Four-loop large-nf contributions to the non-singlet structure functions

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[2211.16485]

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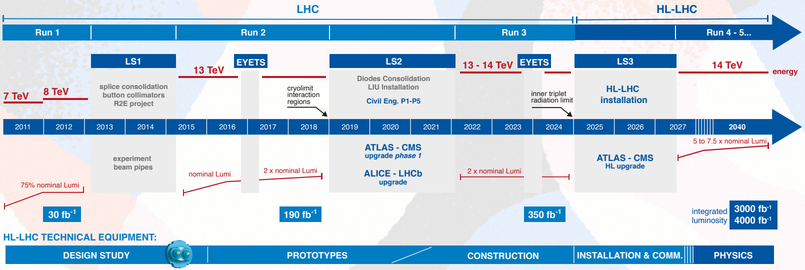
Motivation

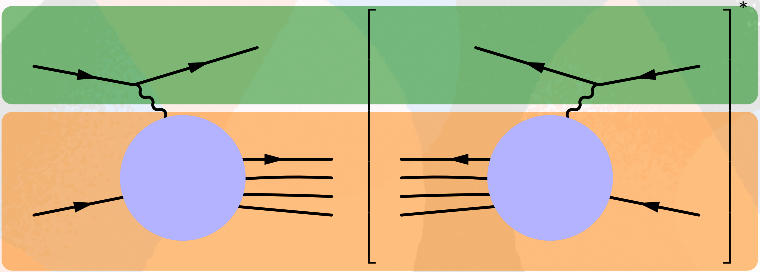
- The computation of contributions to the DIS coefficient functions represents the ideal testing ground for the possible applications for computing splitting functions.
- **Theoretical predictions** need to keep up with the ever-increasing precision of ***experimental measurements***
- Need to understand the SM background in order to resolve **new physics**



- **Example:** Higgs inclusive: $8\% \to 3\%$ expected experimental uncertainty at 3000 fb^{-1}. The PDF uncertainty on the theoretical prediction cannot be neglected anymore.

Deep Inelastic Scattering

Probing the **hadron** structure by mean of high energetic **leptons** :



- Factorize the **leptonic** from the **hadronic** part in the cross-section

$$\frac{\sigma_{DIS}}{dxdy} = \frac{2\pi\alpha_{em}^2}{Q^2} L^{\mu\nu} W_{\mu\nu}$$

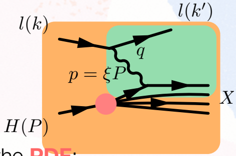
$$Q = -q^2, \qquad x = \frac{Q^2}{2P\cdot q}, \qquad y = \frac{P\cdot q}{P\cdot k}$$

Structure Functions

- The computation of the corresponding 4-loop Wilson coefficients is extremely challenging from the theoretical point of view

$$\frac{\sigma_{DIS}}{dxdy} = \frac{2\pi\alpha_{em}^2}{Q^2} L^{\mu\nu} W_{\mu\nu}$$

$$\begin{aligned} W_{\mu\nu} = & \left(P^\mu - \frac{(P\cdot q)q_\mu}{q^2}\right)\left(P^\nu - \frac{(P\cdot q)q_\nu}{q^2}\right)\frac{F_1(x,Q^2)}{P\cdot q} \\ & + \left(-g_{\mu\nu} + \frac{q^\mu q^\nu}{q^2}\right) F_2(x,Q^2) \\ & + i\epsilon_{\mu\nu\rho\sigma}\frac{P^\rho q^\sigma}{2P\cdot q} F_3(x,Q^2) \end{aligned}$$



- Where the **hadronic** and **partonic** quantities are related via the **PDF**:

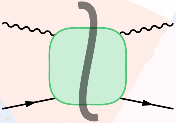
$$F_a(x,Q^2) = \sum_i \left[f_i(\xi) \otimes \mathcal{C}_{a,i}(\xi,Q^2) \right](x)$$

- The problem can be simplified by using the optical theorem for extracting the **Mellin moments** of the process.

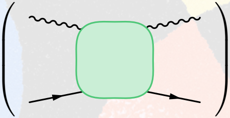
Mellin Moments

Objective:

- Compute the hadronic cross-section $\hat{W}_{\mu\nu}$ using the forward scatting $\hat{T}_{\mu\nu}$



$\approx \mathrm{Disc}_x$



How:

- Compute the **Mellin moments** of the structure functions:

$$F_a(x,Q^2) = \sum_i \left[f_i(\xi) \otimes \mathcal{C}_{a,i}(\xi,Q^2) \right](x)$$

with the **Mellin transform** defined by

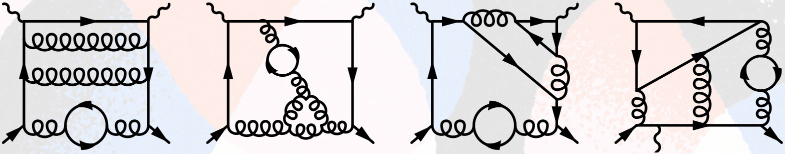
$$M[f(x)](N) = \int_0^1 \mathrm{d}x\; x^{N-1} f(x)$$

- The Mellin moments of **cross-section** correspond to the expansion coefficients around $\omega = \frac{1}{x} = 0$ of the **Forward Scattering Amplitude** :

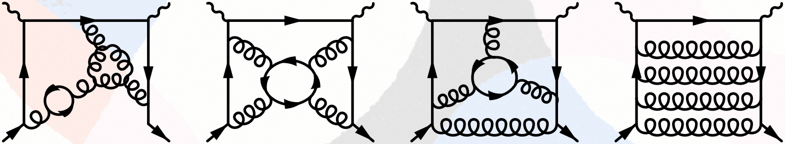
$$M[\hat{W}_{\mu\nu}](N) = \frac{1}{N!}\left[\frac{\mathrm{d}^N T_{\mu\nu}}{\omega^N}\right]_{\omega=0}$$

SYSTEM OF DIFFERENTIAL EQUATIONS

- Contributions to the non-singlet forward amplitude can be separated by **color structure** and n_f order:



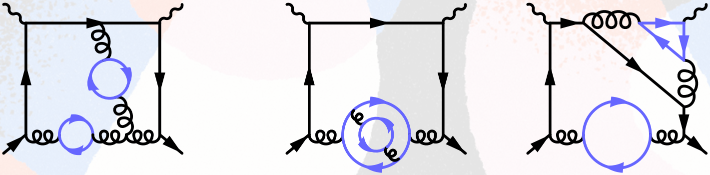
$C_F^3 n_f$ $\quad$ $C_F C_A^2 n_f$ $\quad$ $C_F^2 n_f^2$ $\quad$ $(d_F^{abc})^2 n_f^2$



$d_F^{abc} d_A^{abc} n_f$ $\quad$ $(d_F^{abcd})^2 n_f$ $\quad$ $(d_F^{abc})^2 C_F n_f$ $\quad$ C_F^4

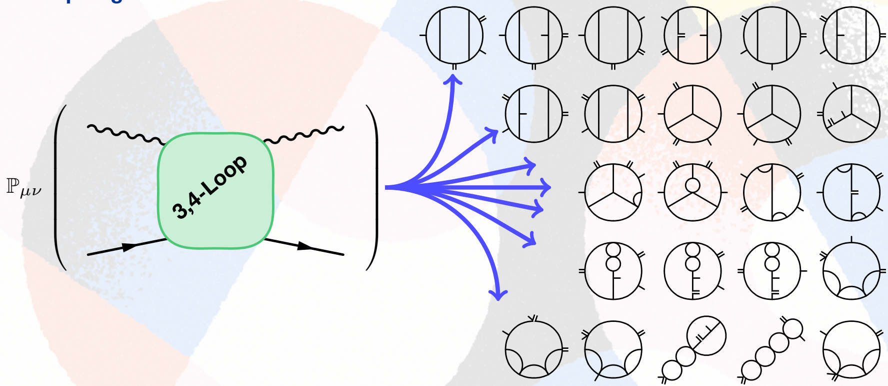
System of differential equations

- We focus our attention on the non-singlet coefficient functions contribution of order $[n_f^3, n_f^2]$ for $q+\gamma \to q+\gamma$:



SYSTEM OF DIFFERENTIAL EQUATIONS:

We process all the diagrams for the relevant process and cast them into 24 different **topologies** :



Still left with $\approx 10^5$ different integrals to be computed!

- We can resize the problem by using **IBP** relations among these integrals.
- Many publicly available programs to perform **reductions to master integrals** each with its strengths and weaknesses.

FIRE [1901.07808]
Reduze [1201.4330]
Kira [1705.05610]

System of Differential Equations

- Within each **topology** we perform a reduction to **master integrals** :

$$I^{(n)}(\omega,\epsilon) = \sum_i c_i(\omega,\epsilon)\cdot \underbrace{\boxed{M_i^{(n)}(\omega,\epsilon)}}_{\text{Expand in }\omega}\,, \qquad \omega = \frac{1}{x}$$

- The master integrals allow to construct a closed system of differential equations:

$$\frac{\partial}{\partial\omega}\vec{M}(\omega,\epsilon) = A(\omega,\epsilon)\cdot\vec{M}(\omega,\epsilon).$$

Assuming:

- The DE matrix has at most a simple pole in ω:

$$A = \frac{A_{-1}}{\omega} + \sum_{k=0}^{\infty} A_k\omega^k$$

Note: Can always be done by applying a linear transformation T for system with **regular singularities** :

$$\vec{M}\to T\cdot\vec{M}, \qquad A \to \frac{\partial T}{\partial\omega}T^{-1} + T\cdot A\cdot T^{-1}$$

J. Moser 1959, J. Henn [1412.2296], Epsilon [1701.00725], Fuchsia [1701.04269], Libra [2012.00279]

System of Differential Equations

- Within each **topology** we perform a reduction to **master integrals** :

Expand in ω

$$I^{(n)}(\omega,\epsilon) = \sum_i c_i(\omega,\epsilon)\cdot \boxed{M_i^{(n)}(\omega,\epsilon)}\,, \qquad \omega = \frac{1}{x}$$

Mellin moments generation

$$\frac{\partial}{\partial\omega}\vec{M}(\omega,\epsilon) = A(\omega,\epsilon)\cdot\vec{M}(\omega,\epsilon).$$

$$A = \frac{A_{-1}}{\omega} + \sum_{k=0}^{\infty} A_k\omega^k \qquad \vec{M} = \sum_{k=0}^{\infty}\vec{m}_k\omega^k$$

$$\underbrace{\left((k+1)\mathbb{1} - A_{-1}\right)}_{:=B_k}\cdot\vec{m}_{k+1} = \sum_{j=0}^{k} A_j\vec{m}_{k-j}$$

$\det(B_k) \neq 0$ $\qquad$ $\det(B_k) = 0$

Recursive Expression:

$$\vec{m}_{k+1} = B_k^{-1}\cdot\left(\sum_{j=0}^{k} A_j\vec{m}_{k-j}\right)$$

Gaussian Elimination:

Required by a **finite number** of k

Expansion

- We need to fix the **boundary** condition for $p \to 0$ **FORCER [1704.06650]**
- In our case the **transformation** matrix T consists of a simple rescaling of the master integrals

$$T = \text{diag}\left(\omega^{\vec{a}}\right), \qquad \vec{a} \in \mathbb{N}_0^{\#\text{ of masters}}$$

- We can perform a simultaneous expansion in the **dimensional regulator** ϵ in order to speed up the computation provide the independence of the ϵ and ω poles

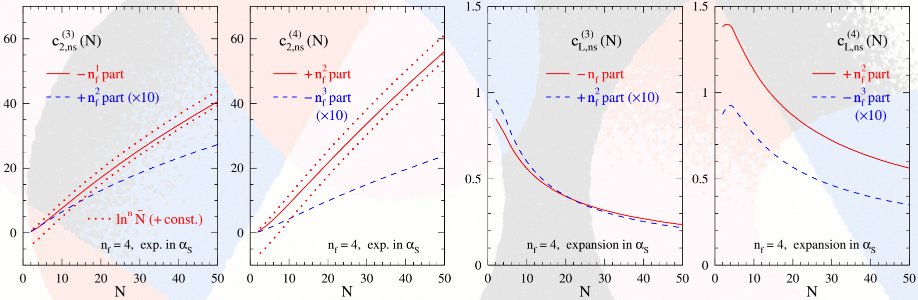
Smirnov^2 [2002.08042], J. Usovitsch [2002.08173]

- Results can be expanded to high order in **Mellin moments**
- Faster than an expansion at the integrand level (Tensor decomposition)
- The **reductions** (FIRE) to master integrals remain the main bottleneck of the computation

Mellin moments at 4-loop

$c_{3,ns}$ Coming Soon!

- Starting to explore the DIS expression for a simple subgroup **[n_f^3,n_f^2]** for $q+\gamma \to q+\gamma$:



Plots from: [2211.16485]

- **Expansion** in ω is possible to **high orders** within a day:

$$\mathcal{O}(\omega^{1500})$$

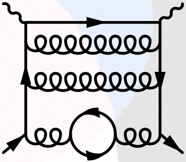
- Allows for the reconstruction of the **structure functions** in **x**-space at **all orders**

Harmonic series: $S_{\vec{m}}(N)$ $\rightarrow$ Harmonic Polylogarithms: $H_{\vec{n}}(x)$

Adventuring further into 4-loop

The **new** goal is to push the same technique to new horizons:

- Consider the diagrams contributing to $C_F^3 n_f$



- Effectively 3-loop topologies with **bubble insertions**!
- The added degrees of freedom start to become a real problem for the reduction to **master integrals:**
  - Moving from 11 to 12 propagators out of 18 degrees of freedom
  - Higher powers in the numerator
- Implement a tailored reduction routine for this specific problem

Tackling the Problems

General Reduction programs :

- Reliable on a wide range of problems
- Thoroughly checked through years of usage and feedbacks
- Parallelization of the reduction problem
- Multiple ways to solve the problem and implementation of general optimization
- Already too slow for the integrals we are dealing with

Problem we are facing :

- Relatively contained number of integrals to be reduced (compared with the d.o.f)
- Very few integrals have the highest complexity (numerator/denominator powers)

Tackling the Problems

Problem :

- With the current available reduction programs it would be impossible to obtain the necessary DEs in a reasonable ammount of time

Idea :

- We want to use out taylor reduction as a **complementary** tool of the full reduction

How:

Obtain a fast partial reduction (not necessarily master integrals) to be able to give simpler problems to the public reduction programs

We then turn to FIRE for eventually refine the reduction and obtain a factorized base for the system of DEs

Necessary for $C_F^3 n_f$ and for the computation of $\mathcal{C}_3$ and n_f^2

Summary

- Use IBP identities for a **reduction to master integrals** and build a system of differential equations
- **Transform** the system to allow for an efficient recursive expression for the extraction of the series coefficients
- Tested the method by computing high numbers of Mellin moments for the DIS **Wilson coefficients** $\mathcal{C}_L$, $\mathcal{C}_2$ and $\mathcal{C}_3$ at **3-loop**
- Generated 1500 Mellin moments for the non-singlet n_f^2 contribution at **4-loop** to obtain for the first time the corresponding **Wilson coefficients**
- Implemented a taylored **reduction** program to be usued together with publicly available tools

Upcoming:

- Reconstruct the expression for $\mathcal{C}_3$ (11 extra topologies)

Future:

- Apply the same method for extracting Mellin moments at **4-loop** for the $C_F^3 n_f$ to extract the **splitting functions**

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Thank you!

Nikhef

Backup Slides

THE WONDERFUL WORLD OF FINITE FIELDS

Consider the simple case of a family of **rational numbers** of the form:

$$q = \frac{a}{b}, \qquad a \in \{-20, .., 20\},\ b \in \{1, .., 20\}$$

0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15
16	17	18	19	20	21	22	23	24	25	26	27	28	29	30	31
32	33	34	35	36	37	38	39	40	41	42	43	44	45	46	47
48	49	50	51	52	53	54	55	56	57	58	59	60	61	62	63
64	65	66	67	68	69	70	71	72	73	74	75	76	77	78	79
80	81	82	83	84	85	86	87	88	89	90	91	92	93	94	95
96	97	98	99	100	101	102	103	104	105	106	107	108	109	110	111
112	113	114	115	116	117	118	119	120	121	122	123	124	125	126	127
128	129	130	131	132	133	134	135	136	137	138	139	140	141	142	143
144	145	146	147	148	149	150	151	152	153	154	155	156	157	158	159
160	161	162	163	164	165	166	167	168	169	170	171	172	173	174	175
176	177	178	179	180	181	182	183	184	185	186	187	188	189	190	191
192	193	194	195	196	197	198	199	200	201	202	203	204	205	206	207
208	209	210	211	212	213	214	215	216	217	218	219	220	221	222	223
224	225	226	227	228											

Perform the operation over a **Finite Field** where all q are **well defined**:

- modulo: $m = 229$

Multiple rational number mapped to the same element of the Finite Field.
Example:

$$-\frac{17}{9}, -\frac{1}{14}, \frac{15}{19}, \frac{16}{5} \quad \rightarrow \quad 49$$

NOT SURPRISING :
there are 511 unique q's

single q mapped   **multiple** q's mapped   **unused** component of FF

The wonderful world of Finite Fields

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0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20	21	22
23	24	25	26	27	28	29	30	31	32	33	34	35	36	37	38	39	40	41	42	43	44	45
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69	70	71	72	73	74	75	76	77	78	79	80	81	82	83	84	85	86	87	88	89	90	91
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483	484	485	486	487	488	489	490	491	492	493	494	495	496	497	498	499	500	501	502	503	504	505
506	507	508	509	510	511	512	513	514	515	516	517	518	519	520								

A **Finite Field** **larger** then the number of possible q's (i.e. 511):

- modulo: $m = 521$

Multiple rational number mapped to the same element of the Finite Field. **Example**:

$$-\frac{18}{19}, \frac{17}{11} \rightarrow 191$$

- Need a larger Field!

Evolution for different field sizes

single q mapped — **multiple** q's mapped — **unused** component of FF

THE WONDERFUL WORLD OF FINITE FIELDS

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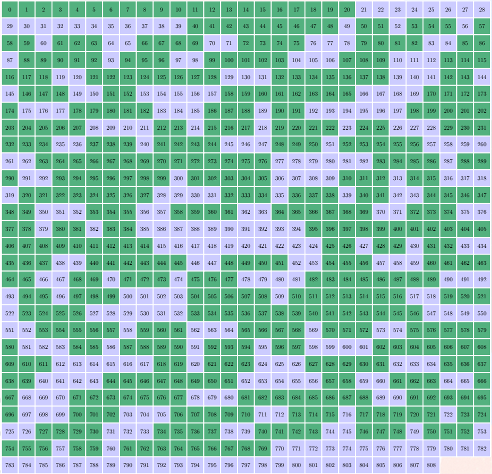
$$q = \frac{a}{b}, \qquad a \in \{-20, .., 20\},\ b \in \{1, .., 20\}$$



Choose the size of the **Finite Field** to allow for a **unique mapping** of each q:

- modulo: $m = 809$

Generally for a successful reconstruction one needs:

$$m > 2 \cdot \max\{a^2, b^2\}$$

$$-\frac{7}{20} \rightleftarrows 121$$

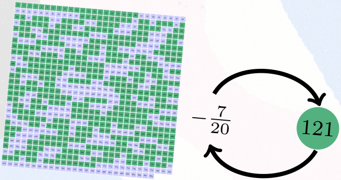
single q mapped   **multiple** q's mapped   **unused** component of FF

Using Finite Fields

The aim is to recover the reduction coefficients :

$$I(\mathbf{x}) = \mathbf{c_i}(\mathbf{x}) M_i(\mathbf{x})$$

- Sample the $\mathbf{c_i}(\mathbf{x})$ over some Finite Field and **interpolate** the corresponding **rational function** with coefficients in $\mathbb{Z}_{p_1}$
- Use the Chinese Reminder Theorem (CRT) to **combine** several interpolated solution over distinct $\mathbb{Z}_{p_i}$
- Recover **rational reconstruct** the original coefficients starting from from their representation in $\mathbb{Z}_{p_1 \ldots p_n}$



Advantages:

- Fast numerical evaluation by keeping the individual primes p_i below machine size arithmetic
- Full control over the IBP sampling

Reduction Overview

We have implemented our own reduction procedure to try to improve the main bottleneck of our approach. The reductions is organized into three levels:

Finite Fields

- Numerical Gaussian Elimination

Solve **IBP** relations using **Finite Fields** by evaluating the variables at some arbitrary points

- Generate instructions table

Create a **logfile** to keep track of all the arithmetic operations performed

Algebraic

- Optimize logfile

Keep only the instructions relevant to the **reduction coefficients**

- Algebraic evaluation

Read out the **exact coefficients** from the logfile by using the un-replaced variables

Reconstruction

If the algebraic evaluation **fails** than we use rational reconstruction

- Interpolate
- Reconstruct

Nikhef

Regular Singularities

Considre a matrix **A** appearing in the DE and its Laurent series

$$A = \frac{1}{\omega^p} \sum_{n=0}^{\infty} A_n \, \omega^n$$

- $p \leq 1$

  The system has manifestly at most regular singularities

- $p > 1$

  There is a transformation T that maps the system to a representation which fulfills the manifest condition $p \leq 1$ provided that

$$\omega^r \det\left(\frac{A_0}{\omega} + A_1 - \lambda \cdot \mathrm{Id}\right)\Big|_{\omega=0} = 0$$

where $r = \mathrm{rank}(A_0)$.