Triple-collinear splitting functions at one loop in QCD

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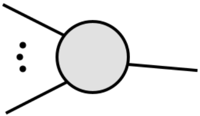
Building blocks of N3LO amplitudes

- Born level



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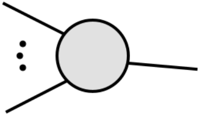
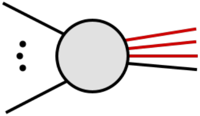
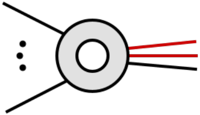
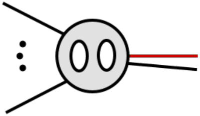


- N3LO







↑

triple collinear limit at one loop

General definitions

The amplitude

$$\mathcal{A} \equiv (\mu^{-\epsilon} g_s^B)^n \Big(\mathcal{A}^{(0)} + \frac{\mu^{-2\epsilon}\alpha_s^B}{(4\pi)^{1-\epsilon}}\, \mathcal{A}^{(1)} + \mathcal{O}\big(\alpha_s^2\big) \Big) \,, \qquad \alpha_s^B \equiv \frac{(g_s^B)^2}{4\pi} \,,$$

where

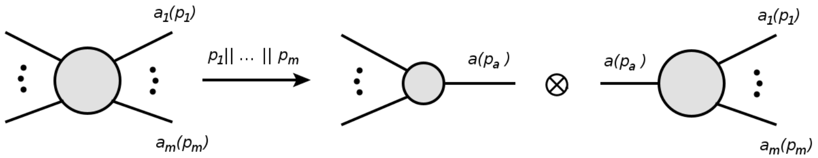
$$g_s^B \quad - \quad \text{bare coupling constant}$$

We work in d dimensions

$$d = 4 - 2\epsilon$$

Our results are not renormalized in UV - not essential - splitting operators renormalize as ordinary amplitudes

Collinear factorization in QCD: tree level



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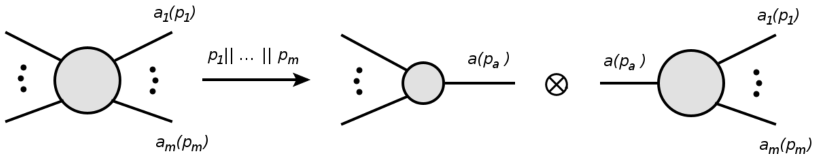

$$\mathcal{A}^{(0)}_{a_1\ldots a_m\ldots}(p_1,\ldots,p_m,\ldots) \xrightarrow{p_1||p_2||\ldots||p_m} \mathcal{A}_{a\ldots}(p_a,\ldots)\,\textbf{Split}^{(0)}_{a\to a_1\ldots a_m}(p_1,\ldots,p_m)$$

$$\sim \left(\frac{1}{\sqrt{s_{1\ldots m}}}\right)^{m-1} \quad \text{when} \quad s_{1\ldots m}\to 0$$

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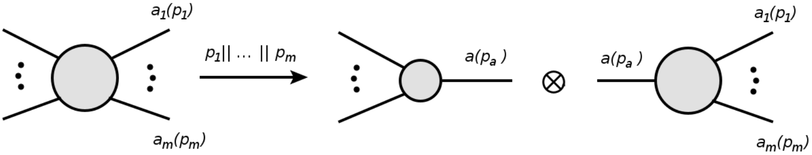
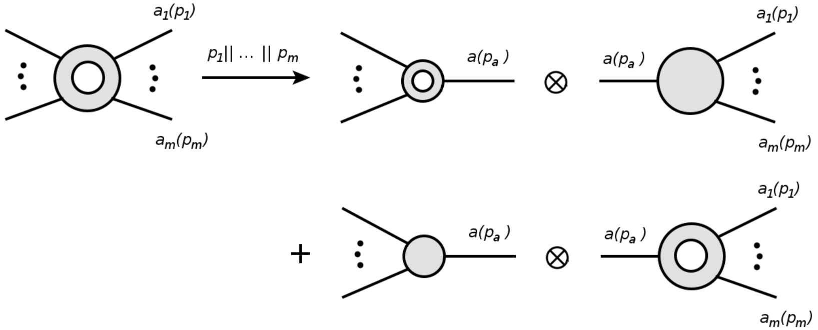


$$\mathcal{A}^{(0)}_{a_1\ldots a_m\ldots}(p_1,\ldots,p_m,\ldots) \overset{p_1||p_2||\ldots||p_m}{\longrightarrow} \mathcal{A}_{a\ldots}(p_a,\ldots)\,\textbf{Split}^{(0)}_{a\to a_1\ldots a_m}(p_1,\ldots,p_m)$$

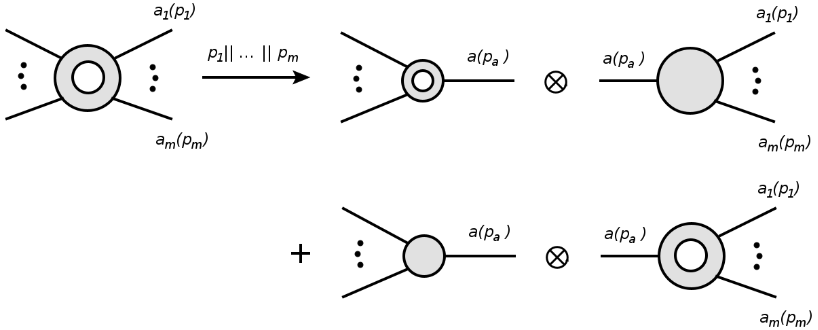
$$\sim \left(\frac{1}{\sqrt{s_{1\ldots m}}}\right)^{m-1} \quad \text{when} \quad s_{1\ldots m}\to 0$$

- $\textbf{Split}^{(0)}_{a\to a_1\ldots a_m}(p_1,\ldots,p_m)$ is the splitting operator at tree level

Collinear factorization in QCD: one-loop



Collinear factorization in QCD: one-loop



$$\mathcal{A}^{(1)}_{a_1 \dots a_m \dots}(p_1, \dots, p_m, \dots) \overset{p_1||p_2||\dots||p_m}{\longrightarrow} \mathcal{A}^{(1)}_{a\dots}(p_a, \dots)\, \textbf{Split}^{(0)}_{a\to a_1\dots a_m}(p_1, \dots, p_m)$$

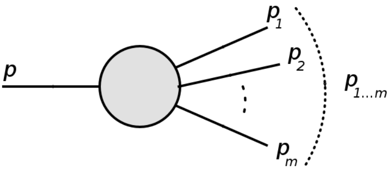
$$+ \; \mathcal{A}^{(0)}_{a\dots}(p_a, \dots)\textbf{Split}^{(1)}_{a\to a_1\dots a_m}(p_1, \dots, p_m)$$

$$\sim \left(\frac{1}{\sqrt{s_{1\dots m}}}\right)^{m-1}\left(\frac{s_{1\dots m}}{\mu^2}\right)^{-\epsilon} \text{ when } \; s_{1\dots m} \to 0$$

Collinear splitting functions

$$p_{1\ldots m} \equiv p + \frac{s_{1\ldots m}}{2\, p_{1\ldots m}\cdot q}\, q\,,$$

$$p^2 = q^2 = 0\,, \quad p\cdot q \neq 0\,,$$



where q is an auxiliary light-like vector

In the collinear limit: $\quad p_i \to z_i p\,, \qquad z_i = \dfrac{p_i\cdot n}{p\cdot n}$

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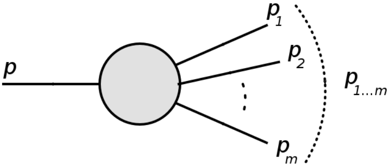

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The splitting functions and the averaged splitting functions are defined as

$$\hat{\boldsymbol{P}}_{a_1\dots a_m} \equiv \left(\frac{s_{1\dots m}}{2}\right)^2 \mathbf{Split}^\dagger_{a_1\dots a_m}\mathbf{Split}_{a_1\dots a_m}, \quad \langle \hat{P}_{a_1\dots a_m}\rangle \equiv \frac{1}{n_a^{\rm col}\, n_a^{\rm spin}}\,{\rm Tr}\Big[\hat{\boldsymbol{P}}_{a_1\dots a_m}\Big]$$

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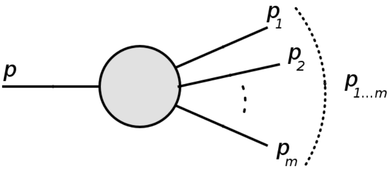

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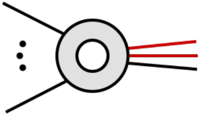
$$\hat{\boldsymbol{P}}^{(1)}_{a_1\ldots a_m} \equiv \left(\frac{s_{1\ldots m}}{2}\right)^2 \left(\textbf{Split}^{(0)\,\dagger}_{a_1\ldots a_m}\textbf{Split}^{(1)}_{a_1\ldots a_m} + \textbf{Split}^{(1)\,\dagger}_{a_1\ldots a_m}\textbf{Split}^{(0)}_{a_1\ldots a_m}\right)$$

Requirements from a subtraction scheme at N3LO



Based on our discussion so far, we can see that the triple collinear singularities are encoded in the expression

$$-\left(\frac{2}{s_{123}}\right)^2 \left[\left\langle \mathcal{A}^{(0)}_{a\dots} \middle| \hat{\boldsymbol{P}}^{(1)}_{a_1 a_2 a_3} \middle| \mathcal{A}^{(0)}_{a\dots} \right\rangle + 2\,\mathrm{Re} \left\langle \mathcal{A}^{(0)}_{a\dots} \middle| \hat{\boldsymbol{P}}^{(0)}_{a_1 a_2 a_3} \middle| \mathcal{A}^{(1)}_{a\dots} \right\rangle \right]$$

- The above can be used as a subtraction term when constructing a scheme for N3LO cross section, as it removes singularities from

$$2\,\mathrm{Re} \left\langle \mathcal{A}^{(0)}_{a_1 a_2 a_3 \dots} \middle| \mathcal{A}^{(1)}_{a_1 a_2 a_3 \dots} \right\rangle$$

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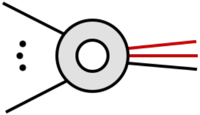

At NLO, we have

$$\int_0^1 \frac{\mathrm{d}\eta}{\eta^{1+\epsilon}} f(\eta) = \Big[\int_0^1 \frac{\mathrm{d}\eta}{\eta^{1+\epsilon}} \big(f(\eta) - f(0)\big)\Big] + \Big[f(0) \int_0^1 \frac{\mathrm{d}\eta}{\eta^{1+\epsilon}} \Big]$$

$$= \Big[\int_0^1 \frac{\mathrm{d}\eta}{\eta^{1+\epsilon}} \big(f(\eta) - f(0)\big)\Big] + \Big[-\frac{1}{\epsilon} f(0) \Big]$$

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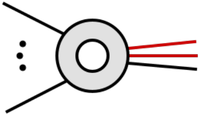

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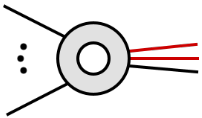
Hence, we need our triple-collinear splitting function at least to $\mathcal{O}(\epsilon)$.

Requirements from a subtraction scheme at N3LO



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$$= \left[\int_0^1 \frac{\mathrm{d}\eta}{\eta^{1+\epsilon}} \big(f(\eta) - f(0)\big)\right] + \left[-\frac{1}{\epsilon} f(0)\right]$$

Hence, we need our triple-collinear splitting function at least to $\mathcal{O}(\epsilon)$.

- We also need approximations up to $\mathcal{O}\big(\epsilon^4\big)$ to the triple-collinear one-loop splitting functions, valid in various additional limits (iterated single-collinear, soft, etc.).

Triple-collinear splitting functions - state of the art

- $q \to q\, q'\, \bar{q}'$ asymmetric part only, $\mathcal{O}\left(\epsilon^0\right)$
  [Catani, de Florian, Rodrigo '04]
- $q \to q\, q\, \bar{q}$ missing
- $q \to q\, g\, g$ missing
- $g \to g\, q\, \bar{q}$ $\mathcal{O}\left(\epsilon^0\right)$
  [Badger, Buciuni, Peraro '15]
- $g \to g\, g\, g$ $\mathcal{O}\left(\epsilon^0\right)$
  [Badger, Buciuni, Peraro '15]

Our aim is to get all the above splitting functions to $\mathcal{O}(\epsilon)$

Two approaches

- top-down

  Use ordinary Feynman rules, calculate the matrix element for the process

  $$\gamma^*/H \to 4\ \text{partons}\,,$$

  and take the collinear limit.

- bottom-up

  Use modified Feynman rules and calculate the amplitude for the process

  $$q^*/g^* \to 3\ \text{partons}\,.$$

Derivation of bottom-up approach

[Bern, Del Duca, Kilgore, Schmidt '99]

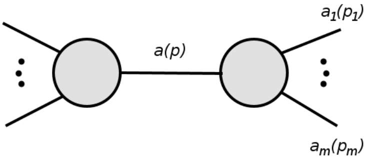
$$(\cdots)\,\frac{D(p)}{p^2}\,(\cdots)$$

$$\downarrow$$

$$(\cdots)\,\frac{\sum_{\rm pol} a(p)\bar{a}(p)}{s_{1\ldots m}}\,(\cdots)$$

$$\sum_{\rm pol}(\cdots)\,a(p)\,\frac{\bar{a}(p)}{s_{1\ldots m}}\,(\cdots)$$

$$\sum_{\rm pol}\mathcal{A}_{n-m}(\ldots,p,\ldots)\,\textbf{Split}_{a\to a_1\ldots a_m}(p_1,\ldots,p_m)$$



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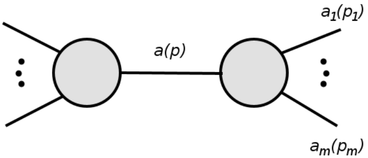
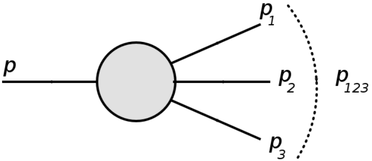


- Splitting function is derived by contracting the incoming off-shell line of the splitting parton with massless spinor (for a quark) or a massless transverse polarization vector (for a gluon)

$$\mathbf{Split}_{a\to a_1\ldots a_m} = \frac{\bar{a}(p)}{s_{1\ldots m}}\,A(a^*\to a_1,\ldots,a_m)$$

Triple-collinear splitting functions - kinematic variables



Triple-collinear splitting functions depend on

$$x_1 \equiv \frac{s_{23}}{s_{123}} \,, \quad x_2 \equiv \frac{s_{13}}{s_{123}} \,, \quad x_3 \equiv \frac{s_{12}}{s_{123}} \,, \qquad z_i \equiv \frac{p_i \cdot q}{p_{123} \cdot q} \,,$$

where

$$x_i \in (0,1) \,, \qquad z_i \in (0,1) \,,$$

and

$$\sum_{i=1}^{3} x_i = \sum_{i=1}^{3} z_i = 1$$

- $\frac{1}{\epsilon^2}$ and $\frac{1}{\epsilon}$ singular terms known

  [Catani, Dittmaier, Trocsanyi '01; Catani, de Florian, Rodrigo '04]

Bottom-up procedure

- Generation of $1^* \to 3$ diagrams (private software)

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- Generation of $1^* \to 3$ diagrams (private software)
- Simplifications of expressions (Form)
- Passarino-Veltman reduction (Fermat)
- Integration By Parts (IBP) reduction (Kira)
  - bubbles, triangles, boxes and “pentagon” (only at $\mathcal{O}(\epsilon)$)
  - 34 master integrals, most of which related by permutations of external momenta
  - at the end: 9 master integrals

Top-down procedure

- Generation of diagrams $1^* \to 4$ (FeynArts)

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    - all standard Feynman integrals known from literature

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- Taking the collinear limit of the final expression

$$s_{123} \to 0\,, \quad s_{12} \to 0\,, \quad s_{13} \to 0\,, \quad s_{23} \to 0\,,$$

with the finite ratios: $\frac{s_{12}}{s_{123}}\,, \frac{s_{13}}{s_{123}}\,, \frac{s_{23}}{s_{123}}\,, \frac{s_{12}}{s_{13}}\,, \frac{s_{12}}{s_{23}}\,, \frac{s_{13}}{s_{23}}$

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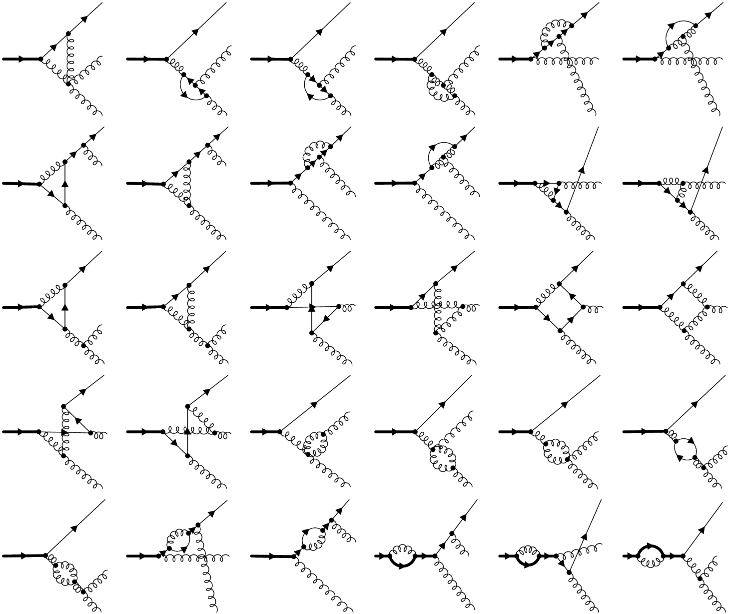
- The masters above available from [Bern, Dixon, Kosower '94] up to box at $\mathcal{O}(\epsilon^0)$.

Channels

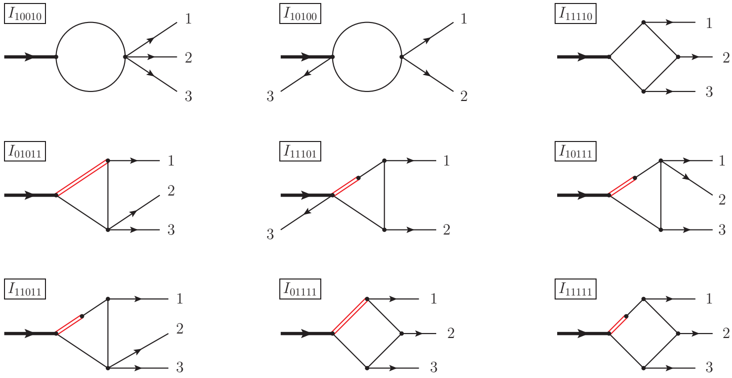
- $q \to q\, q'\, \bar{q}'$ (9 diagrams)
- $q \to q\, q\, \bar{q}$ (18 diagrams)
- $q \to q\, g\, g$ (30 diagrams)

- $g \to g\, q\, \bar{q}$ (33 diagrams)
- $g \to g\, g\, g$ (68 diagrams)

Example set: $q \to qgg$



Master integrals

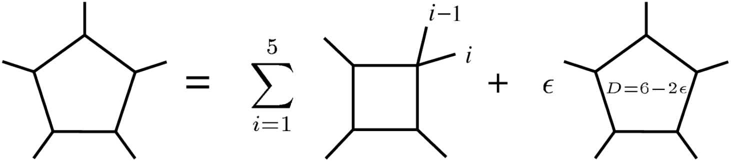


$$I^{(d)}_{a_1 a_2 a_3 a_4 a_5} \equiv$$

$$\mu^{2\epsilon} \int \frac{\mathrm{d}^d l}{i\pi^{d/2}} \frac{1}{\left(l^2\right)^{a_1} \left((l+p_1)^2\right)^{a_2} \left((l+p_1+p_2)^2\right)^{a_3} \left((l+p_1+p_2+p_3)^2\right)^{a_4} \left(l \cdot q\right)^{a_5}}$$

The pentagon

It is well known [Bern, Dixon, Kosower '94] that for the standard pentagon



- Follows from 4-dimensional relations between spin structures
- The same happens for our "pentagon" integral, with four ordinary and one linear propagator

Master results - 8 integrals with full ϵ dependence

- Ordinary Feynman integrals [Bern, Dixon, Kosower '94]

$$\text{bubbles:} \quad I_{10010}^{(4-2\epsilon)}, \quad I_{10100}^{(4-2\epsilon)}$$
$$\text{one-external-mass box:} \quad I_{11110}^{(4-2\epsilon)}$$

Master results - 8 integrals with full ϵ dependence

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$$\text{bubbles:} \quad I^{(4-2\epsilon)}_{10010}, \quad I^{(4-2\epsilon)}_{10100}$$

$$\text{one-external-mass box:} \quad I^{(4-2\epsilon)}_{11110}$$

- Integrals with linear propagator (known) [Sborlini '14]

$$I^{(4-2\epsilon)}_{01011}, \quad I^{(4-2\epsilon)}_{11101}, \quad I^{(4-2\epsilon)}_{10111}$$

Master results - 8 integrals with full ϵ dependence

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$$I^{(4-2\epsilon)}_{01011}, \quad I^{(4-2\epsilon)}_{11101}, \quad I^{(4-2\epsilon)}_{10111}$$

- Integrals with linear propagator (new)

$$I^{(4-2\epsilon)}_{11011}, \quad I^{(4-2\epsilon)}_{01111}$$

Master results - the 9^{th} integral

- one-external-mass box with linear propagator

$$I_{11111}^{(4-2\epsilon)}$$

Master results - the 9th integral

- one-external-mass box with linear propagator

$$I_{11111}^{(4-2\epsilon)}$$

- we derive the following dimension-shift relation

$$2s_{123}\, I_{11111}^{(d)} = \frac{\left(x_1z_1 - x_2z_2 + x_3z_3\right)^2 - 4x_1x_3z_1z_3}{x_1x_3z_1\left(1-x_3-z_3\right)}\,\left(d-4\right)\, I_{11111}^{(d+2)}$$

$$+ \frac{\left(x_1z_1 - x_2z_2 + x_3z_3\right)\left(z_1+z_2\right) - 2x_3z_1z_3}{x_3z_1\left(1-x_3-z_3\right)}\, I_{01111}^{(d)}$$

$$- \frac{x_1z_1 - x_2z_2 + x_3z_3}{x_1x_3z_1}\, I_{10111}^{(d)}$$

$$+ \frac{\left(x_1z_1 - x_2z_2 + x_3z_3\right) - 2x_1x_3}{x_1x_3\left(1-x_3-z_3\right)}\, I_{11011}^{(d)}$$

$$- \frac{\left(x_1z_1 - x_2z_2 + x_3z_3\right) - 2x_1\left(z_1+z_2\right)}{x_1\left(1-x_3-z_3\right)}\, I_{11101}^{(d)}$$

$$+ \frac{\left(x_1z_1 - x_2z_2 + x_3z_3\right) - 2z_1\left(x_1+x_2\right)}{z_1\left(1-x_3-z_3\right)}\left(\frac{s_{123}}{p_{123}\cdot q}\right) I_{11110}^{(d)}$$

Master results - the 9^{th} integral

Hence, we need to evaluate

$$I^{(6-2\epsilon)}_{11111}$$

- Feynman representation
- Rescalings Feynman parameters

$$\alpha_3 \;\rightarrow\; \alpha_3\, y_1 \;, \qquad \alpha_4 \;\rightarrow\; \alpha_4\, z_1$$

- Defining

$$y_1 \equiv \frac{z_1}{z_1 + z_2} \in (0,1) \;, \qquad u_3 \equiv \frac{x_3}{1 - z_3} \in (0,1)$$

- Integration with PolyLogTools in the order $\alpha_3, \alpha_2, \alpha_4$
- The result up to $\mathcal{O}\big(\epsilon^0\big)$ and up to $\mathcal{O}(\epsilon)$ in double-soft limit, is expressed in terms of multiple polylogarithms

$$G(a_1,\ldots,a_n,z) \equiv \int_0^z \frac{\mathrm{d}t}{t-a_1}\, G(a_2,\ldots,a_n,t) \;, \quad G(\underbrace{0,\ldots,0}_{n},z) \equiv \frac{1}{n!}\ln^n(z)$$

Checks

1. comparison of the predicted singularity structure of the splitting operators [Catani et al.] with that obtained from our direct calculation
2. comparison of the anti-symmetric part of the splitting function for $q \to qq'\bar{q}'$ with the result given in [Catani, de Florian, Rodrigo '04]
3. comparison of the splitting functions for $q \to qq'\bar{q}'$ and $q \to qq\bar{q}$ expanded to $\mathcal{O}(\epsilon^0)$ between the top-down and the bottom-up approaches
4. numerical comparison of the triple-collinear limits of one-loop matrix-elements squared at $\mathcal{O}(\epsilon^0)$ for the processes $V \to q\bar{q}gg$, $H \to q\bar{q}gg$ and $H \to gggg$ with the predicted asymptotics
5. comparison of the values of the master integrals obtained from analytic formulae and from Mellin-Barnes representations up to the provided orders of ϵ-expansion

Conclusions and outlook

- We have completed the study of one-loop triple-collinear splitting functions in QCD
- We used two strategies of calculations and performed extensive analytic and numerical checks
- Our results are sufficient in ϵ expansion in order to be used for construction of a N3LO subtraction scheme
- The complete set of the splitting functions and splitting operators is provided in the form of MATHEMATICA files