Recent results on massless and massive Wilson coefficients up to 3-loop

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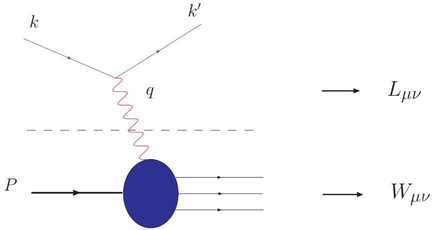
Outline

Theory of Deep Inelastic Scattering



- Kinematic invariants:

$$Q^2 = -q^2, \qquad x = \frac{Q^2}{2P.q}$$

- The cross section factorizes into leptonic and hadronic tensor:

$$\frac{d^2\sigma}{dQ^2 dx} \sim L_{\mu\nu} W^{\mu\nu}$$

- The hadronic tensor can be expressed through structure functions:

$$W_{\mu\nu} = \frac{1}{4\pi} \int d^4\xi \exp(iq\xi) \langle P, | \left[J_\mu^{em}(\xi), J_\nu^{em}(\xi)\right] |P\rangle$$

$$= \frac{1}{2x}\left(g_{\mu\nu} + \frac{q_\mu q_\nu}{Q^2}\right) F_L(x, Q^2) + \frac{2x}{Q^2}\left(P_\mu P_\nu + \frac{q_\mu P_\nu + q_\nu P_\mu}{2x} - \frac{Q^2}{4x^2} g_{\mu\nu}\right) F_2(x, Q^2)$$

$$+ i\epsilon_{\mu\nu\rho\sigma} \frac{q^\rho S^\sigma}{q \cdot P} g_1(x, Q^2) + i\epsilon_{\mu\nu\rho\sigma} \frac{q^\rho \left(q \cdot PS^\sigma - q \cdot SP^\sigma\right)}{(q \cdot P)^2} g_2(x, Q^2)$$

- F_L, F_2, g_1 and g_2 contain contributions from both, charm and bottom quarks.

Introduction

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Why are Heavy Flavor Contributions important?

- They form a significant contribution to all structure functions particularly at small x and high Q^2.
- Concise 3-loop corrections are needed to determine $\alpha_s(M_Z)$, m_c and perhaps m_b.
- The accuracy of measurements at the LHC reaches a level of precision requiring 3-loop VFNS matching.

NNLO: [S. Alekhin, J. Blümlein, S. Moch and R. Placakyte (Phys. Rev. D96 (2017))]

$$\alpha_s(M_Z^2) = 0.1147 \pm 0.0008$$

$$m_c(m_c) = 1.252 \pm 0.018(exp) \begin{array}{l} +0.03 \\ -0.02 \end{array} (scale), \begin{array}{l} +0.00 \\ -0.07 \end{array} (thy)\text{GeV} \qquad (\overline{\text{MS}}\text{-scheme})$$

Yet approximate NNLO treatment for F_2 (non-negligible error) [H. Kawamura et al. (Nucl. Phys. B864 (2012))]

NS & PS corrections are exact [J. Ablinger et al. (Nucl. Phys. B886 & B890 (2014))]

Factorization of the Structure Functions

At leading twist the structure functions factorize in terms of a Mellin convolution

$$F_{(2,L)}(x, Q^2) = \sum_j \underbrace{\mathbb{C}_{j,(2,L)}\left(x, \frac{Q^2}{\mu^2}, \frac{m^2}{\mu^2}\right)}_{perturbative} \otimes \underbrace{f_j(x, \mu^2)}_{nonpert.}$$

into (pert.) Wilson coefficients and (nonpert.) parton distribution functions (PDFs).
$\otimes$ denotes the Mellin convolution

$$f(x) \otimes g(x) \equiv \int_0^1 dy \int_0^1 dz \, \delta(x - yz) f(y) g(z) \, .$$

The subsequent calculations are performed in Mellin space, where $\otimes$ reduces to a multiplication, due to the Mellin transformation

$$\hat{f}(N) = \int_0^1 dx \, x^{N-1} f(x) \, .$$

Wilson coefficients:

$$\mathbb{C}_{j,(2,L)}\left(N,\frac{Q^2}{\mu^2},\frac{m^2}{\mu^2}\right)=C_{j,(2,L)}\left(N,\frac{Q^2}{\mu^2}\right)+H_{j,(2,L)}\left(N,\frac{Q^2}{\mu^2},\frac{m^2}{\mu^2}\right) .$$

At $Q^2 \gg m^2$ the heavy flavor part

$$H_{j,(2,L)}\left(N,\frac{Q^2}{\mu^2},\frac{m^2}{\mu^2}\right)=\sum_i C_{i,(2,L)}\left(N,\frac{Q^2}{\mu^2}\right)A_{ij}\left(\frac{m^2}{\mu^2},N\right)$$

[Buza, Matiounine, Smith, van Neerven (Nucl.Phys.B (1996))]

factorizes into the light flavor Wilson coefficients C and the massive operator matrix elements (OMEs) of local operators O_i between partonic states j

$$A_{ij}\left(\frac{m^2}{\mu^2},N\right)=\langle j \mid O_i \mid j\rangle .$$

$\rightarrow$ additional Feynman rules with local operator insertions for partonic matrix elements.

For $F_2(x,Q^2)$: at $Q^2 \gtrsim 10m^2$ the asymptotic representation holds at the 1% level.

Massless Wilson Coefficients

based on: Blümlein, Marquard, Schneider, Schönwald, JHEP (2022) (arXiv:2208.14325)

Massless Wilson Coefficients

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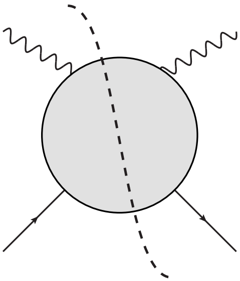
- The massless Wilson coefficients can be calculated by evaluating the forward compton amplitude.
- The unpolarized v coefficients are known up to N^{3}LO ($\mathcal{O}(\alpha_s^3)$):
  - NLO: [Furmanski, Petronzio '82; ...]
  - NNLO: [van Neerven, Zijlstra '91,'92; ... ; Moch, Vermaseren '00]
  - N^{3}LO: [Moch, Vermasern, Vogt, Nucl.Phys.B '05,'09'; Moch, Rogal, Vogt '08]
- The calculating at N^{3}LO was performed by one group only.
- The polarized Wilson coefficients have been known up to N^{2}LO ($\mathcal{O}(\alpha_s^2)$):
  - NLO: [Furmanski, Petronzio '82; ...]
  - NNLO: [van Neerven, Zijlstra '94; Vogt, Moch, Rogal, Vermaseren '08; ...]



$\Rightarrow$ Cross check the results for unpolarized scattering and extend to polarized structure function g_1.

Massless Wilson Coefficients

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Technical Aspects:

- Mellin moments can be calculated by expansion in $y = 2p.q$: $M[F_i](N) = \frac{1}{N!}[\frac{d^N F_i}{dy^N}]_{y=0}$.
- Diagram generation: `QGRAF` [Nogueira '91]
- Lorentz, Dirac and color algebra: `TFORM` [Ruijl, Ueda, Vermaseren, Tentyukov '17] with `color.h` [Ritbergen, Schellekens, Vermaseren '99]
- IBP reduction and differential equations: `Crusher` [Marquard, Seidel]
- We use two independent methods to compute the Mellin space result:
  1. Method of large moments:
     - Compute a large number of moments: `SolveCoupledSystems` [Blümlein, Schneider '17]
     - Determine a recurrence from the moments: `Guess` [Kauers '09-'15]
     - Solve the recurrence: `Sigma` [Schneider '07]
  2. Analytic computation of the master integrals: [Ablinger, Blümlein, Marquard, Rana, Schneider '18]
     - Decouple systems of differential equations: `OreSys` [Gerhold, Schneider '02]
     - Solve analytically in y via factorization of the differential operator: `HarmonicSums` [Ablinger '10 -]
     - Take the Nth derivative symbolically to obtain the Mellin space expression: `HarmonicSums`
- Both methods lead to the same results.

Massless Wilson Coefficients

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- We agree with the unpolarized structure functions F_2, F_L, F_3 in the literature.
- We extend the calculation to the polarized case of g_1 in the Larin scheme:

$$\gamma_5\gamma_\mu \to \frac{1}{6}\epsilon_{\mu\nu\rho\sigma}\gamma^\nu\gamma^\rho\gamma^\sigma$$

- The non-singlet contribution agrees with the one of F_3 after a finite renormalization (which we independently checked by considering off-shell OMEs).
- The finite renormalization for the singlet piece is not known at $\mathcal{O}(\alpha_s^3)$ yet.
- The master integrals and Wilson coefficients can be expressed by:
  - harmonic sums in N-space: $S_{a,\vec{b}}(N) = \sum_{i=1}^{N} \frac{\text{sign}(a)^i}{i^a} S_{\vec{b}}(i)$
  - harmonic polylogarithms in x-space: $H_{a,\vec{b}}(x) = \int_0^x \mathrm{d}x' f_a(x') H_{\vec{b}}(x')$, with $f_0(x) = \frac{1}{x}$, $f_1(x) = \frac{1}{1-x}$, $f_{-1}(x) = \frac{1}{1+x}$

only. [Vermaseren '99; Blümlein, Kurth '99; Remiddi, Vermaseren '00]

- We provide results in Mellin N-space and in x-space as well as large and small x limits:

$$
\begin{aligned}
\Delta C_{g_1,g}^{(3),d_{abc}} = {} & \frac{d_{abc}d^{abc}N_F^2}{N_A}\Bigg\{-\frac{128(N-2)(3+N)P_{1048}}{45N(1+N)(2+N)}S_5-\frac{64(N-2)(3+N)P_{1048}}{9N(1+N)(2+N)}\zeta_5 \\
& +\frac{128P_{1049}}{(N-1)N(1+N)(2+N)^2}S_3+\frac{32P_{1052}}{45(N-1)N(1+N)(2+N)^2} \\
& -\frac{256P_{1053}}{45N^2(1+N)^2(2+N)}S_{-2,1}-\frac{512P_{1055}}{3(N-1)N^2(1+N)^2(2+N)^2}S_4 \\
& +\frac{1024P_{1055}}{3(N-1)N^2(1+N)^2(2+N)^2}S_{3,1}-\frac{64P_{1058}}{45(N-1)N^2(1+N)^2(2+N)^2}\zeta_3 \\
& +\Bigg[-\frac{128\big(-4+N+N^2\big)P_{1050}}{(N-1)N^2(1+N)^2(2+N)^2}S_3-\frac{64P_{1051}}{45(N-1)N(1+N)(2+N)^2} \\
& +\frac{512P_{1056}}{3(N-1)N^2(1+N)^2(2+N)^2}\zeta_3+\frac{512\big(-4+N+N^2\big)}{N(1+N)(2+N)}S_4 \\
& -\frac{1024\big(-4+N+N^2\big)S_{3,1}}{N(1+N)(2+N)}+\frac{512\big(-8+3N+3N^2\big)}{3N(1+N)(2+N)}S_{-2,1}\Bigg]S_1 \\
& +\Bigg[\frac{128\big(-19+3N+3N^2\big)}{3N(1+N)(2+N)}+\frac{256\big(-4+N+N^2\big)}{N(1+N)(2+N)}S_3-\frac{512\big(-4+N+N^2\big)}{N(1+N)(2+N)} \\
& \times\zeta_3\Bigg]S_1^2-\frac{768\big(-4+N+N^2\big)}{N(1+N)(2+N)}S_2S_3+\Bigg[-\frac{64P_{1054}}{45(N-1)N(1+N)(2+N)^2} \\
& +\frac{256P_{1057}}{45(N-1)N^2(1+N)^2(2+N)^2}S_1-\frac{256\big(-8+3N+3N^2\big)}{3N(1+N)(2+N)}S_1^2 \\
& -\frac{256}{45}(N-1)\big(-21+N+N^2\big)[S_3+S_{-2,1}+\zeta_3]\Bigg]S_{-2}-\frac{128}{2+N}S_{-2}^2 \\
& +\Bigg[\frac{128P_{1053}}{45N^2(1+N)^2(2+N)}-\frac{256\big(-8+3N+3N^2\big)}{3N(1+N)(2+N)}S_1\Bigg]S_{-3} \\
& -\frac{256\big(-4+3N+3N^2\big)}{3N(1+N)(2+N)}S_{-4}-\frac{128}{45}(N-1)\big(-21+N+N^2\big)S_{-5} \\
& +\frac{768\big(-4+N+N^2\big)}{N(1+N)(2+N)}[S_{2,3}-S_{4,1}]+\frac{256\big(-8+3N+3N^2\big)}{3N(1+N)(2+N)}S_{-2,2} \\
& +\frac{256}{45}(N-1)\big(-21+N+N^2\big)[S_{-2,3}+S_{-2,1,-2}]+\frac{256\big(-8+3N+3N^2\big)}{3N(1+N)(2+N)}S_{-3,1} \\
& +\frac{1536\big(-4+N+N^2\big)}{N(1+N)(2+N)}S_{3,1,1}-\frac{512\big(-8+3N+3N^2\big)}{3N(1+N)(2+N)}S_{-2,1,1}\Bigg\} \qquad (6.333)
\end{aligned}
$$

$$
\begin{aligned}
\Delta C_{g_1,q}^{(3),\mathrm{PS,L}} \simeq {} & C_F\Bigg\{T_F^2N_F^2\Bigg[-\frac{184}{27}\ln^4(x)-\frac{7312}{81}\ln^3(x)+\Bigg(-\frac{37744}{81}+\frac{64}{9}\zeta_2\Bigg)\ln^2(x) \\
& +\Bigg(-\frac{245504}{243}+\frac{1088}{27}\zeta_2-\frac{64}{9}\zeta_3\Bigg)\ln(x)-\frac{292640}{243}+\frac{14528}{81}\zeta_2-\frac{32}{27}\zeta_3+\frac{224}{15}\zeta_2^2\Bigg] \\
& +C_FT_FN_F\Bigg[4\ln^5(x)+\frac{632}{9}\ln^4(x)+\Bigg(\frac{5156}{9}-\frac{1328}{9}\zeta_2\Bigg)\ln^3(x)+\Bigg(2708 \\
& -\frac{4096\zeta_2}{3}-\frac{256\zeta_3}{3}\Bigg)\ln^2(x)+\Bigg(\frac{23360}{3}-5064\zeta_2-224\zeta_3+\frac{3856}{15}\zeta_2^2\Bigg)\ln(x) \\
& +\frac{31712}{3}-6856\zeta_2+\Bigg(-\frac{6160}{3}+32\zeta_2\Bigg)\zeta_3+\frac{15008}{15}\zeta_2^2+352\zeta_5\Bigg] \\
& +C_AT_FN_F\Bigg[4\ln^5(x)+\frac{2450}{27}\ln^4(x)+\Bigg(\frac{67472}{81}-\frac{688}{9}\zeta_2\Bigg)\ln^3(x)+\Bigg(\frac{325580}{81} \\
& -\frac{6992\zeta_2}{9}-352\zeta_3\Bigg)\ln^2(x)+\Bigg(\frac{2395192}{243}-\frac{74464}{27}\zeta_2-\frac{21136}{9}\zeta_3+\frac{3088}{15}\zeta_2^2\Bigg)\ln(x) \\
& +\frac{2166976}{243}-\frac{315832}{81}\zeta_2+\Bigg(-\frac{93968}{27}+\frac{1808}{3}\zeta_2\Bigg)\zeta_3+\frac{10312}{15}\zeta_2^2-264\zeta_5\Bigg]\Bigg\},
\end{aligned}
$$

Massive Operator Matrix Elements

Status of OME Calculations

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Leading Order: [Witten (1976); Babcock, Sivers, Wolfram (1978); Shifman, Vainshtein, Zakharov (1978); Leveille, Weiler (1979); Glück, Reya (1979); Glück, Hoffmann, Reya (1982)]

Next-to-Leading Order:

full m dependence (numeric) [Laenen, van Neerven, Riemersma, Smith (1993)]

$Q^2 \gg m^2$: via IBP [Buza, Matiounine, Smith, Migneron, van Neerven (1996)]

Compact results via ${}_pF_q$'s [Bierenbaum, Blümlein, Klein (2007)]

$O(\alpha_s^2 \varepsilon)$ (for general N) [Bierenbaum, Blümlein, Klein (2008, 2009)]

Next-to-Next-to-Leading Order: $Q^2 \gg m^2$

- Moments (using `MATAD` [Steinhauser (2000)]):
  - F_2: $N = 2...10(14)$ [Bierenbaum, Blümlein, Klein (2009)]
  - transversity: $N = 1...13$
  - Two masses $m_1 \neq m_2 \to$ Moments N = 2,4,6 [Blümlein, Wißbrock (2011)]
- Analytic solutions for $A_{qq,Q}^{NS}$, $A_{qg,Q}$, $A_{gq,Q}$, $A_{qq,Q}^{PS}$, A_{Qq}^{PS} [Blümlein et al (2010-2023)] , with recent extension to polarized scattering.
- Analytic two mass solutions for $A_{qq,Q}^{NS}$, $A_{qg,Q}$, $A_{gq,Q}$, $A_{qq,Q}^{PS}$, A_{Qq}^{PS}, $A_{gg,Q}$ [Blümlein et al (2017-2020)] , with recent extension to polarized scattering.

The Wilson Coefficients at Large Q^2

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$$
\begin{aligned}
L_{q,(2,L)}^{\rm NS}(N_F+1) &= a_s^2\left[A_{qq,Q}^{(2),\rm NS}(N_F+1)\delta_2 + \hat{C}_{q,(2,L)}^{(2),\rm NS}(N_F)\right] + a_s^3\left[A_{qq,Q}^{(3),\rm NS}(N_F+1)\delta_2 + A_{qq,Q}^{(2),\rm NS}(N_F+1)C_{q,(2,L)}^{(1),\rm NS}(N_F+1) + \hat{C}_{q,(2,L)}^{(3),\rm NS}(N_F)\right] \\
L_{q,(2,L)}^{\rm PS}(N_F+1) &= a_s^3\left[A_{qq,Q}^{(3),\rm PS}(N_F+1)\delta_2 + N_F A_{gq,Q}^{(2),\rm NS}(N_F)\tilde{C}_{g,(2,L)}^{(1),\rm NS}(N_F+1) + N_F\hat{\tilde{C}}_{q,(2,L)}^{(3),\rm PS}(N_F)\right] \\
L_{g,(2,L)}^{\rm S}(N_F+1) &= a_s^2 A_{gg,Q}^{(1)}(N_F+1)N_F\tilde{C}_{g,(2,L)}^{(2)}(N_F+1) + a_s^3\left[A_{qg,Q}^{(3)}(N_F+1)\delta_2 + A_{gg,Q}^{(1)}(N_F+1)N_F\tilde{C}_{g,(2,L)}^{(2)}(N_F+1)\right. \\
&\quad \left. + A_{gg,Q}^{(2)}(N_F+1)N_F\tilde{C}_{g,(2,L)}^{(1)}(N_F+1) + A_{Qg}^{(1)}(N_F+1)N_F\tilde{C}_{q,(2,L)}^{(2),\rm PS}(N_F+1) + N_F\hat{\tilde{C}}_{g,(2,L)}^{(3)}(N_F)\right] \\
H_{q,(2,L)}^{\rm PS}(N_F+1) &= a_s^2\left[A_{Qq}^{(2),\rm PS}(N_F+1)\delta_2 + \tilde{C}_{q,(2,L)}^{(2),\rm PS}(N_F+1)\right] \\
&\quad + a_s^3\left[A_{Qq}^{(3),\rm PS}(N_F+1)\delta_2 + A_{gq,Q}^{(2)}(N_F+1)\tilde{C}_{g,(1,L)}^{(2)}(N_F+1) + A_{Qq}^{(2),\rm PS}(N_F+1)\tilde{C}_{q,(2,L)}^{(1),\rm NS}(N_F+1) + \tilde{C}_{q,(2,L)}^{(3),\rm PS}(N_F+1)\right] \\
H_{g,(2,L)}^{\rm S}(N_F+1) &= a_s\left[A_{Qg}^{(1)}(N_F+1)\delta_2 + \tilde{C}_{g,(2,L)}^{(1)}(N_F+1)\right] \\
&\quad + a_s^2\left[A_{Qg}^{(2)}(N_F+1)\delta_2 + A_{Qg}^{(1)}(N_F+1)\tilde{C}_{q,(2,L)}^{(1)}(N_F+1) + A_{gg,Q}^{(1)}(N_F+1)\tilde{C}_{g,(2,L)}^{(1)}(N_F+1) + \tilde{C}_{g,(2,L)}^{(2)}(N_F+1)\right] \\
&\quad + a_s^3\left[A_{Qg}^{(3)}(N_F+1)\delta_2 + A_{Qg}^{(2)}(N_F+1)\tilde{C}_{q,(2,L)}^{(1)}(N_F+1) + A_{gg,Q}^{(2)}(N_F+1)\tilde{C}_{g,(2,L)}^{(1)}(N_F+1)\right. \\
&\quad \left. + A_{Qg}^{(1)}(N_F+1)\tilde{C}_{q,(2,L)}^{(2),\rm S}(N_F+1) + A_{gg,Q}^{(1)}(N_F+1)\tilde{C}_{g,(2,L)}^{(1)}(N_F+1) + \tilde{C}_{g,(2,L)}^{(3)}(N_F+1)\right]
\end{aligned}
$$

- All first order factorizable contributions and $O(1000)$ fixed moments of $A_{Qg}^{(3)}$ are known.
- The case for two different masses obeys an analogous representation.

The Variable Flavor Number Scheme

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- Matching conditions for parton distribution functions:

$$f_k(N_F+2) + f_{\overline{k}}(N_F+2) = A_{qq,Q}^{\rm NS}\left(N_F+2, \frac{m_1^2}{\mu^2}, \frac{m_2^2}{\mu^2}\right) \cdot \left[f_k(N_F) + f_{\overline{k}}(N_F)\right] + \frac{1}{N_F} A_{qq,Q}^{\rm PS}\left(N_F+2, \frac{m_1^2}{\mu^2}, \frac{m_2^2}{\mu^2}\right) \cdot \Sigma(N_F)$$
$$+ \frac{1}{N_F} A_{qg,Q}\left(N_F+2, \frac{m_1^2}{\mu^2}, \frac{m_2^2}{\mu^2}\right) \cdot G(N_F)\,,$$

$$f_Q(N_F+2) + f_{\overline{Q}}(N_F+2) = A_{Qq}^{\rm PS}\left(N_F+2, \frac{m_1^2}{\mu^2}, \frac{m_2^2}{\mu^2},\right) \cdot \Sigma(N_F) + A_{Qg}\left(N_F+2, \frac{m_1^2}{\mu^2}, \frac{m_2^2}{\mu^2}\right) \cdot G(N_F)\,,$$

$$\Sigma(N_F+2) = \left[A_{qq,Q}^{\rm NS}\left(N_F+2, \frac{m_1^2}{\mu^2}, \frac{m_2^2}{\mu^2}\right) + A_{qq,Q}^{\rm PS}\left(N_F+2, \frac{m_1^2}{\mu^2}, \frac{m_2^2}{\mu^2}\right) + A_{Qq}^{\rm PS}\left(N_F+2, \frac{m_1^2}{\mu^2}, \frac{m_2^2}{\mu^2}\right)\right] \cdot \Sigma(N_F)$$
$$+ \left[A_{qg,Q}\left(N_F+2, \frac{m_1^2}{\mu^2}, \frac{m_2^2}{\mu^2}\right) + A_{Qg}\left(N_F+2, \frac{m_1^2}{\mu^2}, \frac{m_2^2}{\mu^2}\right)\right] \cdot G(N_F)\,,$$

$$G(N_F+2) = A_{gq,Q}\left(N_F+2, \frac{m_1^2}{\mu^2}, \frac{m_2^2}{\mu^2}\right) \cdot \Sigma(N_F) + A_{gg,Q}\left(N_F+2, \frac{m_1^2}{\mu^2}, \frac{m_2^2}{\mu^2}\right) \cdot G(N_F)\,.$$

The Variable Flavor Number Scheme

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- Matching conditions for parton distribution functions:

$$f_k(N_F+2)+f_{\overline{k}}(N_F+2)=A_{qq,Q}^{\rm NS}\left(N_F+2,\frac{m_1^2}{\mu^2},\frac{m_2^2}{\mu^2}\right)\cdot\left[f_k(N_F)+f_{\overline{k}}(N_F)\right]+\frac{1}{N_F}A_{qq,Q}^{\rm PS}\left(N_F+2,\frac{m_1^2}{\mu^2},\frac{m_2^2}{\mu^2}\right)\cdot\Sigma(N_F)$$
$$+\frac{1}{N_F}A_{qg,Q}\left(N_F+2,\frac{m_1^2}{\mu^2},\frac{m_2^2}{\mu^2}\right)\cdot G(N_F)\,,$$

$$f_Q(N_F+2)+f_{\overline{Q}}(N_F+2)=A_{Qq}^{\rm PS}\left(N_F+2,\frac{m_1^2}{\mu^2},\frac{m_2^2}{\mu^2},\right)\cdot\Sigma(N_F)+A_{Qg}\left(N_F+2,\frac{m_1^2}{\mu^2},\frac{m_2^2}{\mu^2}\right)\cdot G(N_F)\,,$$

$$\Sigma(N_F+2)=\left[A_{qq,Q}^{\rm NS}\left(N_F+2,\frac{m_1^2}{\mu^2},\frac{m_2^2}{\mu^2}\right)+A_{qq,Q}^{\rm PS}\left(N_F+2,\frac{m_1^2}{\mu^2},\frac{m_2^2}{\mu^2}\right)+A_{Qq}^{\rm PS}\left(N_F+2,\frac{m_1^2}{\mu^2},\frac{m_2^2}{\mu^2}\right)\right]\cdot\Sigma(N_F)$$
$$+\left[A_{qg,Q}\left(N_F+2,\frac{m_1^2}{\mu^2},\frac{m_2^2}{\mu^2}\right)+A_{Qg}\left(N_F+2,\frac{m_1^2}{\mu^2},\frac{m_2^2}{\mu^2}\right)\right]\cdot G(N_F)\,,$$

$$G(N_F+2)=A_{gq,Q}\left(N_F+2,\frac{m_1^2}{\mu^2},\frac{m_2^2}{\mu^2}\right)\cdot\Sigma(N_F)+A_{gg,Q}\left(N_F+2,\frac{m_1^2}{\mu^2},\frac{m_2^2}{\mu^2}\right)\cdot G(N_F)\,.$$

The Massive OME $A_{gg,Q}^{(3)}$ [Ablinger et al JHEP (2022) (arXiv:2211.05462)]

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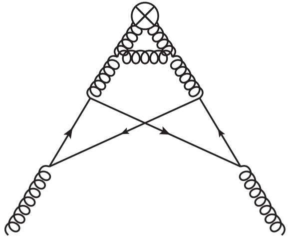
Technical Aspects:

- The diagrams are given by gluon propagators with operator insertions.
- To deal with the operators we can resum into propagator structures:

$$(\Delta.k)^N \to \sum_{N=0}^{\infty} t^N (\Delta.k)^N = \frac{1}{1 - t\,\Delta.k}$$

- The formal parameter t now resembles the variable y in the computation of the structure functions: $M[F_i](N) = \frac{1}{N!}[\frac{d^N F_i}{dt^N}]_{t=0}$
  $\Rightarrow$ All calculational techniques as mentioned before apply.



Operators induce additional Feynman rules, e.g.:

$$\frac{1+(-1)^N}{2}\delta^{ab}(\Delta\cdot p)^{N-2}$$
$$\left[g_{\mu\nu}(\Delta\cdot p)^2 - (\Delta_\mu p_\nu + \Delta_\nu p_\mu)\Delta\cdot p + p^2\Delta_\mu\Delta_\nu\right], \quad N \geq 2$$

- We find much more involved analytical structures than in the massless case:
  - Binomially weighted sums in Mellin space, e.g.:

$$\mathrm{BS}_3(N) = \sum_{\tau_1=1}^{N} \frac{4^{-\tau_1}(2\tau_1)!}{(\tau_1!)^2 \tau_1}, \qquad \mathrm{BS}_8(N) = \sum_{\tau_1=1}^{N} \frac{\sum_{\tau_2=1}^{\tau_1} \frac{4^{\tau_2}(\tau_2!)^2}{(2\tau_2)!\tau_2^2}}{\tau_1}$$

  - Iterated integrals over square root valued letters in x-space, i.e. over the alphabet:

$$\left\{\frac{1}{x}, \frac{1}{1-x}, \frac{1}{1+x}, \frac{\sqrt{1-x}}{x}, \sqrt{x(1-x)}, \frac{1}{\sqrt{1-x}}\right\}$$

  - The (inverse) Mellin transformations can be calculated analytically with `HarmonicSums`:

$$\mathbf{M}^{-1}[\mathrm{BS}_8(N)](x) = \left[-\frac{4(1-\sqrt{1-x})}{1-x} + \left(\frac{2(1-\ln(2))}{1-x} + \frac{\mathrm{H}_0(x)}{\sqrt{1-x}}\right)\mathrm{H}_1(x) - \frac{\mathrm{H}_{0,1}(x)}{\sqrt{1-x}} + \frac{\mathrm{H}_1(x)}{2(1-x)}\int_0^x \frac{\mathrm{H}_0(x)}{\sqrt{1-x}}\mathrm{d}x - \frac{1}{2(1-x)}\int_0^x \frac{\mathrm{H}_{0,1}(x)}{\sqrt{1-x}}\mathrm{d}x\right]_+$$

- The x-space representation of some diagrams has been cross-checked with a new method, which allows to go directly from t- to x-space we developed.

Inverse Mellin Transformation by Analytic Continuation

[Behring, Blümlein, Schönwald, arXiv:2303.05943]

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A sketch of the technique:

$$F(N) = \int_0^1 \mathrm{d}x f(x), \qquad \tilde{F}(t) = \sum_{N=1}^{\infty} t^N F(N) = \int_0^1 \mathrm{d}x' \frac{f(x')}{1 - tx'} = \int_0^1 \mathrm{d}x' \frac{f(x')}{x - x'},$$

with $t = 1/x$. Use Sochocki formula:

$$\lim_{\delta \to 0^+} \frac{1}{\xi \pm i\delta} = \mathcal{P}\frac{1}{\xi} \mp \mathrm{i}\pi\delta(\xi)$$

$\Rightarrow$ x-space result can be obtained by analytic continuation of the Function $\tilde{F}(1/x)$ and taking the imaginary part.

There are some more subtleties related to distributions and factors like $(-1)^N$ I glossed over here.

Small and Large *x* Limits of $a_{gg,Q}^{(3)}$

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- We considered unpolarized and polarized scattering.
- The analytic results allow to obtain the small and large *x* expansion.
- Despite the iterated integrals over square roots only well known constants occur.

$$
\begin{aligned}
&a_{gg,Q}^{x\to 0}(x) \propto \\
&\frac{1}{x}\Bigg\{\ln(x)\Bigg[C_A^2 T_F\left(-\frac{11488}{81}+\frac{224\zeta_2}{27}+\frac{256\zeta_3}{3}\right)+C_A C_F T_F\left(-\frac{15040}{243}-\frac{1408\zeta_2}{27}\right.\\
&\left.-\frac{1088\zeta_3}{9}\right)\Bigg]+C_A T_F^2\left[\frac{112016}{729}+\frac{1288}{27}\zeta_2+\frac{1120}{27}\zeta_3+\left(\frac{108256}{729}+\frac{368\zeta_2}{27}-\frac{448\zeta_3}{27}\right)\right.\\
&\left.\times N_F\right]+C_F\Bigg[T_F^2\left(-\frac{107488}{729}-\frac{656}{27}\zeta_2+\frac{3904}{27}\zeta_3+\left(\frac{116800}{729}+\frac{224\zeta_2}{27}-\frac{1792\zeta_3}{27}\right)N_F\right)\\
&+C_A T_F\left(-\frac{5538448}{3645}+\frac{1664\mathrm{B}_4}{3}-\frac{43024\zeta_4}{9}+\frac{12208}{27}\zeta_2+\frac{211504}{45}\zeta_3\right)\Bigg]\\
&+C_A^2 T_F\left(-\frac{4849484}{3645}-\frac{352\mathrm{B}_4}{3}+\frac{11056\zeta_4}{9}-\frac{1088}{81}\zeta_2-\frac{84764}{135}\zeta_3\right)\\
&+C_F^2 T_F\left(\frac{10048}{5}-640\mathrm{B}_4+\frac{51104\zeta_4}{9}-\frac{10096}{9}\zeta_2-\frac{280016}{45}\zeta_3\right)\Bigg\}\\
&+\left[-\frac{4}{3}C_F C_A T_F+\frac{2}{15}C_F^2 T_F\right]\ln^5(x)+\Bigg[-\frac{40}{27}C_A^2 T_F+\frac{4}{9}C_F^2 T_F+C_F\Bigg(-\frac{296}{27}C_A T_F\\
&+\left(\frac{28}{27}+\frac{56}{27}N_F\right)T_F^2\Bigg)\Bigg]\ln^4(x)+\Bigg[\frac{112}{81}C_A(1+2N_F)T_F^2+C_F\Bigg(\left(\frac{1016}{81}+\frac{496}{81}N_F\right)T_F^2\\
&+C_A T_F\left(-\frac{10372}{81}-\frac{328\zeta_2}{9}\right)\Bigg)+C_F^2 T_F\left[-\frac{2}{3}+\frac{4\zeta_2}{9}\right]+C_A^2 T_F\left[-\frac{1672}{81}+8\zeta_2\right]\Bigg]\ln^3(x)\\
&+\Bigg[\frac{8}{81}C_A(155+118N_F)T_F^2+C_F\Bigg[T_F^2\left(-\frac{32}{81}+N_F\left(\frac{3872}{81}-\frac{16\zeta_2}{9}\right)+\frac{232\zeta_2}{9}\right)\\
&+C_A T_F\left(-\frac{70304}{81}-\frac{680\zeta_2}{9}+\frac{80\zeta_3}{3}\right)\Bigg]+C_A^2 T_F\left[\frac{4684}{81}+\frac{20\zeta_2}{3}\right]+C_F^2 T_F\Bigg[56\\
&+\frac{8\zeta_2}{3}-40\zeta_3\Bigg]\Bigg]\ln^2(x)+\Bigg[C_F\Bigg[T_F^2\left(\frac{140992}{243}+N_F\left(\frac{182528}{243}-\frac{400\zeta_2}{27}-\frac{640\zeta_3}{9}\right)\right.\\
&\left.-\frac{728}{27}\zeta_2-\frac{224}{9}\zeta_3\right)+C_A T_F\left(-\frac{514952}{243}+\frac{152\zeta_4}{3}-\frac{21140\zeta_2}{27}-\frac{2576\zeta_3}{9}\right)\Bigg]\\
&+C_A T_F^2\left[\frac{184}{27}+N_F\left(\frac{656}{27}-\frac{32\zeta_2}{27}\right)+\frac{464\zeta_2}{27}\right]+C_A^2 T_F\Bigg[-\frac{42476}{81}-92\zeta_4+\frac{4504\zeta_2}{27}\\
&+\frac{64\zeta_3}{3}\Bigg]+C_F^2 T_F\left[-\frac{1036}{3}-\frac{976\zeta_4}{3}-\frac{58\zeta_2}{3}+\frac{416\zeta_3}{3}\right]\Bigg]\ln(x),
\end{aligned}
$$

Small and Large x Limits of $a_{gg,Q}^{(3)}$

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- We considered unpolarized and polarized scattering.
- The analytic results allow to obtain the small and large x expansion.
- Despite the iterated integrals over square roots only well known constants occur.

$$
\begin{aligned}
a_{gg,Q}^{(3),x\to1}(x) \propto{}& a_{gg,Q,\delta}^{(3)}\delta(1-x)+a_{gg,Q,\mathrm{plus}}^{(3)}(x)+\Bigg[-\frac{32}{27}C_AT_F^2(17+12N_F)+C_AC_FT_F\left(56-\frac{32\zeta_2}{3}\right)\\
&+C_A^2T_F\left(\frac{9238}{81}-\frac{104\zeta_2}{9}+16\zeta_3\right)\Bigg]\ln(1-x)+\Bigg[-\frac{8}{27}C_AT_F^2(7+8N_F)\\
&+C_A^2T_F\left(\frac{314}{27}-\frac{4\zeta_2}{3}\right)\Bigg]\ln^2(1-x)+\frac{32}{27}C_A^2T_F\ln^3(1-x). \qquad (4.11)
\end{aligned}
$$

$$
\begin{aligned}
(\Delta)a_{gg,Q,\delta}^{(3)} ={}& T_F\Bigg\{C_F\Bigg[C_A\left(\frac{16541}{162}-\frac{64\mathrm{B}_4}{3}+\frac{128\zeta_4}{3}+52\zeta_2-\frac{2617\zeta_3}{12}\right)+T_F\Bigg(-\frac{1478}{81}\\
&+N_F\left(-\frac{1942}{81}-\frac{20\zeta_2}{3}\right)-\frac{88\zeta_2}{3}-7\zeta_3\Bigg)\Bigg]+C_A^2\Bigg[\frac{34315}{324}+\frac{32\mathrm{B}_4}{3}-\frac{3778\zeta_4}{27}\\
&+\frac{992}{27}\zeta_2+\left(\frac{20435}{216}+24\zeta_2\right)\zeta_3-\frac{304}{9}\zeta_5\Bigg]+C_AT_F\Bigg[\frac{2587}{135}+N_F\left(-\frac{178}{9}+\frac{196\zeta_2}{27}\right)\\
&+\frac{572\zeta_2}{27}-\frac{291\zeta_3}{10}\Bigg]+C_F^2\left[\frac{274}{9}+\frac{95\zeta_3}{3}\right]+\frac{64}{27}T_F^2\zeta_3\Bigg\}, \qquad (4.6)
\end{aligned}
$$

$$
\begin{aligned}
(\Delta)a_{gg,Q,\mathrm{plus}}^{(3)} ={}& \frac{T_F}{1-x}\Bigg\{C_AT_F\left[\frac{35168}{729}+N_F\left(\frac{55552}{729}+\frac{160\zeta_2}{27}-\frac{448\zeta_3}{27}\right)+\frac{560}{27}\zeta_2+\frac{1120}{27}\zeta_3\right]\\
&+C_A^2\left[-\frac{32564}{729}-\frac{32\mathrm{B}_4}{3}+104\zeta_4-\frac{3248\zeta_2}{81}-\frac{1796\zeta_3}{27}\right]+C_AC_F\Bigg[-\frac{6152}{27}+\frac{64\mathrm{B}_4}{3}\\
&-96\zeta_4-40\zeta_2+\frac{1208\zeta_3}{9}\Bigg]\Bigg\}. \qquad (4.7)
\end{aligned}
$$

Representations of the OME

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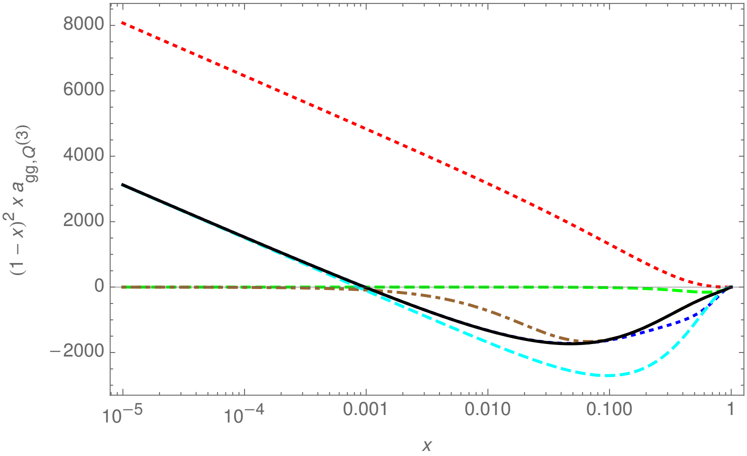
- The logarithmic parts of $(\Delta)A_{gg}^{(3)}$ have been computed before [Behring et al., (2014)], [Blümlein et al. (2021)].
- ***N* space**
  - Recursions available for all building blocks: $N \to N + 1$.
  - Asymptotic representations available.
  - Contour integral around the singularities of the problem at the non-positive real axis.
- ***x* space**
  - All constants occurring in the transition $t \to x$ can be calculated in terms of ζ-values.
  - This can be proven analytically by first rationalizing and then calculating the obtained cyclotomic harmonic polylogarithms.
  - Separate the $\delta(1 - x)$ and $+$-function terms first.
  - Series representations to 50 terms around $x = 0$ and $x = 1$ can be derived for the regular part analytically (12 digits).
  - The accuracy can be easily enlarged, if needed.



The non–N_F terms of $a_{gg,Q}^{(3)}(N)$ (rescaled) as a function of x. Full line (black): complete result; upper dotted line (red): term $\propto \ln(x)/x$, BFKL limit; lower dashed line (cyan): small x terms $\propto 1/x$; lower dotted line (blue): small x terms including all $\ln(x)$ terms up to the constant term; upper dashed line (green): large x contribution up to the constant term; dash-dotted line (brown): complete large x contribution.

Conclusions and Outlook

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Conclusions

- We recomputed the massless Wilson coefficients F_2, F_L and F_3 for DIS up to $\mathcal{O}(\alpha_s^3)$.
- We computed the massless Wilson coefficient g_1 for polarized scattering for the first time at $\mathcal{O}(\alpha_s^3)$.
- We calculated the gluonic massive Operator matrix elements $(\Delta)A_{gg}^{(3)}$ analytically in N- and x-space.
- With this, all operator matrix elements except of $(\Delta)A_{Qg}^{(3)}$ have been calculated for general values of N.

Outlook

- Study of the VFNS in the polarized case.
- Calculation of $(\Delta)A_{Qg}^{(3)}$ with new method to obtain x-space representation is in progress.
- Calculation of two mass contributions to $(\Delta)A_{Qg}^{(3)}$ in an expansion in m_c/m_b underway.

Backup

Calculation of the 3-loop Operator Matrix Elements

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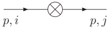
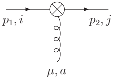
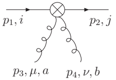
The OMEs are calculated using the QCD Feynman rules together with the following operator insertion Feynman rules:



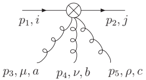
$$\delta^{ij}\not\Delta\gamma_\pm(\Delta\cdot p)^{N-1}\,, \quad N\geq 1$$



$$g t^a_{ji}\Delta^\mu\not\Delta\gamma_\pm\sum_{j=0}^{N-2}(\Delta\cdot p_1)^j(\Delta\cdot p_2)^{N-j-2}\,, \quad N\geq 2$$



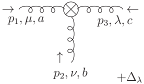
$$g^2\Delta^\mu\Delta^\nu\not\Delta\gamma_\pm\sum_{j=0}^{N-3}\sum_{l=j+1}^{N-2}(\Delta p_2)^j(\Delta p_1)^{N-l-2} \Big[(t^at^b)_{ji}(\Delta p_1+\Delta p_4)^{l-j-1}+(t^bt^a)_{ji}(\Delta p_1+\Delta p_3)^{l-j-1}\Big]\,,$$
$$N\geq 3$$



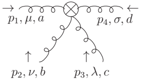
$$\begin{aligned}
&g^3\Delta_\mu\Delta_\nu\Delta_\rho\not\Delta\gamma_\pm\sum_{j=0}^{N-4}\sum_{l=j+1}^{N-3}\sum_{m=l+1}^{N-2}(\Delta.p_2)^j(\Delta.p_1)^{N-m-2}\\
&\Big[\ (t^at^bt^c)_{ji}(\Delta.p_4+\Delta.p_5+\Delta.p_1)^{l-j-1}(\Delta.p_5+\Delta.p_1)^{m-l-1}\\
&\ +(t^at^ct^b)_{ji}(\Delta.p_4+\Delta.p_5+\Delta.p_1)^{l-j-1}(\Delta.p_4+\Delta.p_1)^{m-l-1}\\
&\ +(t^bt^at^c)_{ji}(\Delta.p_3+\Delta.p_5+\Delta.p_1)^{l-j-1}(\Delta.p_5+\Delta.p_1)^{m-l-1}\\
&\ +(t^bt^ct^a)_{ji}(\Delta.p_3+\Delta.p_5+\Delta.p_1)^{l-j-1}(\Delta.p_3+\Delta.p_1)^{m-l-1}\\
&\ +(t^ct^at^b)_{ji}(\Delta.p_3+\Delta.p_4+\Delta.p_1)^{l-j-1}(\Delta.p_4+\Delta.p_1)^{m-l-1}\\
&\ +(t^ct^bt^a)_{ji}(\Delta.p_3+\Delta.p_4+\Delta.p_1)^{l-j-1}(\Delta.p_3+\Delta.p_1)^{m-l-1}\Big]\,,\\
&\qquad N\geq 4
\end{aligned}$$

$$\gamma_+=1\,, \quad \gamma_-=\gamma_5\,.$$

$$\frac{1+(-1)^N}{2}\delta^{ab}(\Delta\cdot p)^{N-2}$$
$$\Big[g_{\mu\nu}(\Delta\cdot p)^2-(\Delta_\mu p_\nu+\Delta_\nu p_\mu)\Delta\cdot p+p^2\Delta_\mu\Delta_\nu\Big]\,, \quad N\geq 2$$



$$\begin{aligned}
&-ig\frac{1+(-1)^N}{2}f^{abc}\Bigg(\\
&\Big[(\Delta_\nu g_{\lambda\mu}-\Delta_\lambda g_{\mu\nu})\Delta\cdot p_1+\Delta_\mu(p_{1,\nu}\Delta_\lambda-p_{1,\lambda}\Delta_\nu)\Big](\Delta\cdot p_1)^{N-2}\\
&+\Delta_\lambda\Big[\Delta\cdot p_1 p_{2,\mu}\Delta_\nu+\Delta\cdot p_2 p_{1,\nu}\Delta_\mu-\Delta\cdot p_1\Delta\cdot p_2 g_{\mu\nu}-p_1\cdot p_2\Delta_\mu\Delta_\nu\Big]\\
&\times\sum_{j=0}^{N-3}(-\Delta\cdot p_1)^j(\Delta\cdot p_2)^{N-3-j}\\
&+\left\{\begin{smallmatrix}p_1\to p_2\to p_3\to p_1\\ \mu\to\nu\to\lambda\to\mu\end{smallmatrix}\right\}+\left\{\begin{smallmatrix}p_1\to p_3\to p_2\to p_1\\ \mu\to\lambda\to\nu\to\mu\end{smallmatrix}\right\}\Bigg)\,, \quad N\geq 2
\end{aligned}$$



$$\begin{aligned}
&g^2\frac{1+(-1)^N}{2}\Bigg(f^{abe}f^{cde}O_{\mu\nu\lambda\sigma}(p_1,p_2,p_3,p_4)\\
&+f^{ace}f^{bde}O_{\mu\lambda\nu\sigma}(p_1,p_3,p_2,p_4)+f^{ade}f^{bce}O_{\mu\sigma\nu\lambda}(p_1,p_4,p_2,p_3)\Bigg)\,,
\end{aligned}$$

$$\begin{aligned}
&O_{\mu\nu\lambda\sigma}(p_1,p_2,p_3,p_4)=\Delta_\nu\Delta_\lambda\Bigg\{-g_{\mu\sigma}(\Delta\cdot p_3+\Delta\cdot p_4)^{N-2}\\
&+[p_{4,\mu}\Delta_\sigma-\Delta\cdot p_4 g_{\mu\sigma}]\sum_{i=0}^{N-3}(\Delta\cdot p_3+\Delta\cdot p_4)^i(\Delta\cdot p_4)^{N-3-i}\\
&-[p_{1,\sigma}\Delta_\mu-\Delta\cdot p_1 g_{\mu\sigma}]\sum_{i=0}^{N-3}(-\Delta\cdot p_1)^i(\Delta\cdot p_3+\Delta\cdot p_4)^{N-3-i}\\
&+[\Delta\cdot p_1\Delta\cdot p_4 g_{\mu\sigma}+p_1\cdot p_4\Delta_\mu\Delta_\sigma-\Delta\cdot p_4 p_{1,\sigma}\Delta_\mu-\Delta\cdot p_1 p_{4,\mu}\Delta_\sigma]\\
&\times\sum_{i=0}^{N-4}\sum_{j=0}^{i}(-\Delta\cdot p_1)^{N-4-i}(\Delta\cdot p_3+\Delta\cdot p_4)^{i-j}(\Delta\cdot p_4)^j\Bigg\}\\
&-\left\{\begin{smallmatrix}p_1\leftrightarrow p_2\\ \mu\leftrightarrow\nu\end{smallmatrix}\right\}-\left\{\begin{smallmatrix}p_3\leftrightarrow p_4\\ \lambda\leftrightarrow\sigma\end{smallmatrix}\right\}+\left\{\begin{smallmatrix}p_1\leftrightarrow p_2,\ p_3\leftrightarrow p_4\\ \mu\leftrightarrow\nu,\ \lambda\leftrightarrow\sigma\end{smallmatrix}\right\}\,, \quad N\geq 2
\end{aligned}$$

N Space Evaluation

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$$
\begin{aligned}
\mathrm{BS}_8(N) - \mathrm{BS}_8(N-1) &= \frac{1}{N}\mathrm{BS}_4(N), \\
\mathrm{BS}_4(N) &= \sum_{\tau_1=1}^{N} \frac{4^{\tau_1}\left(\tau_1!\right)^2}{\left(2\tau_1\right)!\tau_1^2} \\
\mathrm{BS}_8(N) &\propto -7\zeta_3 + \left[+3(\ln(N)+\gamma_E) + \frac{3}{2N} - \frac{1}{4N^2} + \frac{1}{40N^4} - \frac{1}{84N^6} + \frac{1}{80N^8} - \frac{1}{44N^{10}}\right]\zeta_2 \\
&\quad + \sqrt{\frac{\pi}{N}}\left[4 - \frac{23}{18N} + \frac{1163}{2400N^2} - \frac{64177}{564480N^3} - \frac{237829}{7741440N^4} + \frac{5982083}{166526976N^5}\right. \\
&\quad + \frac{5577806159}{438593126400N^6} - \frac{12013850977}{377864847360N^7} - \frac{1042694885077}{90766080737280N^8} \\
&\quad \left. + \frac{6663445693908281}{127863697547722752N^9} + \frac{23651830282693133}{1363413316298342400N^{10}}\right] \qquad (1)
\end{aligned}
$$

Function Spaces

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Sums

Harmonic Sums

$$\sum_{k=1}^{N} \frac{1}{k} \sum_{l=1}^{k} \frac{(-1)^l}{l^3}$$

gen. Harmonic Sums

$$\sum_{k=1}^{N} \frac{(1/2)^k}{k} \sum_{l=1}^{k} \frac{(-1)^l}{l^3}$$

Cycl. Harmonic Sums

$$\sum_{k=1}^{N} \frac{1}{(2k+1)} \sum_{l=1}^{k} \frac{(-1)^l}{l^3}$$

Binomial Sums

$$\sum_{k=1}^{N} \frac{1}{k^2} \binom{2k}{k} (-1)^k$$

Integrals

Harmonic Polylogarithms

$$\int_0^x \frac{dy}{y} \int_0^y \frac{dz}{1+z}$$

gen. Harmonic Polylogarithms

$$\int_0^x \frac{dy}{y} \int_0^y \frac{dz}{z-3}$$

Cycl. Harmonic Polylogarithms

$$\int_0^x \frac{dy}{1+y^2} \int_0^y \frac{dz}{1-z+z^2}$$

root-valued iterated integrals

$$\int_0^x \frac{dy}{y} \int_0^y \frac{dz}{z\sqrt{1+z}}$$

iterated integrals on ${}_2F_1$ functions

$$\int_0^z dx \frac{\ln(x)}{1+x} {}_2F_1 \frac{4}{3}, \frac{5}{3} 2 \frac{x^2(x^2-9)^2}{(x^2+3)^3}$$

Special Numbers

multiple zeta values

$$\int_0^1 dx \frac{\mathrm{Li}_3(x)}{1+x} = -2\mathrm{Li}_4(1/2) + \ldots$$

gen. multiple zeta values

$$\int_0^1 dx \frac{\ln(x+2)}{x-3/2} = \mathrm{Li}_2(1/3) + \ldots$$

cycl. multiple zeta values

$$\mathbf{C} = \sum_{k=0}^{\infty} \frac{(-1)^k}{(2k+1)^2}$$

associated numbers

$$\mathrm{H}_{8,w_3} = 2\mathrm{arccot}(\sqrt{7})^2$$

associated numbers

$$\int_0^1 dx \, {}_2F_1 \frac{4}{3}, \frac{5}{3} 2 \frac{x^2(x^2-9)^2}{(x^2+3)^3}$$

shuffle, stuffle, and various structural relations $\Longrightarrow$ algebras

Except the last line integrals, all other ones stem from 1st order factorizable equations.