

# The Compton amplitude and nucleon structure functions in lattice QCD

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in collaboration with QCDSF/UKQCD:

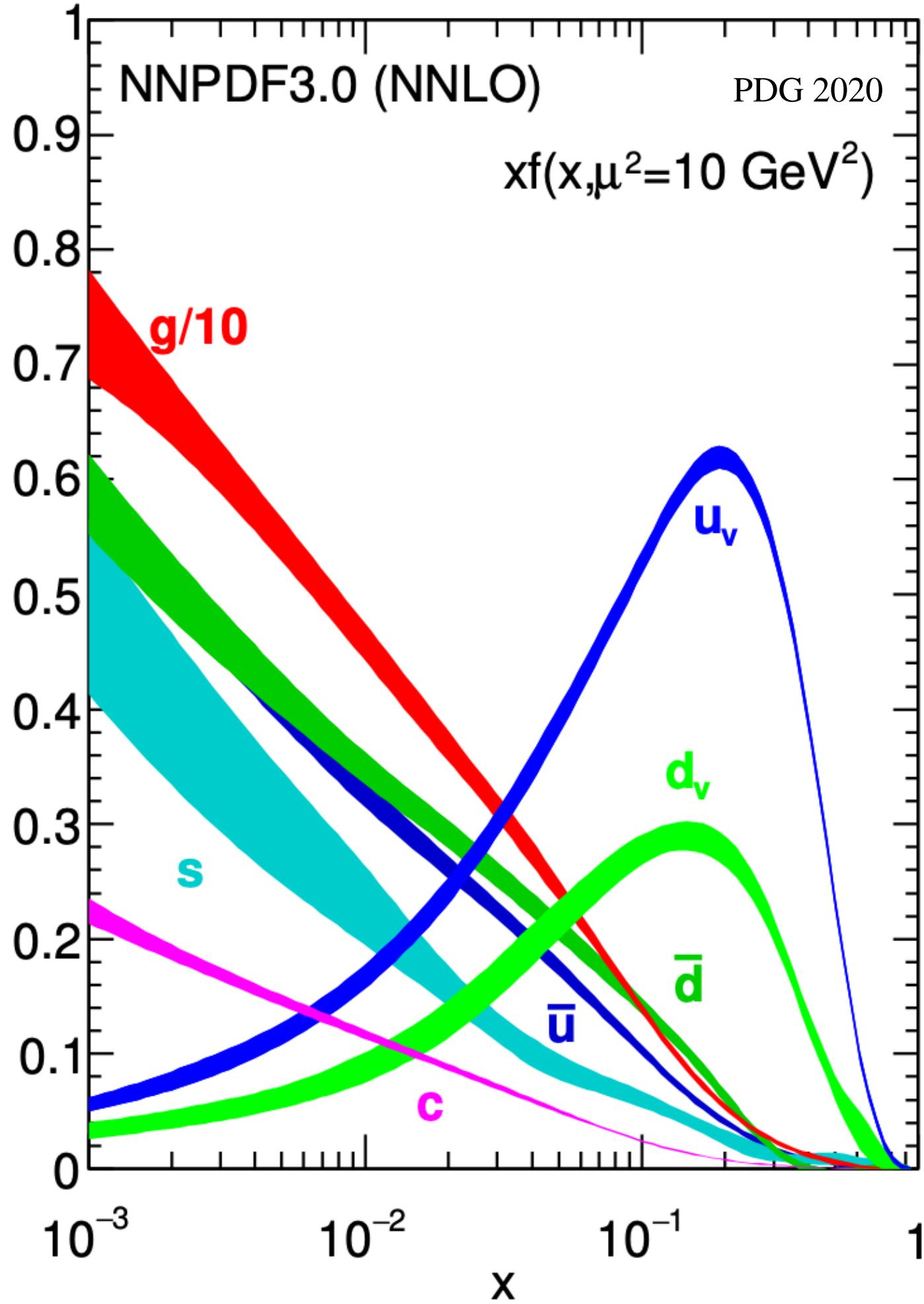
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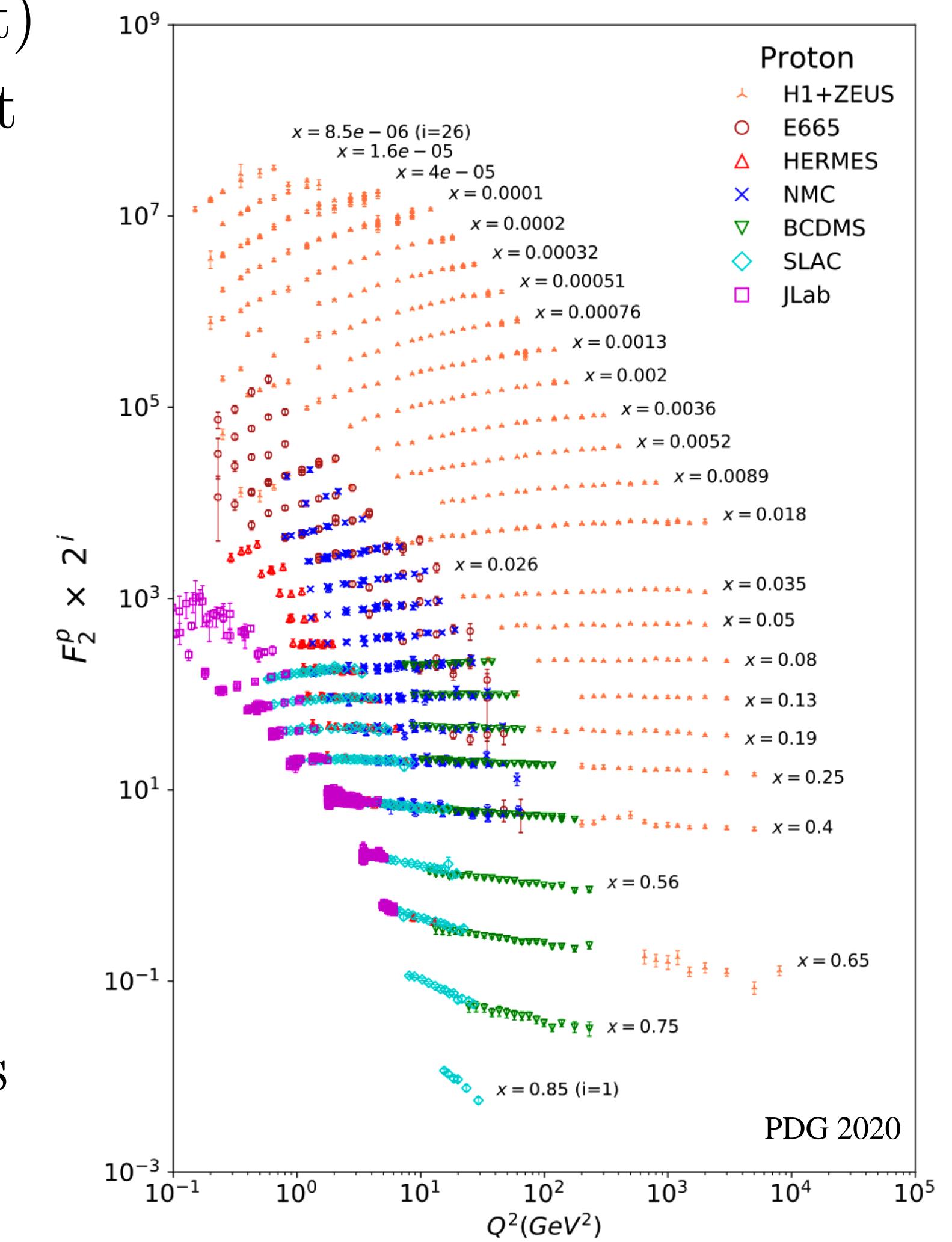
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The numerical configuration generation (using the BQCD lattice QCD program) and data analysis (using the Chroma software library) was carried out on the IBM BlueGene/Q and HP Tesseract using DIRAC 2 resources (EPCC, Edinburgh, UK), the IBM Blue-Gene/Q (NIC, Jülich, Germany) and the Cray XC40 at HLRN (The North-German Supercomputer Alliance), the NCI National Facility in Canberra, Australia (supported by the Australian Commonwealth Government) and Phoenix (University of Adelaide). RH is supported by STFC through grant ST/P000630/1. HP is supported by DFG Grant No. PE 2792/2-1. PELR is supported in part by the STFC under contract ST/G00062X/1. GS is supported by DFG Grant No. SCHI 179/8-1. KUC, RDY and JMZ are supported by the Australian Research Council grants DP190100297 and DP220103098.

# Motivation

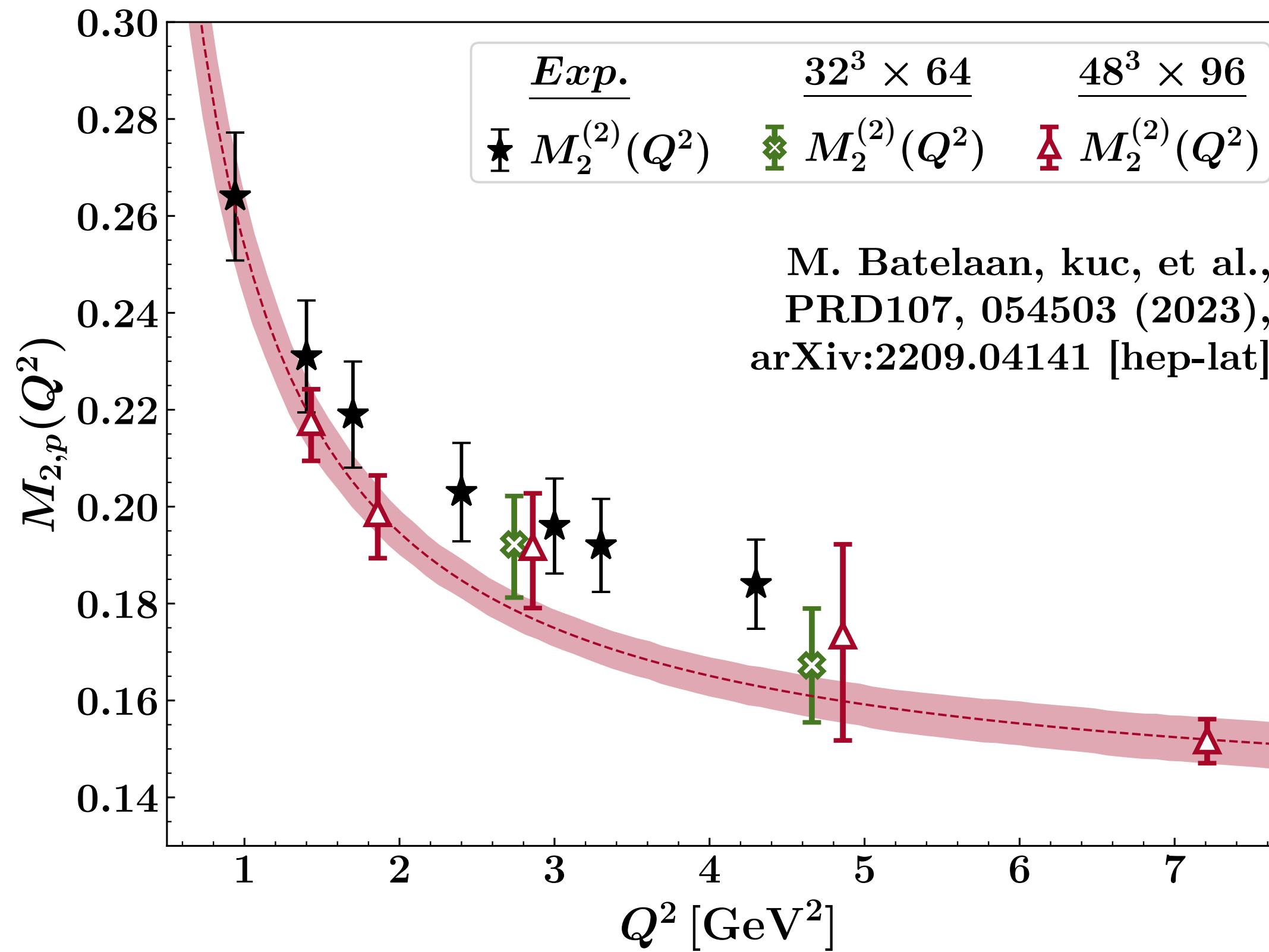


- Nucleon structure (leading twist)
  - Structure functions from first principles
  - Constraining the high- and low- $x$  regions better
- Scaling and Power corrections
  - $Q^2$  cuts of global QCD analyses
  - Large- $x$ , low- $Q^2$ :
    - Higher-twist contributions
    - Target mass corrections

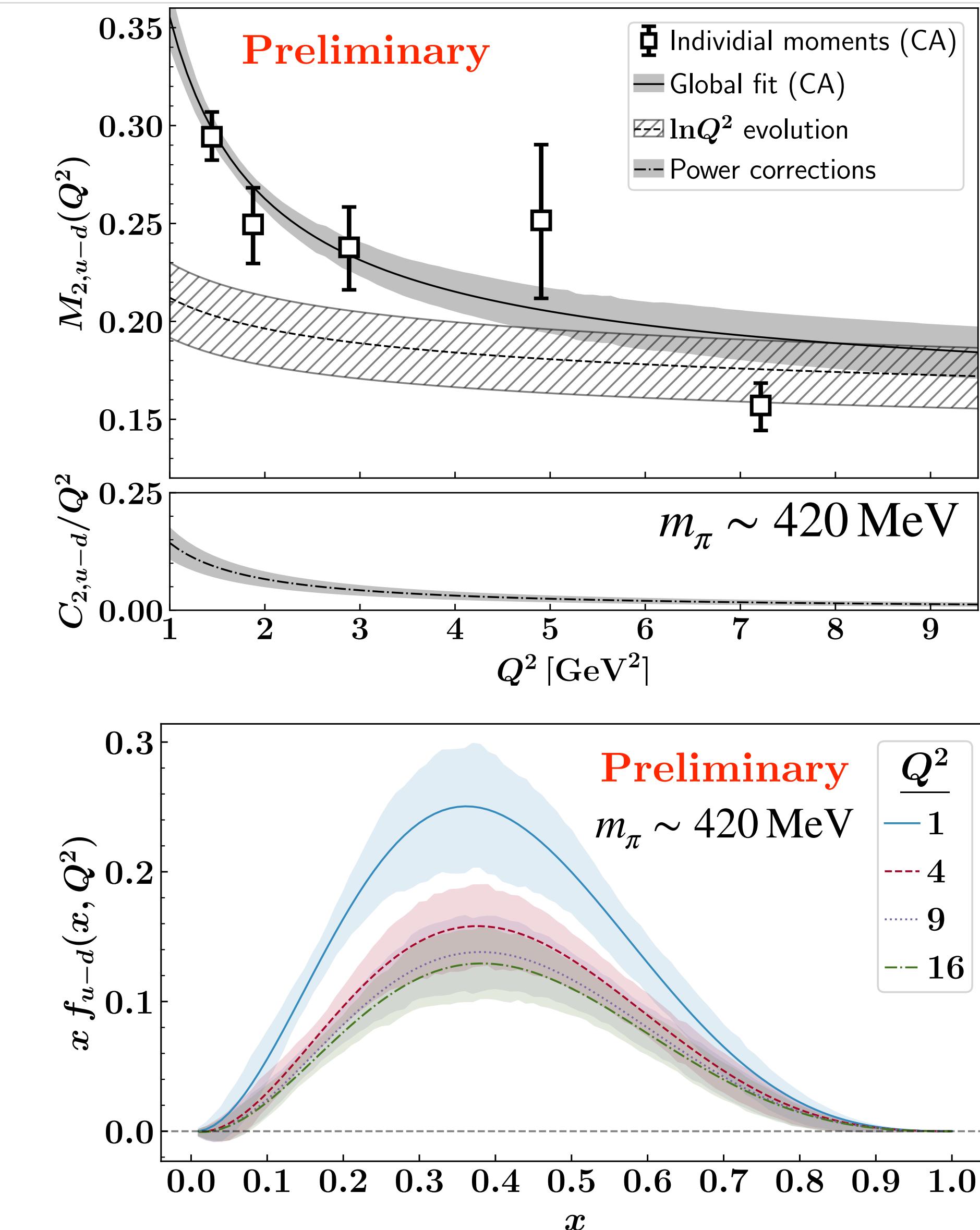


# Highlights

- Lowest moments of proton  $F_2$  vs.  $Q^2$
- Estimation of power corrections



- Estimation of the PDFs, including scaling and power corrections, via a global fit to the lattice Compton amp.



# Forward Compton Amplitude

$$\text{Diagram 1} = \text{Diagram 2} + \mathcal{O}\left(\frac{M_N}{Q^2}, \frac{1}{Q^2}\right)$$

The equation illustrates the factorisation theorem for the forward Compton amplitude. The left-hand side shows a single Feynman diagram of a nucleon (oval) interacting with two incoming and two outgoing particles (wavy lines). This is set equal to the sum of two terms: the first term is a diagram where the nucleon is shown with a quark-gluon loop (horizontal line with arrows) passing through it, and the second term is a correction given by the order of the perturbation theory,  $\mathcal{O}(M_N/Q^2, 1/Q^2)$ .

- Factorisation theorem
- Leading twist:
  - local matrix elements (ME)  
or quasi/pseudo-distributions
- Power corrections  
relevant at low- $Q^2$
- Difficult to calculate  
from local ME

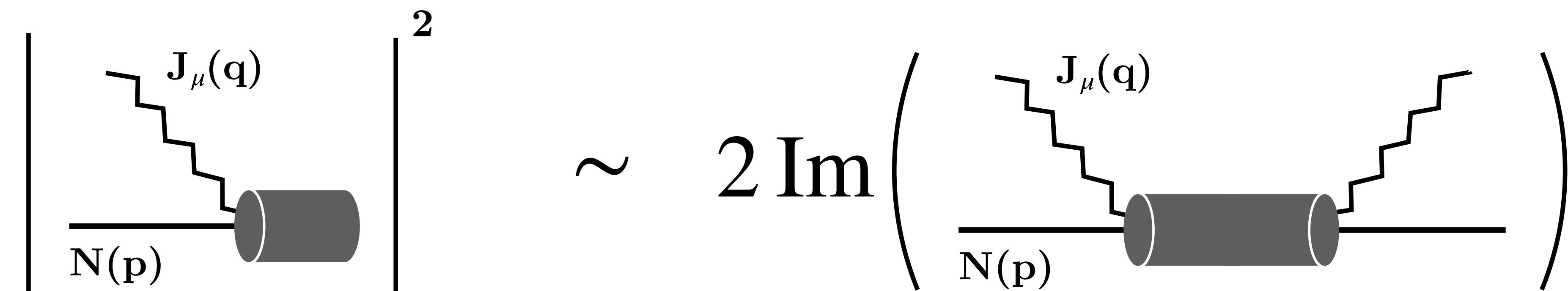
# Forward Compton Amplitude

$$T_{\mu\nu}(p, q) = i \int d^4z e^{iq \cdot z} \rho_{ss'} \langle p, s' | \mathcal{T}\{J_\mu(z) J_\nu(0)\} | p, s \rangle \quad , \text{ spin avg. } \rho_{ss'} = \frac{1}{2} \delta_{ss'} \quad \underline{\omega} = \frac{2p \cdot q}{Q^2}$$

Same Lorentz decomposition as the Hadronic Tensor

$$= \left( -g_{\mu\nu} + \frac{q_\mu q_\nu}{q^2} \right) \mathcal{F}_1(\omega, Q^2) + \left( p_\mu - \frac{p \cdot q}{q^2} q_\mu \right) \left( p_\nu - \frac{p \cdot q}{q^2} q_\nu \right) \frac{\mathcal{F}_2(\omega, Q^2)}{p \cdot q}$$

Compton Structure Functions (SF)



DIS Cross Section ~ Hadronic Tensor

Forward Compton Amplitude ~ Compton Tensor

# Nucleon Structure Functions

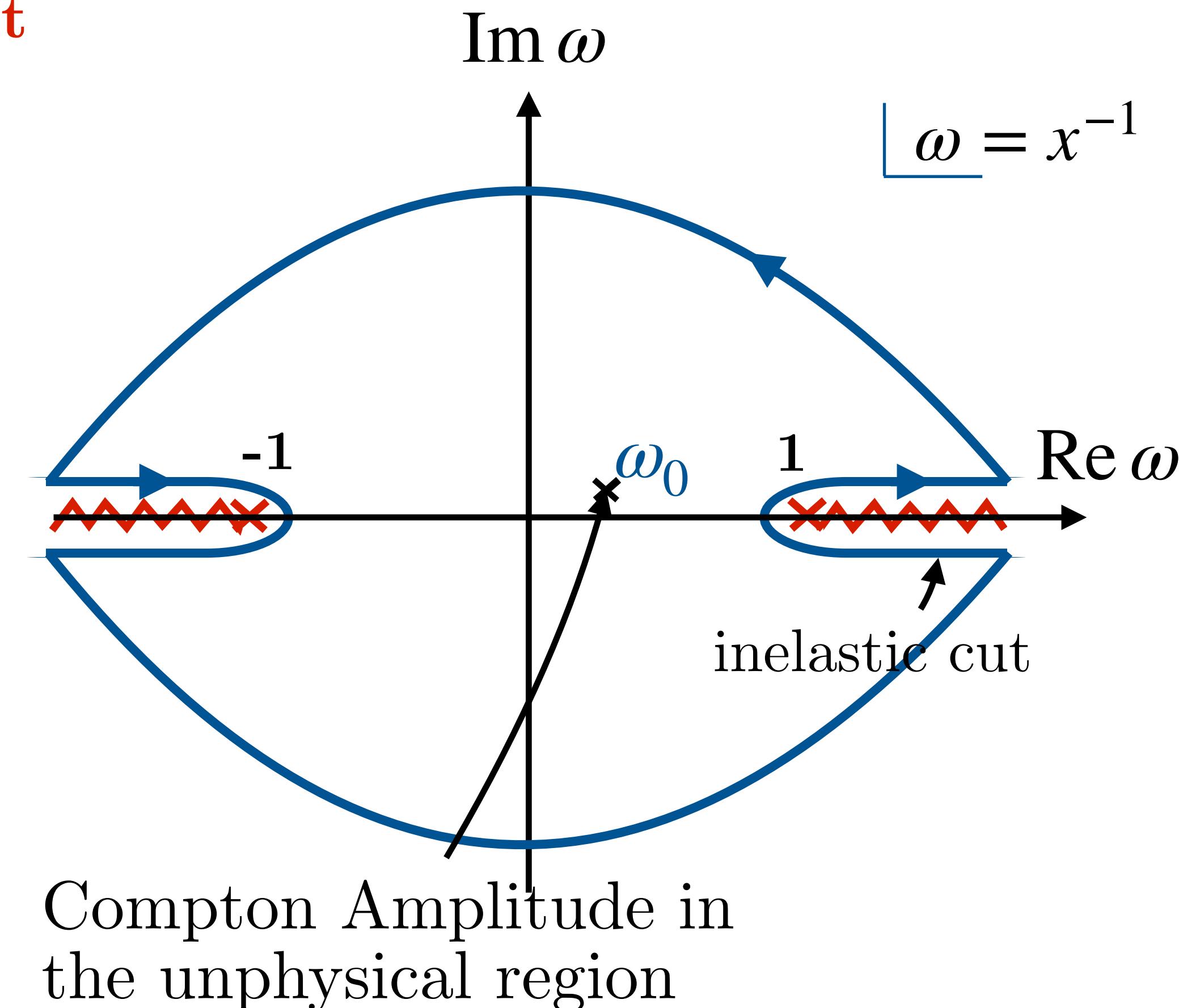
- we can write down dispersion relations and connect Compton SFs to DIS SFs:

$$\underbrace{\mathcal{F}_1(\omega, Q^2) - \mathcal{F}_1(0, Q^2)}_{\equiv \overline{\mathcal{F}}_1(\omega, Q^2)} = 2\omega^2 \int_0^1 dx \frac{2x F_1(x, Q^2)}{1 - x^2 \omega^2 - i\epsilon}$$

$$\mathcal{F}_2(\omega, Q^2) = 4\omega^2 \int_0^1 dx \frac{F_2(x, Q^2)}{1 - x^2 \omega^2 - i\epsilon}$$

$$\underbrace{\mathcal{F}_L(\omega, Q^2) + \mathcal{F}_1(0, Q^2)}_{\equiv \overline{\mathcal{F}}_L(\omega, Q^2)} = \frac{8M_N^2}{Q^2} \int_0^1 dx F_2(x, Q^2)$$

$$+ 2\omega^2 \int_0^1 dx \frac{F_L(x, Q^2)}{1 - x^2 \omega^2 - i\epsilon}$$



# Nucleon Structure Functions

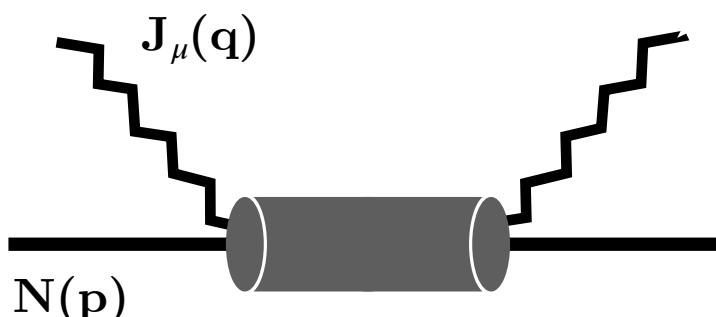
- using the Taylor expansion,  $\frac{1}{1 - (x\omega)^2} = \sum_{n=1}^{\infty} (x\omega)^{2n-2}$   $\underline{\omega} = \frac{2p \cdot q}{Q^2} \equiv x^{-1}$
- $\overline{\mathcal{F}}_{1,L}(\omega, Q^2) = \sum_{n=0}^{\infty} 2\omega^{2n} M_{2n}^{(1,L)}(Q^2)$ , where  $M_{2n}^{(1)}(Q^2) = 2 \int_0^1 dx x^{2n-1} F_1(x, Q^2)$ , and  $M_0^{(1)}(Q^2) = 0$
- $\mathcal{F}_2(\omega, Q^2) = \sum_{n=1}^{\infty} 4\omega^{2n-1} M_{2n}^{(2)}(Q^2)$ , where  $M_{2n}^{(2,L)}(Q^2) = \int_0^1 dx x^{2n-2} F_{2,L}(x, Q^2)$ , and  $M_0^{(L)}(Q^2) = \frac{4M_N^2}{Q^2} M_2^{(2)}(Q^2)$
- $\mu = \nu = 3$  and  $p_3 = q_3 = 0 \implies \mathcal{F}_1(\omega, Q^2) = T_{33}(p, q)$
- $\mu = \nu = 0$  and  $p_3 = q_3 = q_0 = 0 \implies \frac{\mathcal{F}_2(\omega, Q^2)}{\omega} = [T_{00}(p, q) + T_{33}(p, q)] \frac{Q^2}{2E_N^2}$

Once we have the Compton amplitude,  $T_{\mu\nu}(p, q)$ ,  
we can extract the Mellin moments!

# Compton Amplitude from FHT at 2<sup>nd</sup> order

- unpolarised Compton Amplitude

$$T_{\mu\mu}(p, q) = \int d^4z e^{i\mathbf{q}\cdot\mathbf{z}} \langle N(p) | \mathcal{T} \{ J_\mu(z) J_\mu(0) \} | N(p) \rangle$$



- Action modification

$$S \rightarrow S(\lambda) = S + \lambda \int d^4z (e^{i\mathbf{q}\cdot\mathbf{z}} + e^{-i\mathbf{q}\cdot\mathbf{z}}) J_\mu(z)$$

local EM current  
 $J_\mu(z) = \sum_q e_q \bar{q}(z) \gamma_\mu q(z)$

- 2<sup>nd</sup> order derivatives of the 2-pt correlator,  $G_\lambda^{(2)}(\mathbf{p}; t)$ , in the presence of the external field

$$\left. \frac{\partial^2 G_\lambda^{(2)}(\mathbf{p}; t)}{\partial \lambda^2} \right|_{\lambda=0} = \left( \frac{\partial^2 A_\lambda(\mathbf{p})}{\partial \lambda^2} - t A(\mathbf{p}) \frac{\partial^2 E_{N_\lambda}(\mathbf{p})}{\partial \lambda^2} \right) e^{-E_N(\mathbf{p})t}$$

from spectral decomposition

$$\left. \frac{\partial^2 G_\lambda^{(2)}(\mathbf{p}; t)}{\partial \lambda^2} \right|_{\lambda=0} = \frac{A(\mathbf{p})}{2E_N(\mathbf{p})} t e^{-E_N(\mathbf{p})t} \int d^4z (e^{iq\cdot z} + e^{-iq\cdot z}) \langle N(\mathbf{p}) | \mathcal{T} \{ \mathcal{J}(z) \mathcal{J}(0) \} | N(\mathbf{p}) \rangle$$

from path integral

- equate the time-enhanced terms:

$$\left. \frac{\partial^2 E_{N_\lambda}(\mathbf{p})}{\partial \lambda^2} \right|_{\lambda=0} = -\frac{1}{2E_N(\mathbf{p})} \overbrace{\int d^4z (e^{iq\cdot z} + e^{-iq\cdot z}) \langle N(\mathbf{p}) | \mathcal{T} \{ \mathcal{J}(z) \mathcal{J}(0) \} | N(\mathbf{p}) \rangle}^{T_{\mu\mu}(p, q)} + (q \rightarrow -q)$$

Compton amplitude is related to the second-order energy shift

# Simulation Details

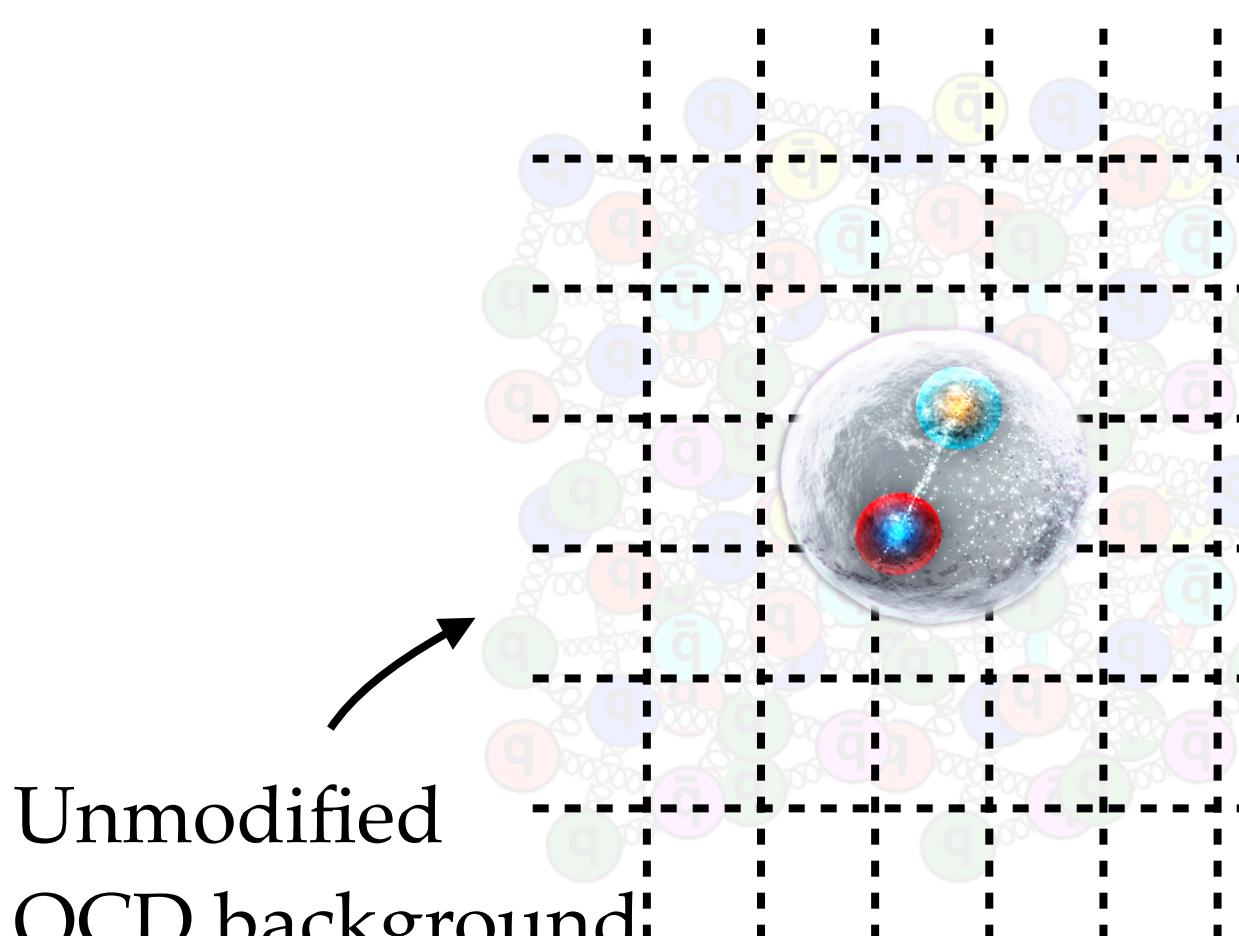
QCDSF/UKQCD configurations  
 $(32^3 \times 64)$ , 2+1 flavour (u/d+s)  
 $(48^3 \times 96)$

$\beta = \begin{pmatrix} 5.50 \\ 5.65 \end{pmatrix}$ , NP-improved Clover action

Phys. Rev. D 79, 094507 (2009), arXiv:0901.3302 [hep-lat]

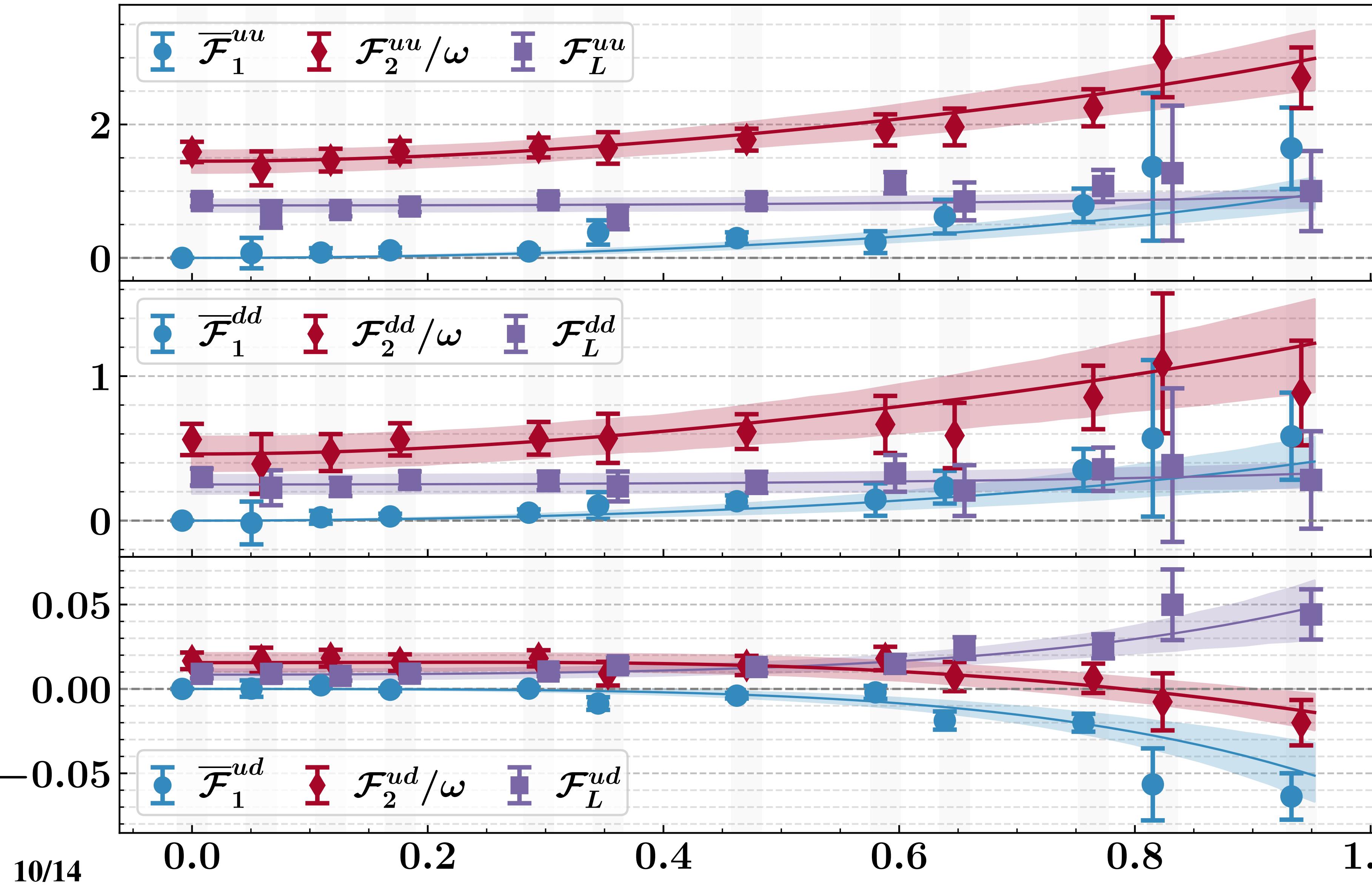
$$m_\pi \sim \begin{bmatrix} 470 \\ 420 \end{bmatrix} \text{ MeV}, \sim \text{SU}(3) \text{ sym.}$$

$$m_\pi L \sim \begin{bmatrix} 5.6 \\ 6.9 \end{bmatrix} \quad a = \begin{bmatrix} 0.074 \\ 0.068 \end{bmatrix} \text{ fm}$$



- FH implementation at the valence quark level
- Valence u/d quark props with modified action,  $S(\lambda)$
- Local EM current insertion,  $J_\mu(x) = Z_V \bar{q}(x) \gamma_\mu q(x)$
- 4 distinct field strengths,  $\lambda = [\pm 0.0125, \pm 0.025]$
- Several current momenta in the range,  $1.5 \lesssim Q^2 \lesssim 7 \text{ GeV}^2$
- Up to  $\mathcal{O}(10^4)$  measurements for each pair of  $Q^2$  and  $\lambda$
- Access to a range of  $\omega = 2 p \cdot q / Q^2$  values for several  $(p, q)$  pairs
  - An inversion for each  $q$  and  $\lambda$ , varying  $p$  is relatively cheap
- Connected diagrams only, no disconnected

# Compton Structure Functions

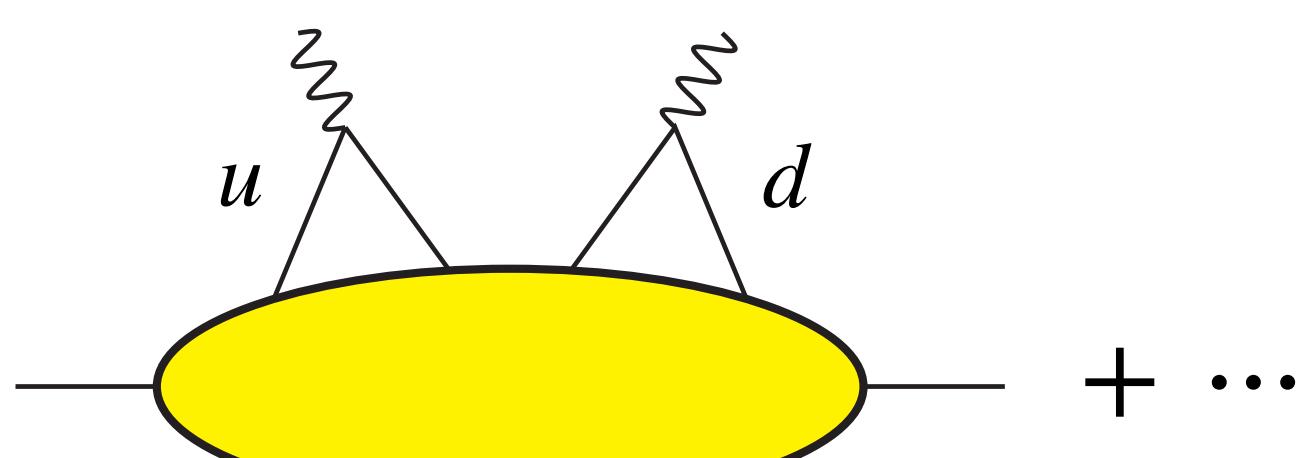
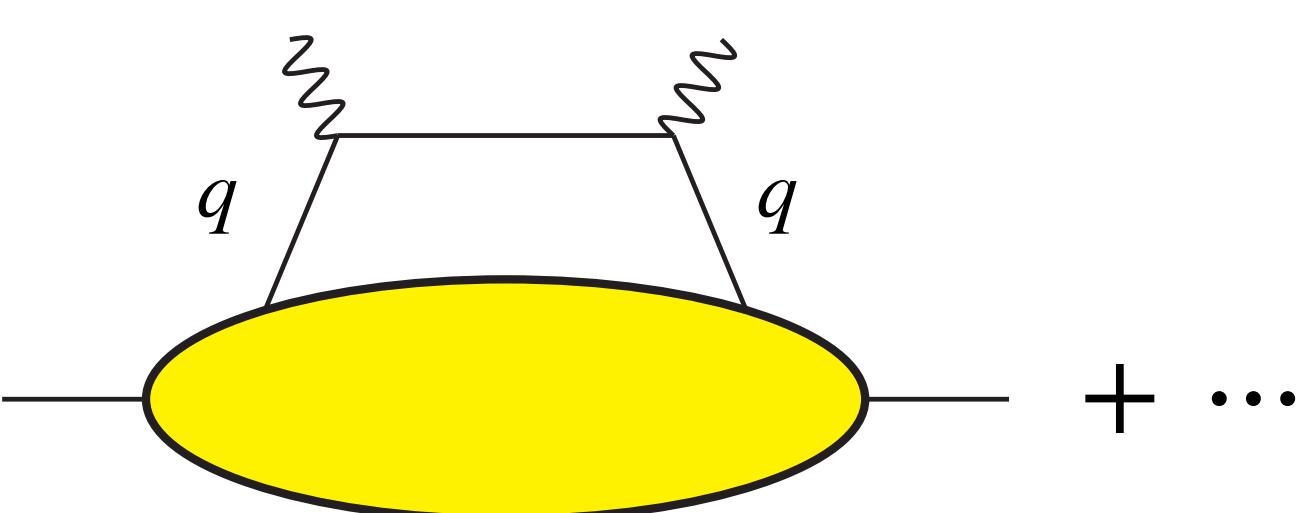


$48^3 \times 96$ , 2+1 flavour

$a = 0.068$  fm

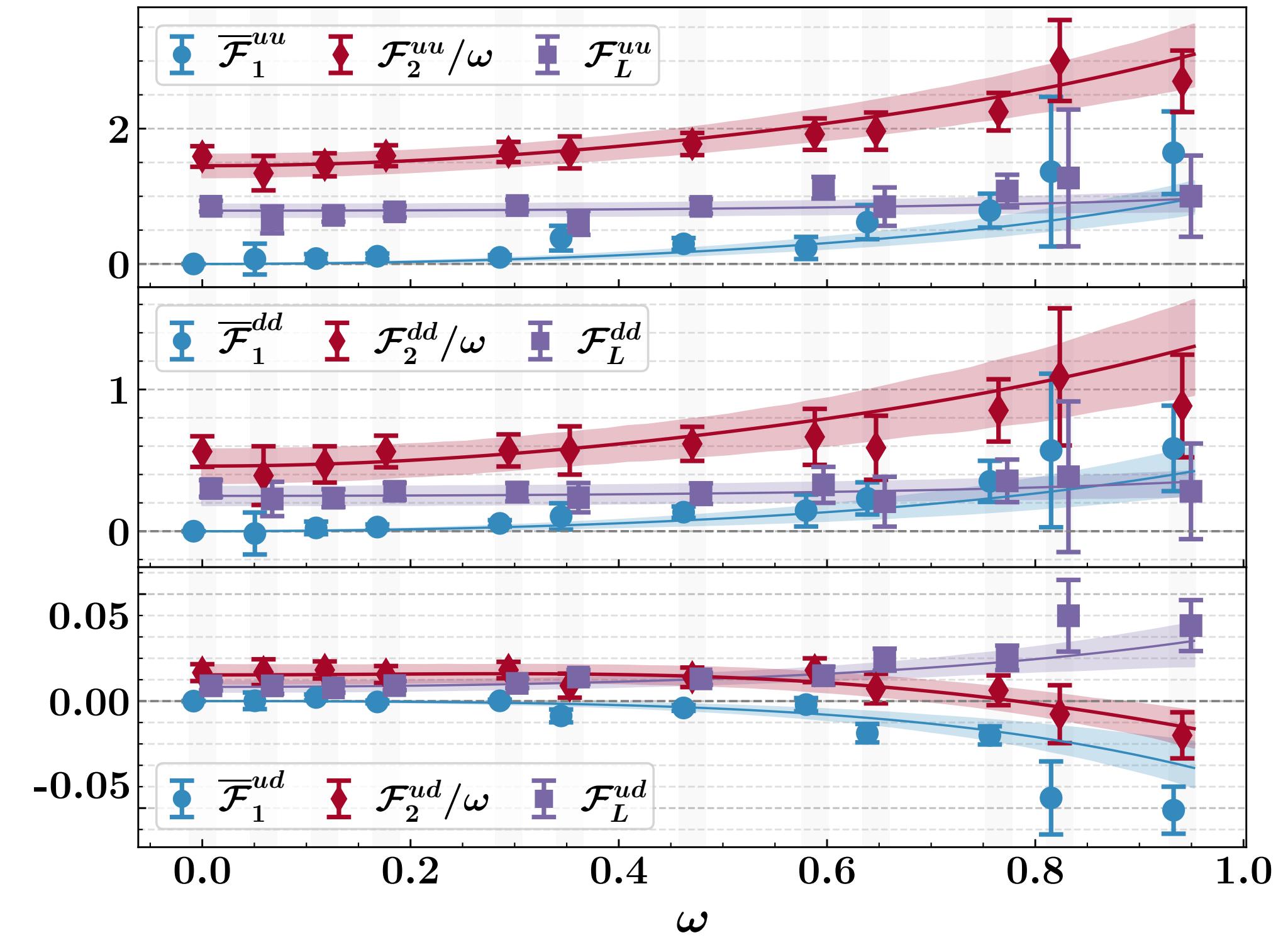
$m_\pi \sim 420$  MeV

$Q^2 = 4.9$  GeV $^2$



$$\omega = 2p \cdot q / Q^2$$

# Moments | Fit details



- Bayesian approach by MCMC method

Sample the moments from Uniform priors  
*individually for u- and d-quark*

$$M_2(Q^2) \sim \mathcal{U}(0, 1)$$

$$M_{2n}(Q^2) \sim \mathcal{U}(0, M_{2n-2}(Q^2))$$

$$\bar{\mathcal{F}}_1^{qq}(\omega, Q^2) = \sum_{n=0}^{\infty} 2\omega^{2n} M_{2n}^{(1)}(Q^2)$$

$$\frac{\mathcal{F}_2^{qq}(\omega, Q^2)}{\omega} = \frac{\tau}{1 + \tau\omega^2} \sum_{n=0}^{\infty} 4\omega^{2n} \left[ M_{2n}^{(1)} + M_{2n}^{(L)} \right](Q^2), \text{ where } \tau = \frac{Q^2}{4M_N^2}$$

- Enforce monotonic decreasing of moments for  $u$  and  $d$  only, not necessarily true for  $u - d$

$$M_2^{(1)}(Q^2) \geq M_4^{(1)}(Q^2) \geq \dots \geq M_{2n}^{(1)}(Q^2) \geq \dots \geq 0$$

We truncate at  $n = 4$  [ $\mathcal{O}(\omega^8)$ ], inclusive  
No dependence to truncation order for  $3 \leq n \leq 10$

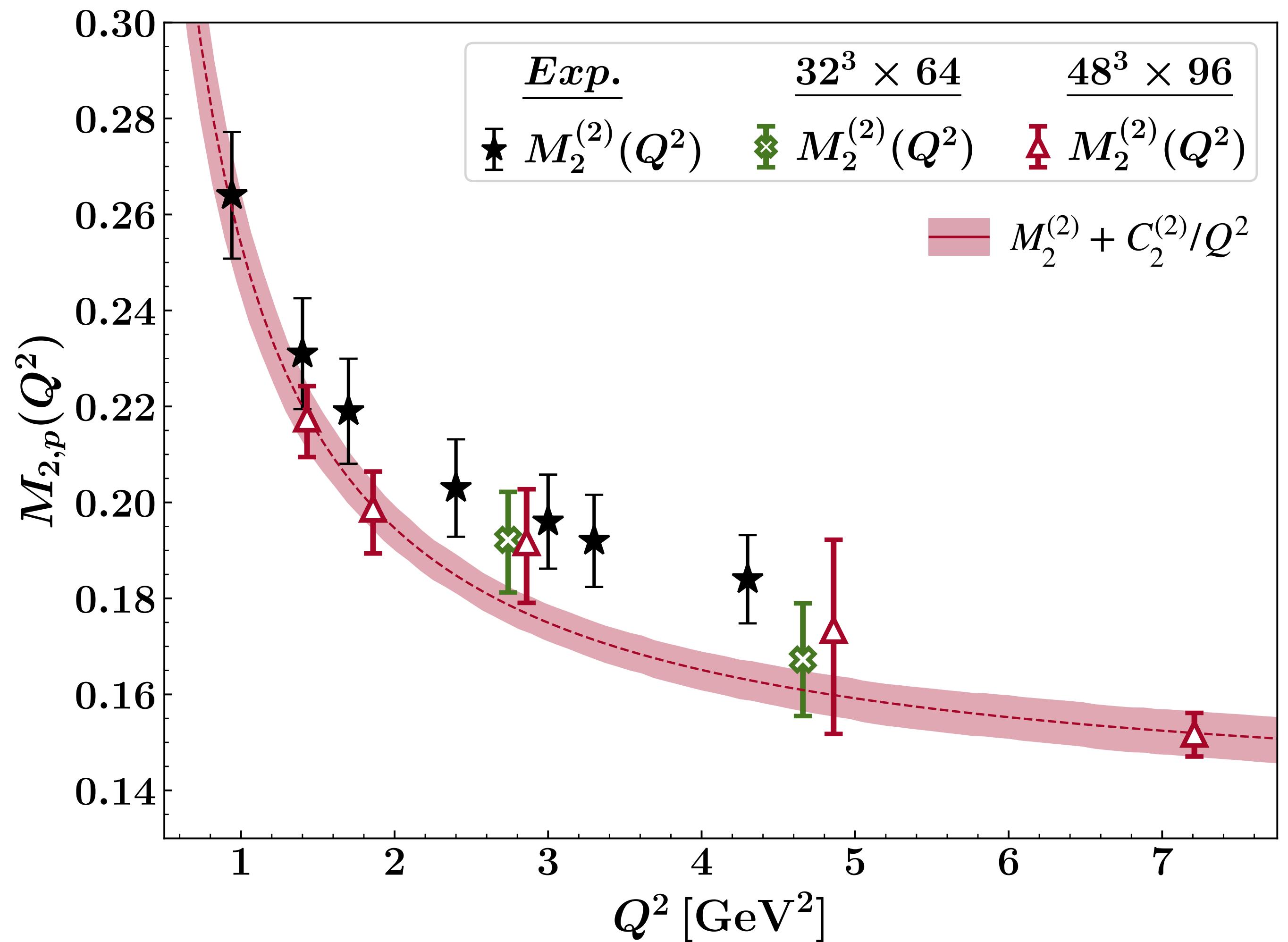
Normal Likelihood function,  $\exp(-\chi^2/2)$

$$\chi^2 = \sum_i \frac{(\bar{\mathcal{F}}_i - \bar{\mathcal{F}}^{obs}(\omega_i))^2}{\sigma_i^2}$$

errors via bootstrap analysis

# Moments | Scaling and Power Corrections

- Unique ability to study the  $Q^2$  dependence of the moments!



- Power corrections below  $\sim 3\text{ GeV}^2$ ?
  - Naive modelling via:
  - $M_2^{(2)}(Q^2) = M_2^{(2)} + C_2^{(2)}/Q^2$
  - $C_2^{(2)}$  is a catch all correction term
- Can we distinguish
  - Target mass corrections,
  - Elastic ( $x = 1$ ),
  - $\ln Q^2$  scaling, and
  - genuine higher twist contributions?

★ Exp  $M_2^{(2)}$ : C. S. Armstrong, R. Ent, C. E. Keppel, S. Liuti, G. Niculescu, and I. Niculescu, Phys. Rev. D 63, 094008 (2001), arXiv:hep-ph/0104055.

# PDFs

- $u_V$  and  $d_V$  quarks, and non-singlet  $u - d$
- Assume a parametric form for the PDFs w/HT
- $f_q(x, Q^2) = a_q x^{b_q} (1 - x)^{c_q} \left( 1 + \frac{d_q x^{e_q} (1 - x)^{f_q}}{Q^2} \right)$
- $a_q, b_q, c_q, d_q, e_q, f_q$  are free fit parameters,  $q = [u, d]$
- $a_q$  is normalised to lowest even moment,  $M_{2,q}$

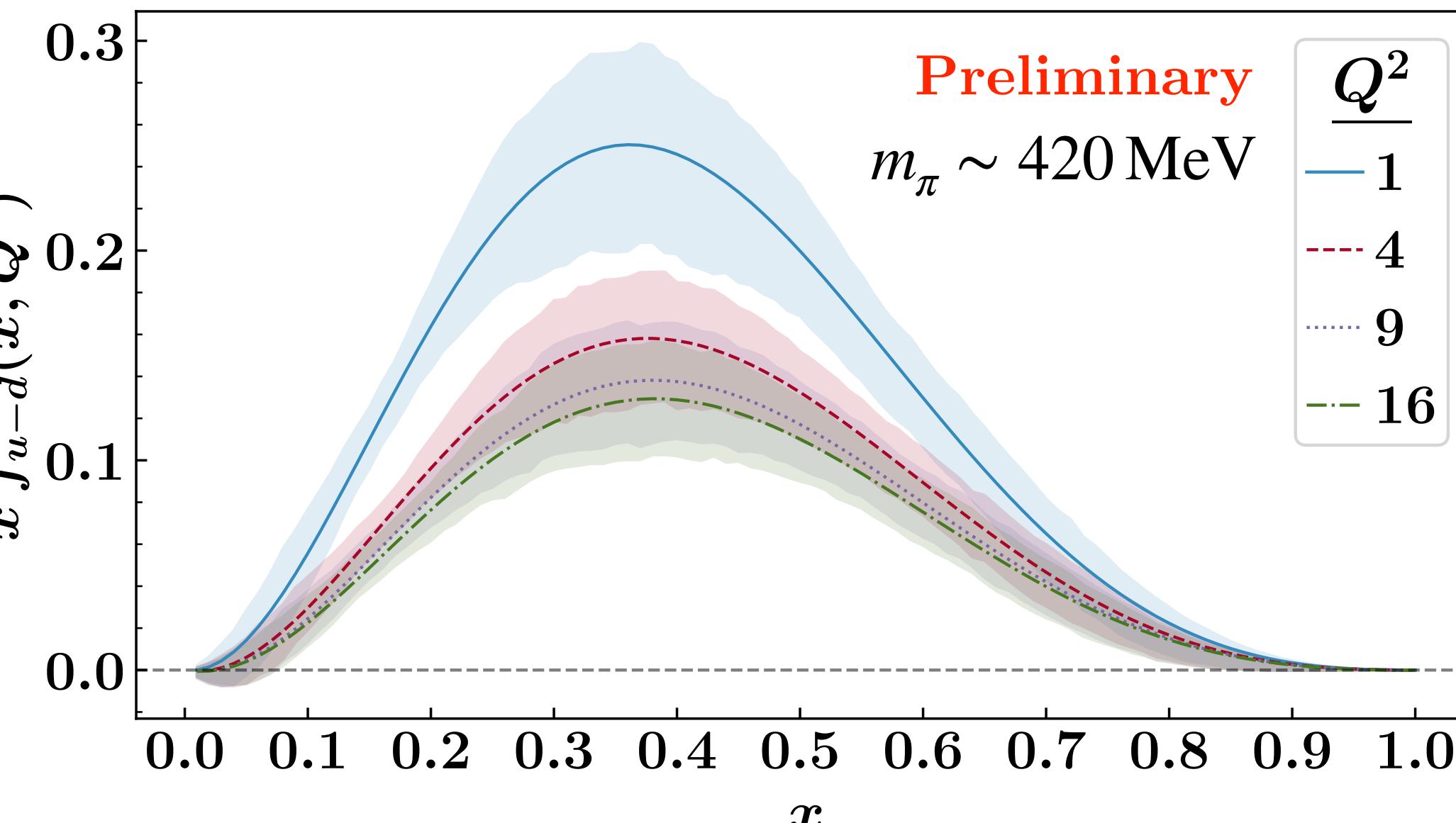
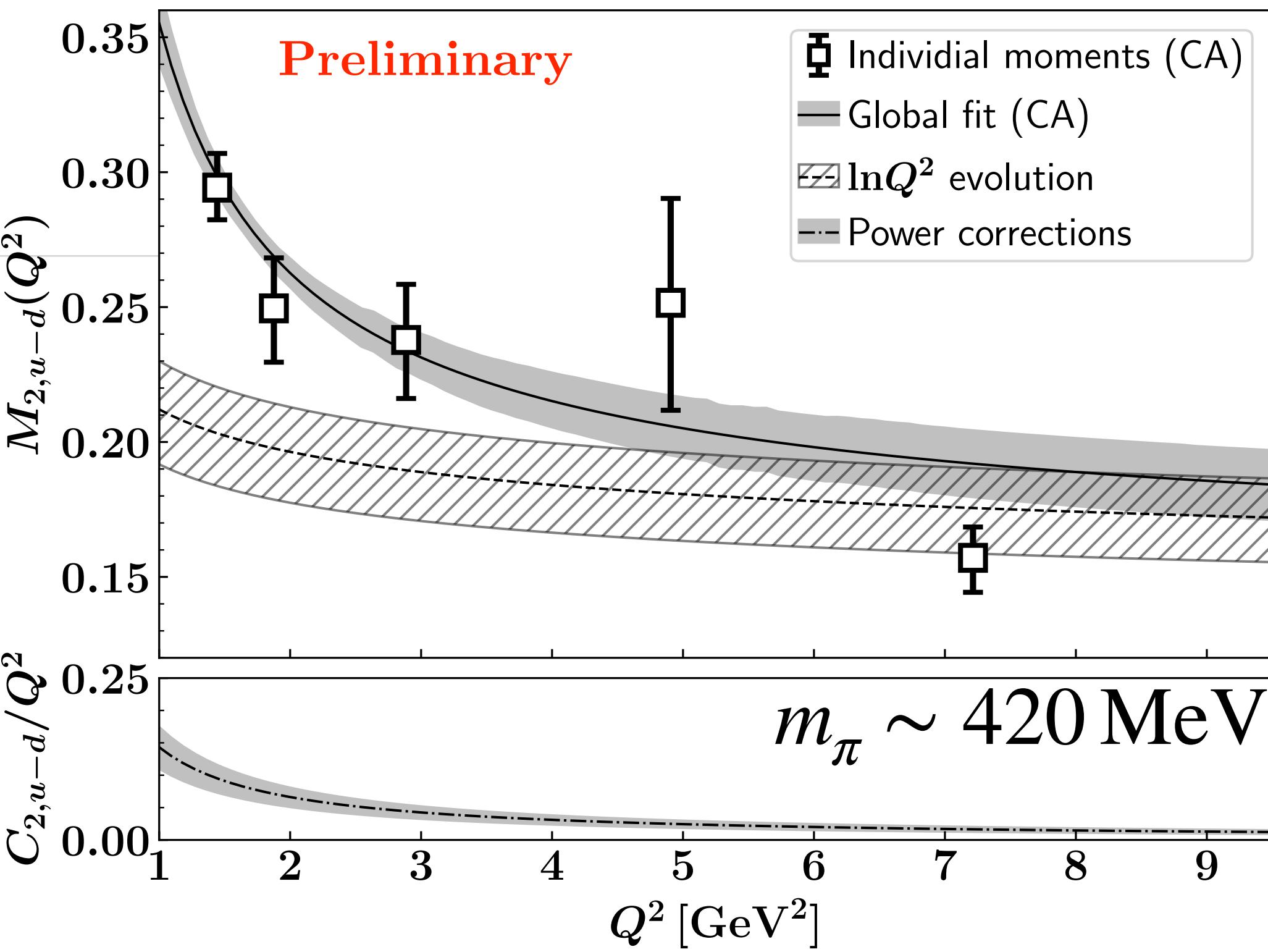
Starting from  $f_q(x, Q^2)$ :

a Compton structure function is described by generalised hypergeometric series,

$$\frac{\mathcal{F}_2(\omega, Q^2)}{\omega} = 4M_2^{LT}(Q_0^2) \sum_{n=1}^N \left( A_{2n}(b, c, Q^2) + \frac{C_{2n}(b, c, d, e, f, Q^2)}{Q^2} \right) \omega^{2n-2}$$

$A_{2n}, C_{2n}$  are known functions  
non-singlet  $\ln Q^2$  evolution at LO embedded in  $A_{2n}, C_{2n}$

$Q_0^2 = 4 \text{ GeV}^2$



# Summary

- A calculation of the moments of unpolarised structure functions
- Systematic investigation of power corrections, higher-twist effects and scaling is within reach
- Exploratory investigation of PDF extraction w/scaling and HT
- Can be extended to:
  - mixed currents, interference terms, e.g.  $\gamma Z$ , giving access to  $F_3$
  - spin-dependent structure functions,  $g_1, g_2$
  - GPDs: A. Hannaford-Gunn et al. Phys. Rev. D 105, 014502 arXiv:2110.11532

