Guiding principles:
  - A twist-two operator has infinite mass dimension when $n \to \infty$
  - Infinite GV operators are required to renormalize the physical operators
  - Some GV operators only contribute starting at higher-loop order
- Extend the naive renormalization of the operator O_g,

$$O_g^{\mathsf{R}} = Z_{gq} O_q^{\mathsf{B}} + Z_{gg} O_g^{\mathsf{B}} + Z_{gA} \left(O_A^{\mathsf{B}} + O_B^{\mathsf{B}} + O_C^{\mathsf{B}} \right) + [ZO]_g^{\mathrm{GV}}$$

$$Z_{gA} = \mathcal{O}(a_s), \ [ZO]_g^{\mathrm{GV}} = \sum_{l=2}^{\infty} a_s^l \, [ZO]_g^{\mathrm{GV},\,(l)}$$

- O_A (gluon fields only), O_B(quark+gluon fileds), O_C (ghost + gluon fields). $[ZO]_g^{\mathrm{GV},\,(l)}$: collection of counterterms
- Similarly for O_q

$$O_q^{\mathsf{R}} \ = \ Z_{qq} O_q^{\mathsf{B}} + Z_{qg} O_g^{\mathsf{B}} + Z_{qA} O_{ABC}^{B} + [ZO]_q^{\mathrm{GV}}, \ O_{ABC} = O_A + O_B + O_C$$

Derive Feynman rules from off-shell OMEs

- Key idea: derive Feynman rules instead of GV operators themselves
- Consider off-shell OMEs with $2j+m$-gluon external states

$$\langle j|O_g^{\mathsf{R}}|j+m\,g\rangle_{\mathsf{1PI}}^{\mu_1\cdots\mu_m} = \langle j|(Z_{gq}O_q^{\mathsf{B}} + Z_{gg}O_g^{\mathsf{B}})|j+m\,g\rangle_{\mathsf{1PI}}^{\mu_1\cdots\mu_m}$$
$$+ \langle j|Z_{gA}O_{ABC}^{\mathsf{B}}|j+m\,g\rangle_{\mathsf{1PI}}^{\mu_1\cdots\mu_m} + \langle j|\,[ZO]_g^{\mathrm{GV}}\,|j+m\,g\rangle_{\mathsf{1PI}}^{\mu_1\cdots\mu_m}\,,\; j=q,g \text{ or c}$$

- Expand OMEs order by order in loops and legs

$$\langle j|O|j+m\,g\rangle^{\mu_1\cdots\mu_m} = \sum_{l=1}^{\infty}\Big[\langle j|O|j+m\,g\rangle^{\mu_1\cdots\mu_m,\,(l),\,(m)}\Big]\,\Big(\frac{\alpha_s}{4\pi}\Big)^l g_s^m$$

- Left: UV renormalized and IR finite $\to$ no poles in ϵ
- Right: Each term is UV divergent, but the sum should be finite
  - Requirement of the finiteness allows to determine couterterm Feynman rules of unknown GV operators order by order

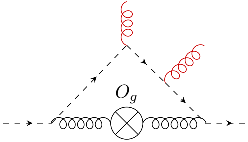
Determine Feynman rules for O_{ABC}

- As an example, consider two ghosts + m-gluon external states and expand to one-loop order

$$Z_{gA}^{(1)} \langle c|O_C|c + m\,g\rangle_{1\text{PI}}^{\mu_1\cdots\mu_m,\,(0),\,(m)} = -\left[\langle c|O_g|c + m\,g\rangle_{1\text{PI}}^{\mu_1\cdots\mu_m,\,(1),\,(m),\,\text{B}}\right]_{1/\epsilon}$$

- $Z_{gA}^{(1)}$ is a m-independent constant and can be determined from $m = 0$

$$Z_{gA}^{(1)} = \frac{-C_A}{\epsilon}\frac{1}{n(n-1)}$$

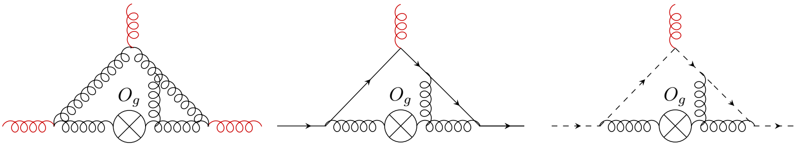


Sample digram to extract Feynman rules for O_C with $m = 2$

Determine Feynman rules for $[ZO]_g^{\mathrm{GV},\,(2)}$

- As an example, consider two ghosts + m-gluon external states and expand to two-loop order

$$\langle c|\,[ZO]_g^{\mathrm{GV},\,(2)}\,|c+m\,g\rangle_{\mathsf{1PI}}^{\mu_1\cdots\mu_m,\,(0),\,(m)} = -\Bigg\{\Bigg[\langle c|O_g|c+m\,g\rangle_{\mathsf{1PI}}^{\mu_1\cdots\mu_m,\,(2),\,(m),\,\mathsf{B}}$$
$$+\left(Z_c^{(1)}+\frac{mZ_g^{(1)}}{2}+Z_{gg}^{(1)}-\frac{\beta_0(m+2)}{2\epsilon}\right)\langle c|O_g|c+m\,g\rangle_{\mathsf{1PI}}^{\mu_1\cdots\mu_m,\,(1),\,(m),\,\mathsf{B}}$$
$$+Z_{gA}^{(1)}\,\langle c|O_{AC}|c+m\,g\rangle_{\mathsf{1PI}}^{\mu_1\cdots\mu_m,\,(1),\,(m),\,\mathsf{B}}+\cdots\Bigg]_{\mathsf{div}}\Bigg\}$$

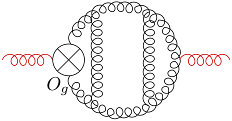


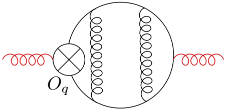
Sample digrams to extract Feynman rules for $[ZO]_g^{\mathrm{GV},\,(2)}$ with $m=1$

Three-loop singlet splitting functions from off-shell OMEs

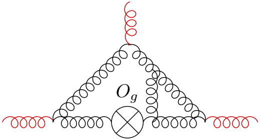
Sample Feynman diagrams

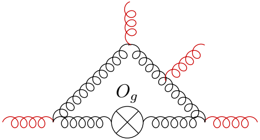
- Two-point diagrams with physical operators insertion



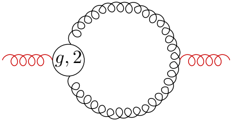


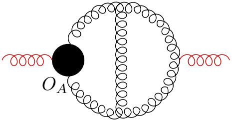
- Multi-leg diagrams to infer GV counterterm Feynman rules





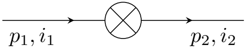
- Two-point diagrams with GV counterterm insertions





Computational methods

- Non-standard terms appearing in the Feynman rules
- Example: Feynman rules for O_q at lowest order



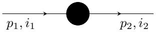
$$\rightarrow \not{\Delta}\,(\Delta \cdot p_1)^{\,n-1}$$

- Sum non-standard term into a linear propagator using a tracing parameter x[J. Ablinger et al. 2012] See also the talk by Kay

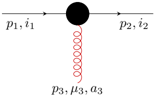
$$(\Delta \cdot p)^{n-1} \rightarrow \sum_{n=1}^{\infty} x^n (\Delta \cdot p)^{n-1} = \frac{x}{1 - x\Delta \cdot p}$$

- Derive differential equation(DE) with respect to the parameter x
- Expand DE to high orders in $x \rightarrow 0$ limit and then reconstruct the final results, or solve the DE in terms of special functions

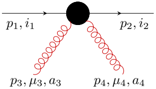
Sample results: Feynman rules for O_B



$$\to 0$$



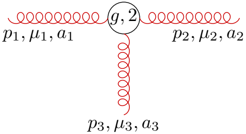
$$\to -\frac{1+(-1)^n}{2}\, g_s\, \Delta^{\mu_3}\, T^{a_3}_{i_2 i_1}\, \not{\Delta}\left(\Delta\cdot(p_1+p_2)\right)^{n-2}$$



$$\to \frac{1+(-1)^n}{-8}\, g_s^2\, \Delta^{\mu_3}\Delta^{\mu_4}\left(T^{a_3}T^{a_4} - T^{a_4}T^{a_3}\right)_{i_2 i_1} \not{\Delta} \sum_{j_1=0}^{n-3} \Bigg(3\left(\Delta\cdot(p_1+p_2)\right)^{-j_1+n-3}\left[(-\Delta\cdot p_3)^{j_1} - (-\Delta\cdot p_4)^{j_1}\right] - (-\Delta\cdot p_4)^{j_1}\,(\Delta\cdot p_3)^{-j_1+n-3}\Bigg)$$

Feynman rules for O_B operator with all momenta flowing into the vertex

Sample results for two-loop counterterm Feynman rules

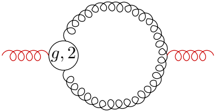


$$
\begin{aligned}
\to\ & 2 i g_s C_A^2 f^{a_1 a_2 a_3} \frac{1+(-1)^n}{256 n(n-1)} \frac{(\Delta\cdot p_1)^{n-2}}{\Delta\cdot p_2} \Bigg(-\Delta^{\mu_1}\Delta^{\mu_2}\Delta^{\mu_3} \sum_{i=1}^{3} p_i^2 \\
& + \Delta^{\mu_2}\Delta^{\mu_3} p_1^{\mu_1} \Delta\cdot p_1 + \Delta^{\mu_1}\Delta^{\mu_3} p_2^{\mu_2} \Delta\cdot p_2 + \Delta^{\mu_1}\Delta^{\mu_2} p_3^{\mu_3} \Delta\cdot p_3 \Bigg) \\
& \times \left\{ \frac{F_{-2,0} + (1-\xi)\, F_{-2,1}}{\epsilon^2} + \frac{F_{-1,0} + (1-\xi)\, F_{-1,1}}{\epsilon} \right\}
\end{aligned}
$$

- ϵ-dependent and ξ-dependent
- $F_{-1,0}$ contains generalized harmonic sums to weight-2
- Weight-3 polylogarithms in x-space

Two-point OMEs with two-loop counterterm insertions

- For a fixed n, normal IBP, but need to reduce integrals with very high numerator degree
- All-n, IBP reduction with polylogarithms?



- Consider a general term of two-loop counterterms with 3-gluon vertex

$$ig_s f^{a_1 a_2 a_3} C_A^2 \Delta^{\mu_1} \Delta^{\mu_2} \Delta^{\mu_3} p_1^2 \sum_{m=0}^{n-3} a_{mn} (\Delta \cdot p_1)^m (\Delta \cdot p_2)^{n-3-m} + \cdots$$

where a_{mn} is known only for fixed m, n

- New idea: replace a_{mn} by another tracing parameter t

$$h(x, t) = \sum_{n=3}^{\infty} x^n \sum_{m=0}^{n-3} t^m (\Delta \cdot p_1)^m (\Delta \cdot p_2)^{n-3-m} = \frac{x^3}{(1 - x\, t \Delta \cdot p_1)(1 - x \Delta \cdot p_2)}$$

- Insert h into two-point diagrams: $\langle g | h(x,t) | g \rangle = \sum_{n=3}^{\infty} x^n \sum_{m=0}^{n-3} t^m c_{mn}$
- $\langle g | \sum_{m=0}^{n-3} a_{mn} (\Delta \cdot p_1)^m (\Delta \cdot p_2)^{n-3-m} | g \rangle = \sum_{m=0}^{n-3} a_{mn} c_{mn}$

Evaluate OMEs to any fixed n efficiently and reconstruct the full-n results

Results

- Combine results for all two-point OMEs (with O_q, O_g and GV counterterm insertions)
- Splitting functions: confirm the ξ-independence explicitly for the first time and recover the well known results in the literature
  - non-singlet:
$$\gamma_{\text{ns}}^{(2)} - \gamma_{\text{ns}}^{(2)}[\text{MVV}] = 0$$
  - singlet:
$$\gamma_{qq}^{(2)} - \gamma_{qq}^{(2)}[\text{VMV}] = 0\,, \gamma_{qg}^{(2)} - \gamma_{qg}^{(2)}[\text{VMV}] = 0$$
$$\gamma_{gq}^{(2)} - \gamma_{gq}^{(2)}[\text{VMV}] = 0\,, \gamma_{gg}^{(2)} - \gamma_{gg}^{(2)}[\text{VMV}] = 0$$

Summary

- For off-shell OMEs, renormalization of physical operators require unknown GV operators
- Developed a new framework to infer splitting functions
  - Two-point OMEs are used to extract splitting functions
  - Multi-point (≥ 3) OMEs are required to determine counterterm Feynman rules of the GV operators
- Applied it to derive 3-loop singlet splitting functions and recovered the well known results in the literature
- Can be directly applied to derive four-loop singlet splitting functions

Loops \ Legs	2	3	4	5
0		$[ZO]_g^{\mathrm{GV},(2)}$	O_{ABC}	O_q, O_g
1	$[ZO]_g^{\mathrm{GV},(2)}$	O_{ABC}	O_g	
2	O_{ABC}	O_g		
3	O_q, O_g			

6

4

Decomposition of splitting functions

- The general structure of quark splitting functions,

$$P_{q_i q_k} = \delta_{ik} P^V_{qq} + P^S_{qq}, \quad P_{q_i \bar{q}_k} = \delta_{ik} P^V_{q\bar{q}} + P^S_{q\bar{q}}$$

- Non-singlet and singlet splitting functions

$$\text{Non-singlet: } P^{\pm}_{\text{ns}} = P^V_{qq} \pm P^V_{q\bar{q}}, \quad P^V_{\text{ns}} = P^- + \overbrace{n_f (P^S_{qq} - P^S_{q\bar{q}})}^{P^s_{\text{ns}}}$$

$$\text{Singlet: } P_{qq} = P^+ + \underbrace{n_f (P^S_{qq} + P^S_{q\bar{q}})}_{P_{\text{ps}}}, \; P_{qg}, \; P_{gq}, \; P_{gg}$$

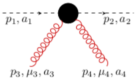
- Evolution of PDFs

$$\frac{dT_i^{\pm}}{d\ln\mu} = 2P^{\pm}_{\text{ns}} \otimes T_i^{\pm}, \; \frac{d\sum_{k=1}^{n_f} q_k^-}{d\ln\mu} = 2P^V_{\text{ns}} \otimes \sum_{k=1}^{n_f} q_k^- \,, \; i = 3, 8, \cdots n_f^2 - 1$$

$$T_3^{\pm} = u^{\pm} - d^{\pm}, T_8^{\pm} = u^{\pm} + d^{\pm} - 2s^{\pm}, \cdots, q_k^{\pm} = q_k \pm \bar{q}_k \,,$$

$$\frac{d}{d\ln\mu} \begin{pmatrix} \Sigma \\ g \end{pmatrix} = \begin{pmatrix} P_{qq} & P_{qg} \\ P_{gq} & P_{gg} \end{pmatrix} \otimes \begin{pmatrix} \Sigma \\ g \end{pmatrix}, \quad \Sigma = \sum_{k=1}^{n_f} q_k^+$$

All-n Feynman rules for O_C



$$
\begin{aligned}
&\to \frac{1}{24}\frac{1+(-1)^n}{2} g_s^2 \Delta^{\mu_3}\Delta^{\mu_4}\Big\{ f^{a_1a_3a}f^{a_2a_4a}\Big(6\,(-\Delta\cdot p_4)^{n-2}+6\,(\Delta\cdot p_3)^{n-2} \\
&+6\,(\Delta\cdot(p_1+p_3))^{n-2}+6\,(\Delta\cdot(p_2+p_3))^{n-2}-\sum_{j_1=0}^{n-2}\Big[\\
&+[(-\Delta\cdot p_3)^{j_1}+(-\Delta\cdot p_4)^{j_1}][3\,(\Delta\cdot p_1)^{n-j_1-2} \\
&+3\,(\Delta\cdot p_2)^{n-j_1-2}+(\Delta\cdot(p_1+p_2))^{n-j_1-2}] \\
&+9[(\Delta\cdot p_1)^{n-j_1-2}+(-\Delta\cdot p_2)^{n-j_1-2}][(\Delta\cdot(-p_2-p_3))^{j_1}+(\Delta\cdot(p_1+p_3))^{j_1}]\Big] \\
&+13\sum_{j_1=0}^{n-2}\sum_{j_2=0}^{j_1}\Big[(-\Delta\cdot p_2)^{j_1-j_2}(\Delta\cdot p_1)^{n-j_1-2}[(\Delta\cdot(-p_2-p_3))^{j_2}+(\Delta\cdot(p_1+p_3))^{j_2}]\Big]\Big) \\
&+f^{a_1a_2a}f^{a_3a_4a}\Big(-6\,(\Delta\cdot p_3)^{n-2}-6\,(\Delta\cdot(p_2+p_3))^{n-2} \\
&+\sum_{j_1=0}^{n-2}\Big[3\,(-\Delta\cdot p_4)^{j_1}(\Delta\cdot p_1)^{n-j_1-2} \\
&+3\,(-\Delta\cdot p_3)^{j_1}(\Delta\cdot p_2)^{n-j_1-2}+[5\,(-\Delta\cdot p_3)^{j_1}-4\,(-\Delta\cdot p_4)^{j_1}](\Delta\cdot(p_1+p_2))^{n-j_1-2} \\
&+9[(\Delta\cdot p_1)^{n-j_1-2}+(-\Delta\cdot p_2)^{n-j_1-2}](\Delta\cdot(-p_2-p_3))^{j_1}\Big]-3\Delta\cdot p_2\sum_{j_1=0}^{n-3}\Big[\\
&3[(-\Delta\cdot p_3)^{j_1}-(-\Delta\cdot p_4)^{j_1}](\Delta\cdot(p_1+p_2))^{n-j_1-3}-(-\Delta\cdot p_4)^{j_1}(\Delta\cdot p_3)^{n-j_1-3}\Big] \\
&+\sum_{j_1=0}^{n-2}\sum_{j_2=0}^{j_1}\Big[(-\Delta\cdot p_2)^{j_1-j_2}(\Delta\cdot p_1)^{n-j_1-2}[(\Delta\cdot(p_1+p_3))^{j_2}-14\,(\Delta\cdot(p_1+p_4))^{j_2}]\Big]\Big) \\
&+\frac{6d_A^{a_1a_2a_3a_4}}{C_A}\Big(-\sum_{j_1=0}^{n-2}\Big[[(-\Delta\cdot p_3)^{j_1}+(-\Delta\cdot p_4)^{j_1}](\Delta\cdot(p_1+p_2))^{-j_1+n-2}\Big] \\
&+\sum_{j_1=0}^{n-2}\sum_{j_2=0}^{j_1}\Big[(-\Delta\cdot p_2)^{j_1-j_2}(\Delta\cdot p_1)^{-j_1+n-2}[(\Delta\cdot(p_1+p_4))^{j_2}+(\Delta\cdot(p_1+p_3))^{j_2}]\Big]\Big)\Big\}.
\end{aligned}
$$

Lorentz structures of a twist-two operator

- Based on the following two properties
  - A twist-two operator has spin-n and mass dimension $n+2$
  - Propagator-type Feynman rules like $1/p^2$ can not appear in a vertex
- A twist-2 operator involving quarks or ghosts has one Lorentz structure only

$$\langle q|O|q+m\,g\rangle_{1\text{PI}}^{\mu_1\cdots\mu_m,\,(0),\,(m)} = c_m \Delta^{\mu_1}\cdots\Delta^{\mu_m}$$

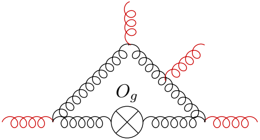
- A twist-two operator involving only gluons
  - Only $1+3/2m(m-1)$ Lorentz structures for m-gluon Feynman rules
  - $m=3$: $a_1\Delta^{\mu_1}\Delta^{\mu_2}\Delta^{\mu_3} + a_2\Delta^{\mu_1}\Delta^{\mu_2}p_1^{\mu_3} + \cdots + a_{10}\Delta^{\mu_3}g^{\mu_1\mu_2}$
  - 19 for $m=4$ and 31 for $m=5$
- Count the mass dimension of a_i: $[a_i] = x_i[\Delta\cdot p_j] + y_i[p_j\cdot p_k](y_i \geq 0)$
  $[a_1] = n-3+y_1[p_j\cdot p_k] = n+2-3 \rightarrow y_1 = 1$(Linear in $p_1^2,\ p_1\cdot p_2\cdots$)
  $[a_2]+[p_1^{\mu_3}] = n-2+y_2[p_j\cdot p_k]+1 = n+2-3 \rightarrow y_2 = 0$
- Why not $a_{11}\Delta^{\mu_1}p_1^{\mu_2}p_2^{\mu_3}$

$$[a_{11}]+2+3 \geq n-1+y_{11}[p_j\cdot p_k]+2+3 = n+4(\text{if } y_{11}=0)$$

where 3 is mass dimension of the external 3 gluons. Twist-4 operators

Computations of single pole for one-loop multi-leg OMEs

- Set all Mandelstam variables $p_1^2,\, p_2^2\, \cdots$ to numerical numbers and reconstruct their linear dependence



- Only two types of integrals are needed, other integrals are finite

$$I_1 = \int \frac{d^d l}{i\pi^{d/2}} \frac{1}{(l-q_1)^2 l^2}, \quad I_2 = \int \frac{d^d l}{i\pi^{d/2}} \frac{1}{(l-q_1)^2 l^2 \big(1 - x\Delta\cdot(l+q_2)\big)}$$

- At most x-dependent logarithms appear in the single pole

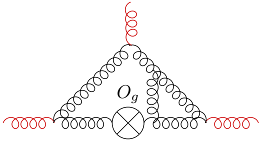
$$I_2 = \frac{1}{\epsilon}\left[\frac{\ln(1-x\Delta\cdot q_1 - x\Delta\cdot q_2) - \ln(1-x\Delta\cdot q_2)}{-x\Delta\cdot q_1}\right] + \mathcal{O}(\epsilon^0)$$

- Logarithms in x-space $\to$ n-space

$$\ln(1-x\Delta\cdot p_1 - x\Delta\cdot p_2) = \sum_{n=1}^{\infty} x^n \left[\frac{-1}{n}(\Delta\cdot p_1 + \Delta\cdot p_2)^n\right]$$

- Factoring out the overall factor $Z_{gA}^{(1)} = -\frac{C_A}{\epsilon}\frac{1}{n(n-1)}$

Computations of two-loop three-leg OMEs



- Set all Mandelstam variables p_1^2, p_2^2 $\cdots$ to numerical numbers and

$$\Delta \cdot p_1 = 1,\ \Delta \cdot p_2 = z_1$$

- Derive DE with respect to x
- Difficult to solve DE in terms of special functions
- Expand DE to x^{100} in the limit of $x \to 0$, with the boundary conditions being two-loop three-leg integrals without operator insertions[T. G. Birthwright, E. W. N. Glover, and P. Marquard, 04]

Reconstruct two-loop counterterm Feynman rules

- Obtain two-loop three-leg OMEs to x^{96} or $n = 96$
- For a fixed n, the result is a polynomial in z_1
- Construct full-x or full-n results from data to $n = 76$ based on ansatz
- Polylogarithms to weight-3, generalized Harmonic sums to weight-2

$$G(1,1,1/(1+z_1);x) = \sum_{n=1}^{\infty} x^n \Bigg[\frac{S_1(z_1+1;n)}{n^2} + \frac{S_2(z_1+1;n)}{n} - \frac{S_{1,1}(1,z_1+1;n)}{n} - \frac{(z_1+1)^n}{n^3} \Bigg]$$

where $S_{1,1}(1,z_1+1;n) = \sum_{t_1=1}^{n} \frac{1}{t_1} \sum_{t_2=1}^{t_1} \frac{(1+z_1)^{t_2}}{t_2}$

- Due to the generalized Harmonic sums, impossible to disentangle
  - renormalization constants (no z_1 dependence)
  - operator Feynman rules (no high-weight (≥ 1) functions)

A counterterm Feynman rule & infinite operator Feynman rules ($N_2 = \infty$)