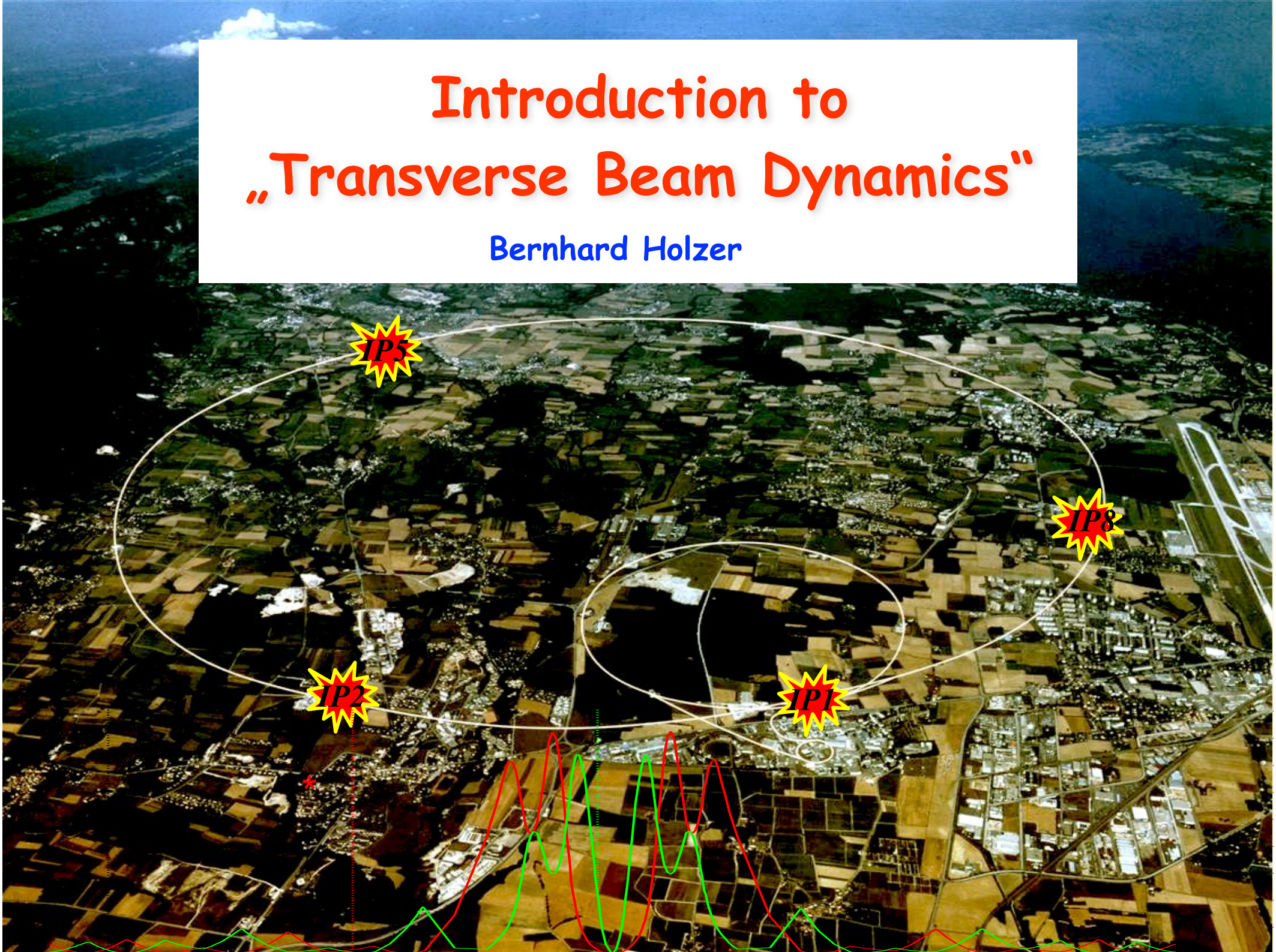
Introduction to „Transverse Beam Dynamics“

Bernhard Holzer



Transverse Beam Dynamics III

I) Linear Beam Optics

- ***Single Particle Trajectories***
- ***Magnets and Focusing Fields***
- ***Tune & Orbit***

II) The State of the Art in High Energy Machines:

- ***The Beam as Particle Ensemble***
- ***Emittance and Beta-Function***
- ***Colliding Beams & Luminosity***

III) Errors in Field and Gradient:

- ***Liouville during Acceleration***
- ***The $\Delta p/p \neq 0$ problem***
- ***Dispersion***
- ***Chromaticity***

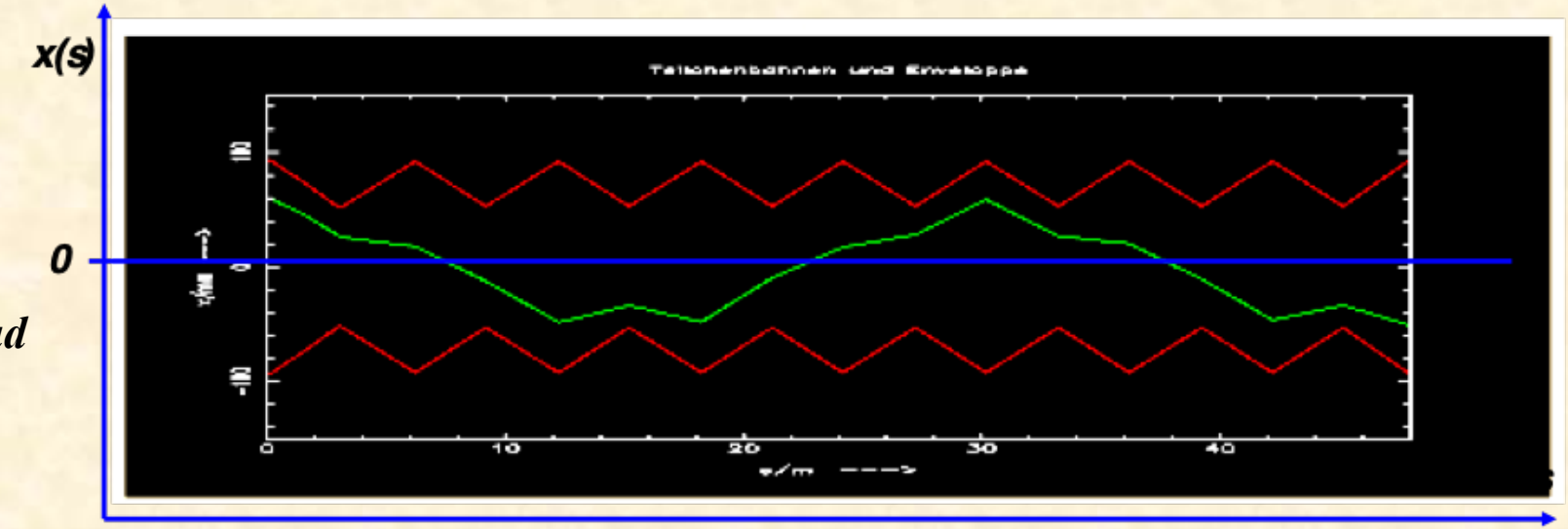
Transverse Beam Dynamics II

The Theory of Synchrotrons: „... how does it work ?"

„...does it ?"

Remember: the "tune" is the oscillation frequency of the beam.

typical values in a strong foc. machine:
$x \approx mm,\ x' \leq mrad$



A short advice about "Resonances":

when working with an oscillating system,
avoid that it "talks" to any (!) external frequency

Most prominent external frequency: Revolution frequency !!

6.) Orbit & Tune:

Tune: number of oscillations per turn

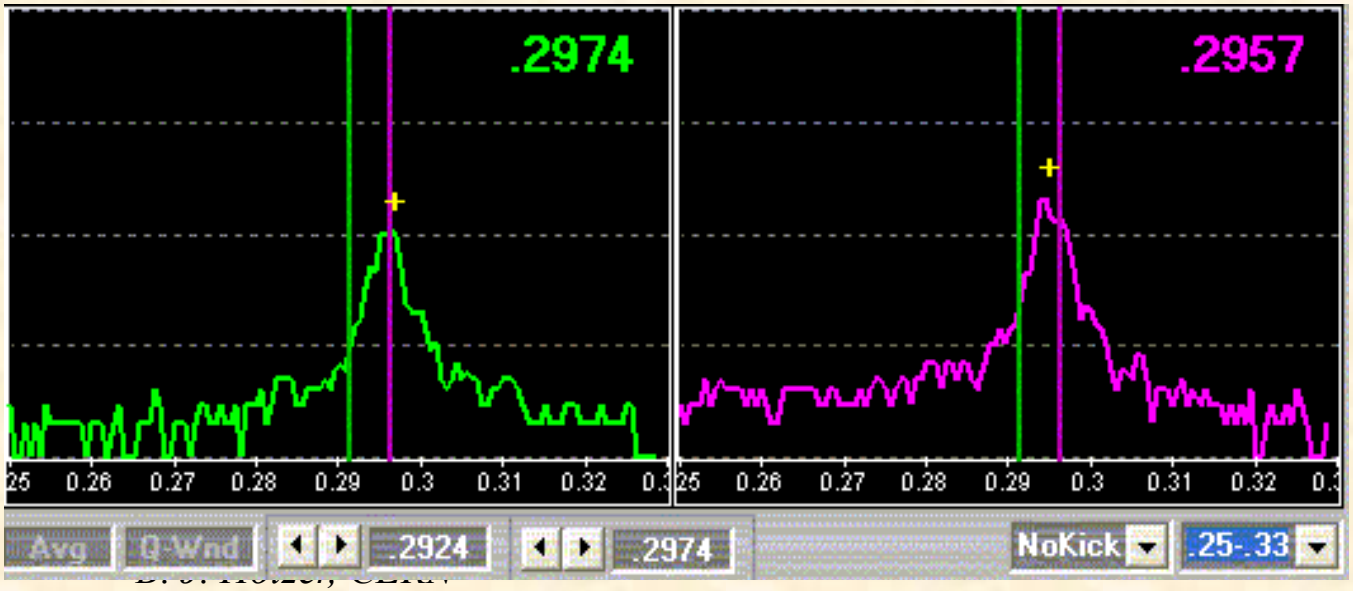
64.31
59.32

***Relevant** for beam stability:*

non integer part

***LHC revolution frequency:** 11.3 kHz*

0.31*11.3 = 3.5*kHz*



... and the tunes in x and y are different.

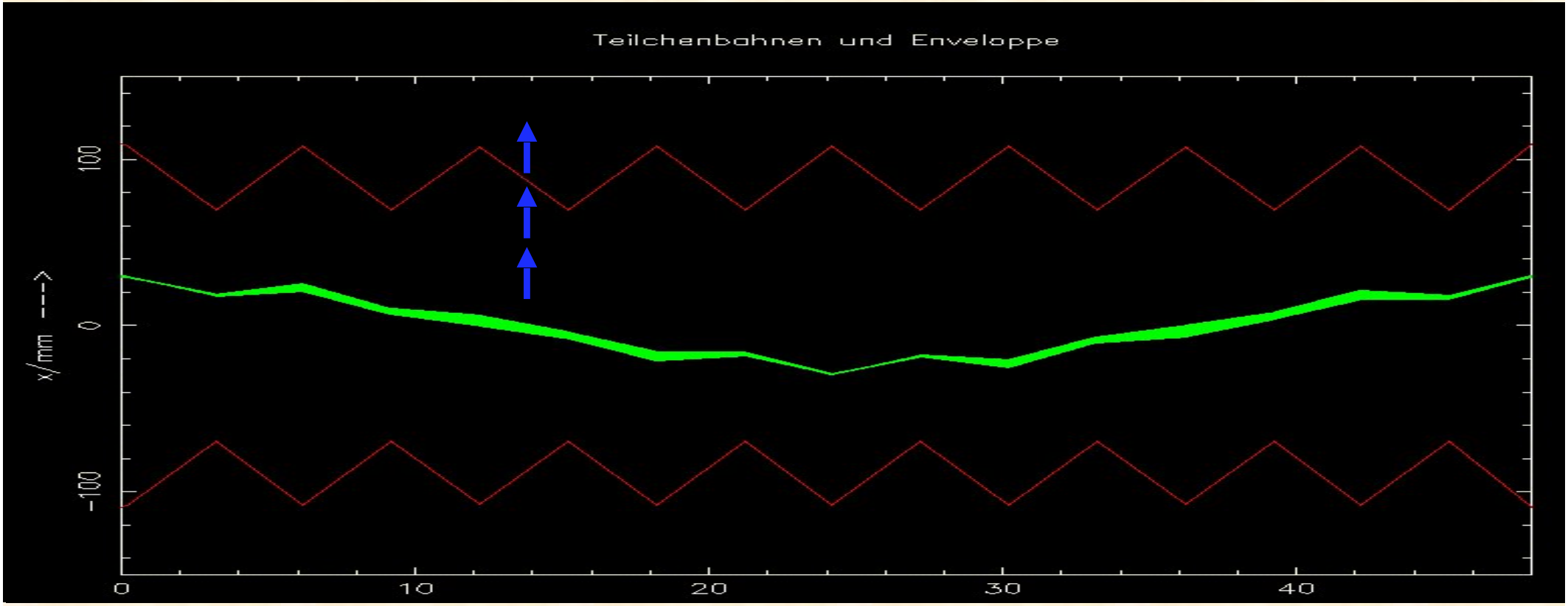
i.e. we can apply different focusing forces in the two planes

i.e. we can create different beam sizes in the two planes

Resonance Problem:

Why do we have so stupid non-integer tunes ?
"Q = 64.0" sounds much better

Qualitatively spoken: ***Integer tunes lead to a resonant increase of the closed orbit amplitude in presence of the smallest dipole field error.***



Orbit in case of a small dipole error:

$$x_{co}(s) = \frac{\sqrt{\beta(s)} * \int \frac{1}{\rho_{sl}} \sqrt{\beta_{sl}} * \cos(\psi_{sl} - \psi_s - \pi Q)\, ds}{\underbrace{2 \sin \pi Q}}$$

Assume: Tune = integer $Q = 1 \rightarrow 0$

Tune and Resonances

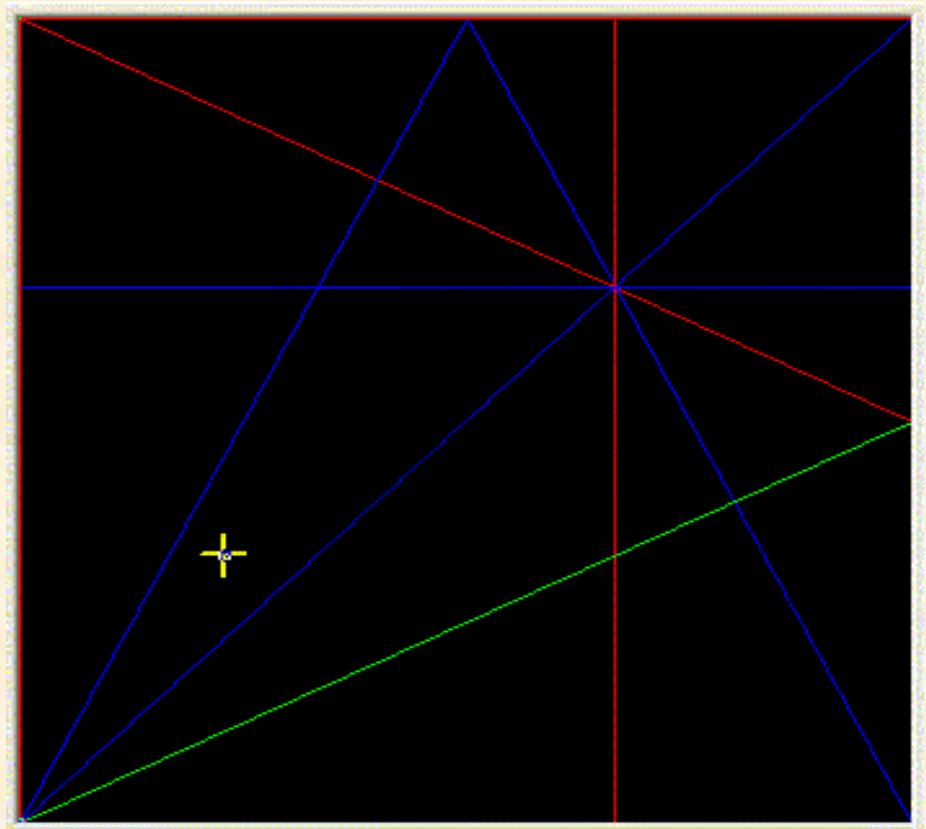
To avoid resonance conditions the frequency of the transverse motion must not be equal to (or a integer multiple of) the revolution frequency

$$1*Q_x = 1 \quad -> \quad Q_x = 1$$
$$2*Q_x = 1 \quad -> \quad Q_x = 0.5$$
$$\vdots \qquad\qquad \vdots$$

in general:

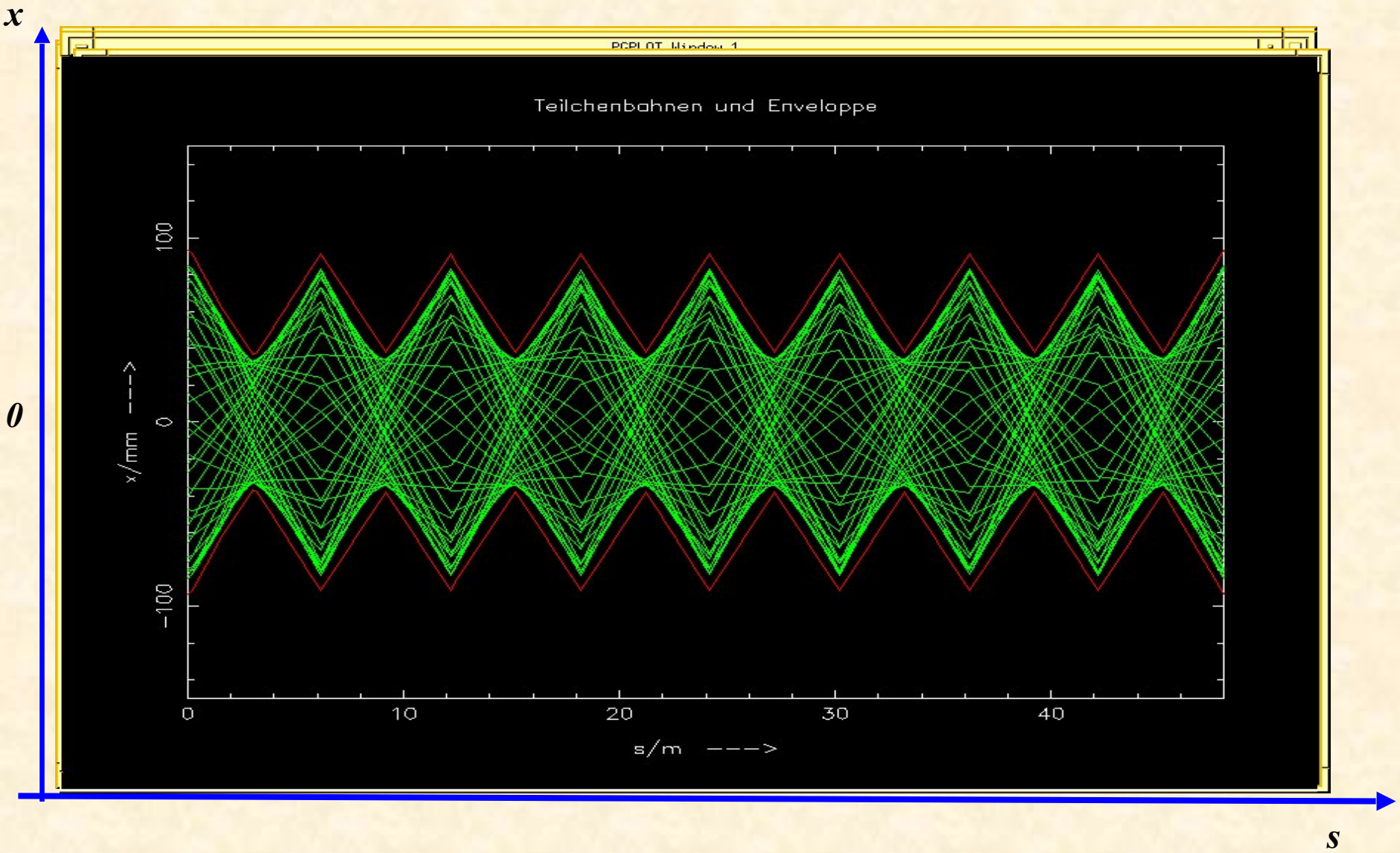
$$m*Q_x+n*Q_y+l*Q_s = integer$$

Tune diagram up to 3rd order



***Question:* what will happen, if the particle performs a second turn ?**

... or a third one or ... 10^{10} turns



Astronomer Hill:

differential equation for motions with periodic focusing properties
„Hill's equation"

Example: particle motion with periodic coefficient

equation of motion: $x''(s) + k(s) * x(s) = 0$

restoring force ≠ const,
k(s) = depending on the position s
k(s+L) = k(s), periodic function

we expect a kind of quasi harmonic oscillation: amplitude & phase will depend on the position s in the ring.

7.) The Beta Function

„it is convenient to see“ … after some beer

... we make two statements:

1.) There exists a mathematical function, that defines the envelope of all particle trajectories and so can act as measure for the beam size. We call it the β – function.

2.) Whow !!

A particle oscillation can then be written in the form

$$x(s) = \sqrt{\varepsilon} * \sqrt{\beta(s)} * \cos(\psi(s) + \phi)$$

ε, Φ = integration constants determined by initial conditions

β(s) periodic function given by focusing properties of the lattice ↔ quadrupoles

$$\beta(s + L) = \beta(s)$$

ε beam emittance = woozilycity of the particle ensemble, intrinsic beam parameter, cannot be changed by the foc. properties.
scientifiquely spoken: area covered in transverse x, x´ phase space

… and it is constant !!!

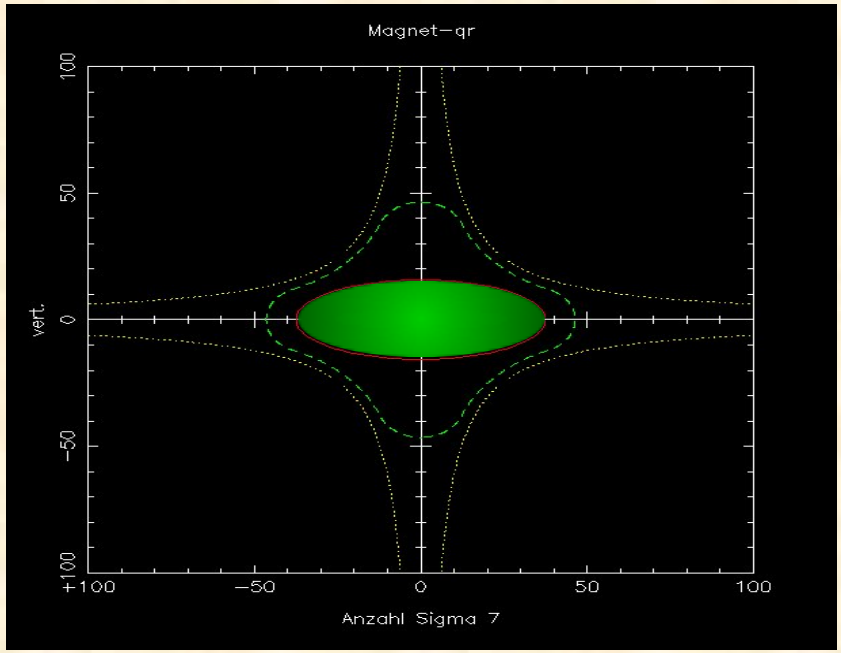
The Beta Function

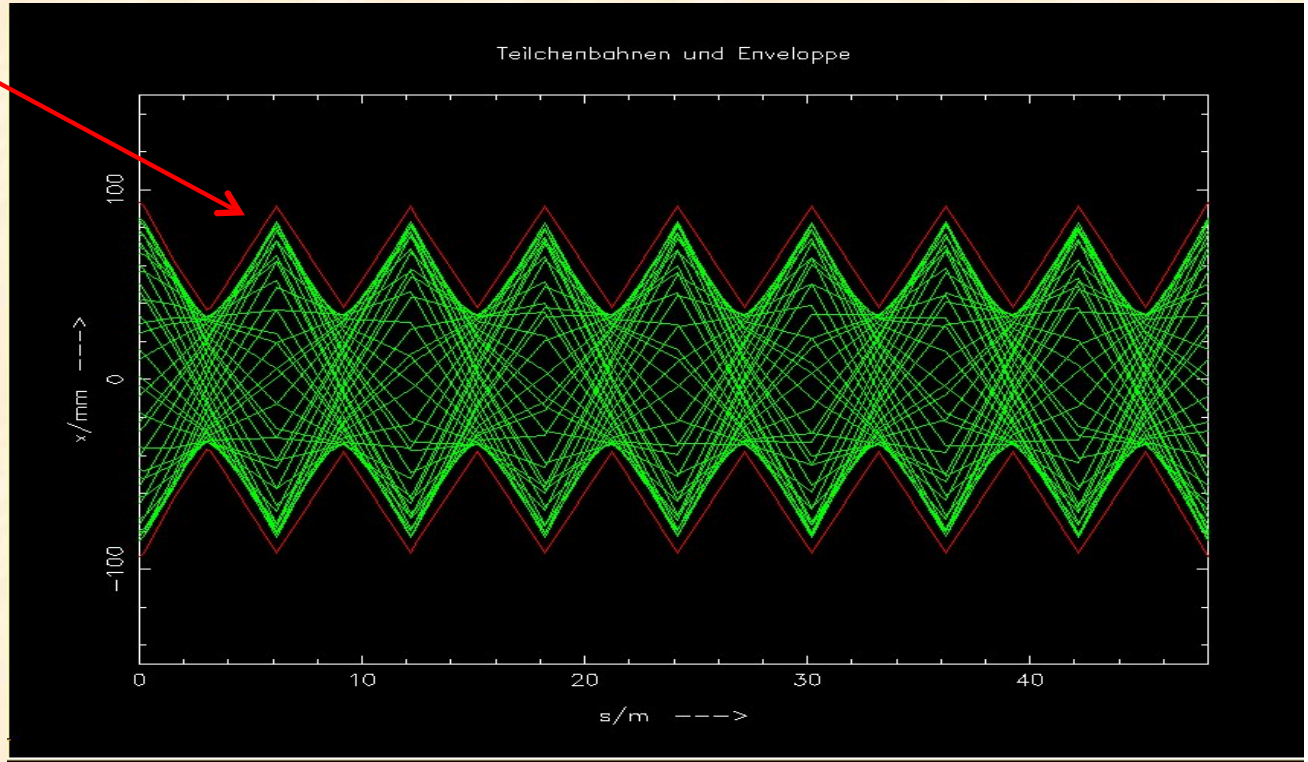
If we obtain the x, x' coordinates of a particle trajectory via

$$\begin{pmatrix} x \\ x' \end{pmatrix}_{s2} = M_{s_1,s_2} * \begin{pmatrix} x \\ x' \end{pmatrix}_{s1}$$

The maximum size of any particle amplitude at a position "s" is given by

$$\hat{x}(s) = \sqrt{\varepsilon}\,\sqrt{\beta(s)}$$





β determines the beam size
... the envelope of all particle trajectories at a given position "s" in the storage ring.

It reflects the periodicity of the magnet structure.

8.) Beam Emittance and Phase Space Ellipse

general solution of Hill equation

$$\begin{cases} (1) \quad x(s) = \sqrt{\varepsilon}\,\sqrt{\beta(s)}\,\cos(\psi(s)+\phi) \\ (2) \quad x'(s) = -\dfrac{\sqrt{\varepsilon}}{\sqrt{\beta(s)}}\left\{\alpha(s)\cos(\psi(s)+\phi)+\sin(\psi(s)+\phi)\right\} \end{cases}$$

from (1) we get

$$\cos(\psi(s)+\phi) = \frac{x(s)}{\sqrt{\varepsilon}\,\sqrt{\beta(s)}}$$

Insert into (2) and solve for ε

introducing two new parameters

$$\alpha(s) = \frac{-1}{2}\beta'(s)$$

$$\gamma(s) = \frac{1+\alpha(s)^2}{\beta(s)}$$

$$\varepsilon = \gamma(s)\,x^2(s) + 2\alpha(s)x(s)x'(s) + \beta(s)\,x'^2(s)$$

**** ε is a constant of the motion … it is independent of „s“***
**** parametric representation of an ellipse in the x x‘ space***
**** shape and orientation of ellipse are given by α, β, γ***

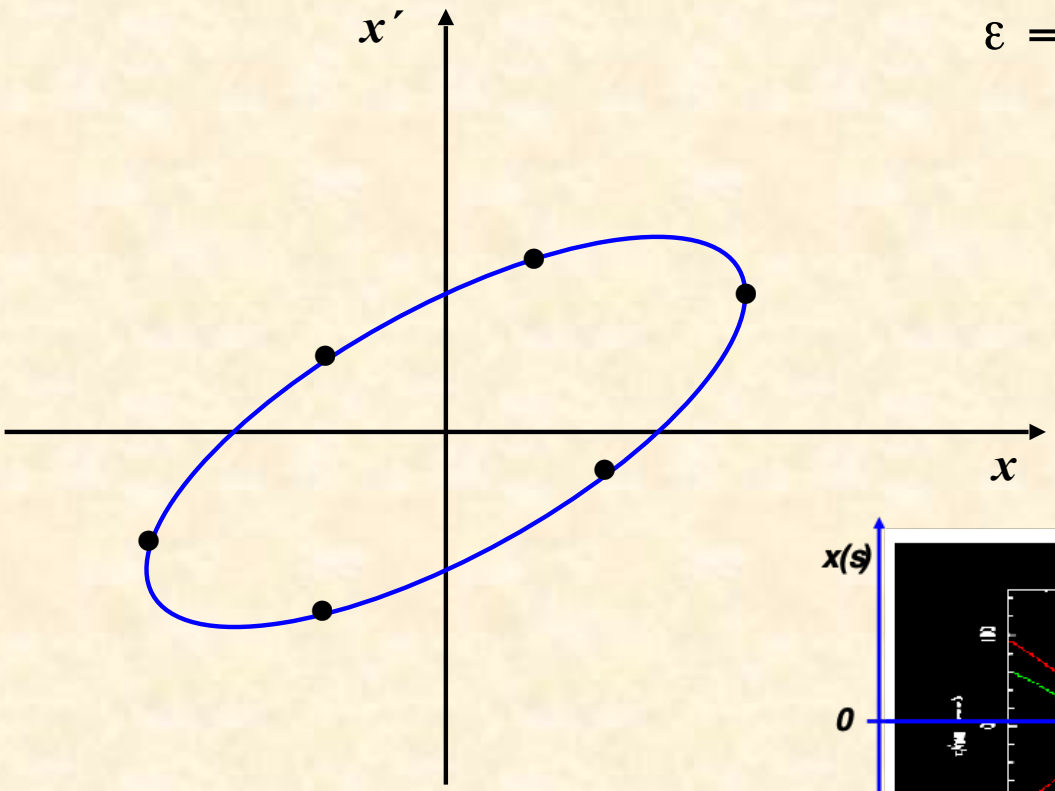
Beam Emittance and Phase Space Ellipse

***In phase space x, x' a particle** oscillation, observed at a given position "s" in the ring **is running on an ellipse** ... making Q revolutions per turn.*

$$x(s) = \sqrt{\varepsilon}\,\sqrt{\beta(s)}\,\cos(\psi(s)+\phi)$$

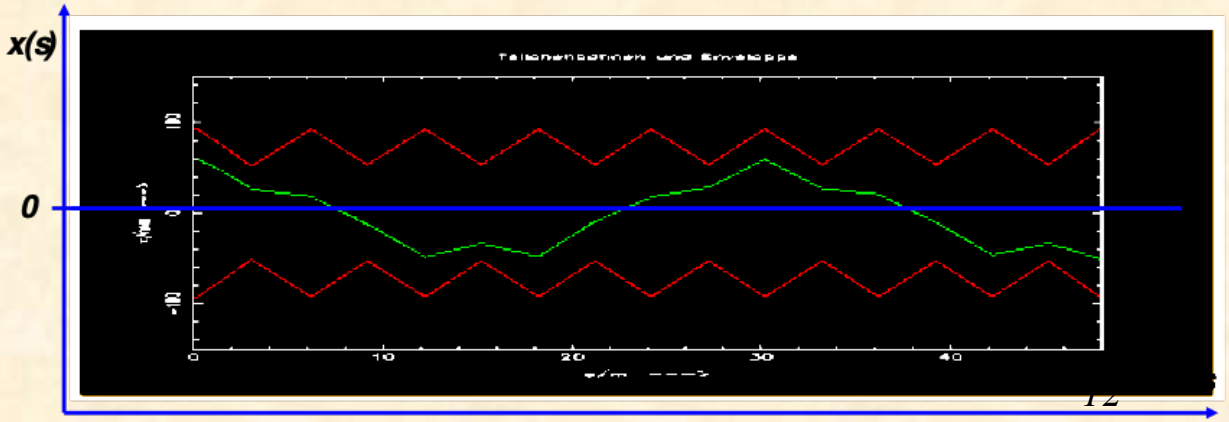
parametric representation of an ellipse in the x x' space

$$\varepsilon = \gamma(s) * x^2(s) + 2\alpha(s)x(s)x'(s) + \beta(s)x'(s)^2$$



Liouville:** in reasonable storage rings **area in phase space is constant.

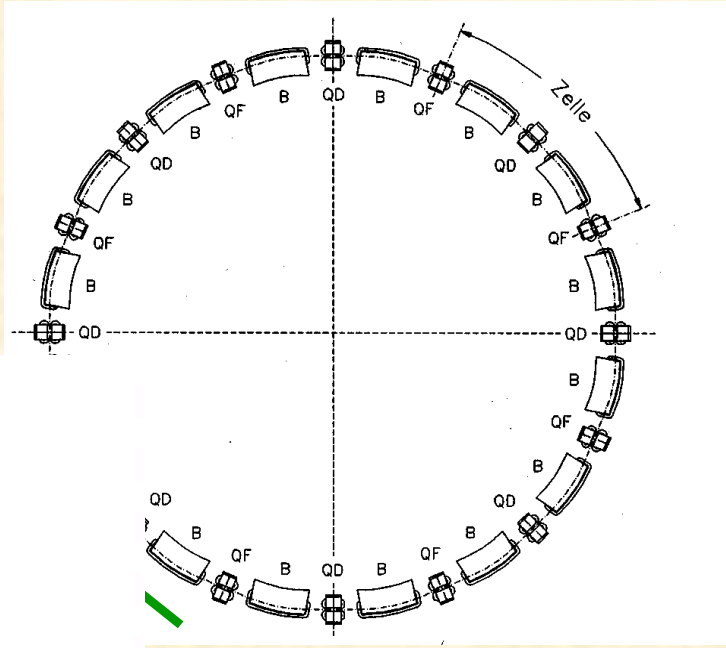
$$A = \pi * \varepsilon = const$$



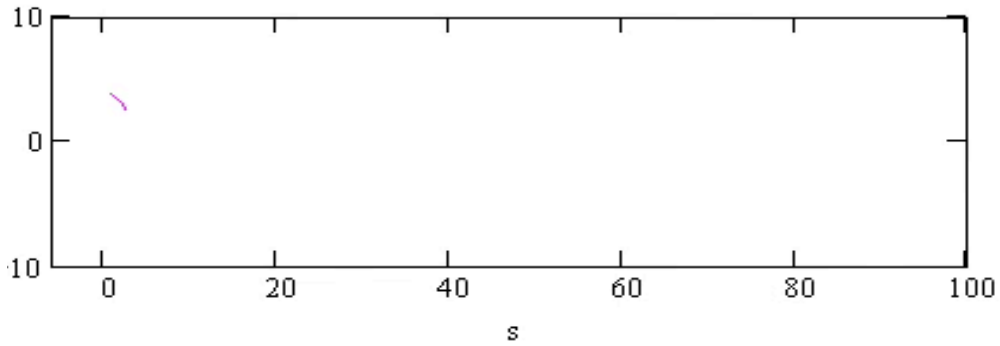
Particle Tracking in a Storage Ring

Calculate x, x´ for each linear accelerator element according to matrix formalism

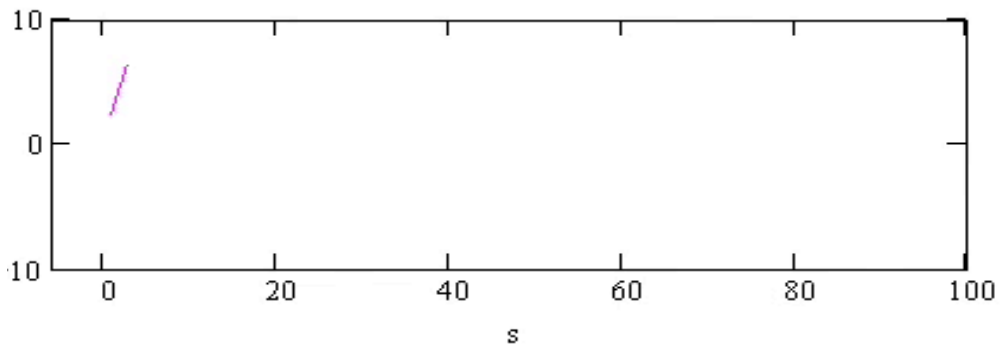
plot x, x´ as a function of „s"



X

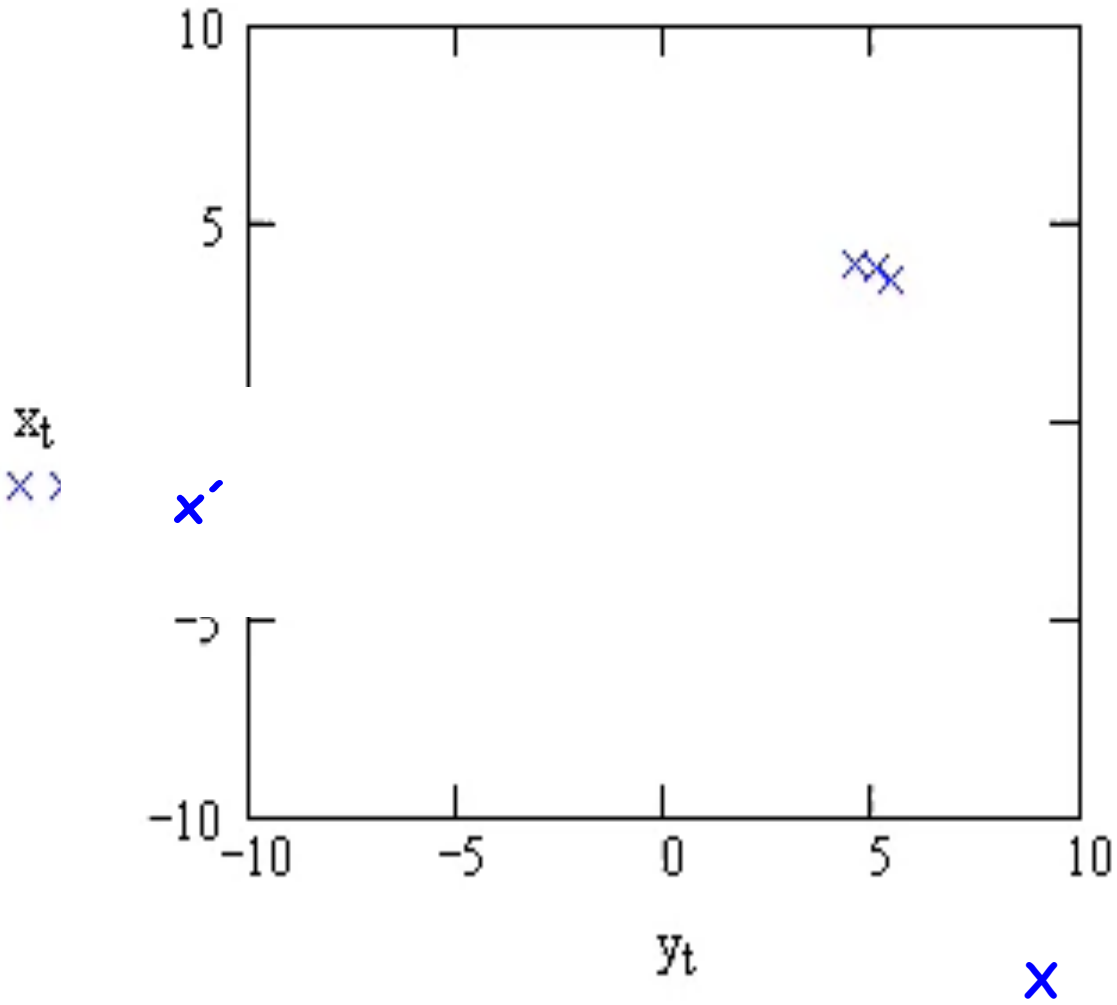


x´



… and now the ellipse:

note for each turn x, x´ at a given position „s_1" and plot in the phase space diagram



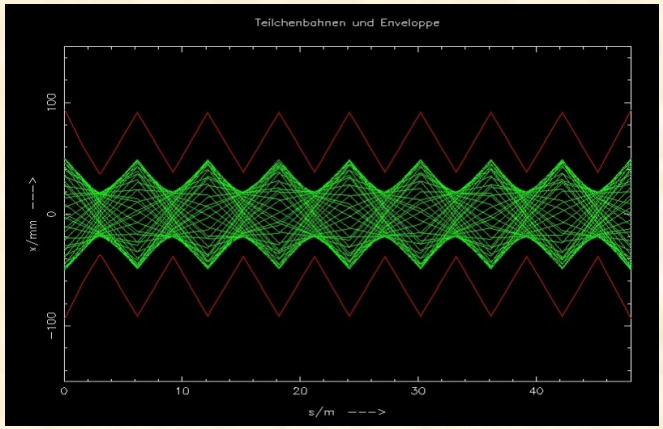
Time for the next blue Slide …

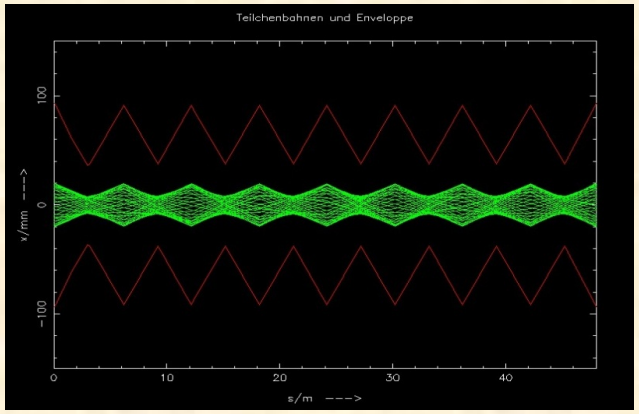
Why do we do that ?

—> the stability of the ellipse tells us about the stability of the particle oscillation, which is … … "the lifetime" of the beam.

—> the size of the ellipse tells us about the particle density, … which is the beam quality in collision.

Emittance of the Particle Ensemble:



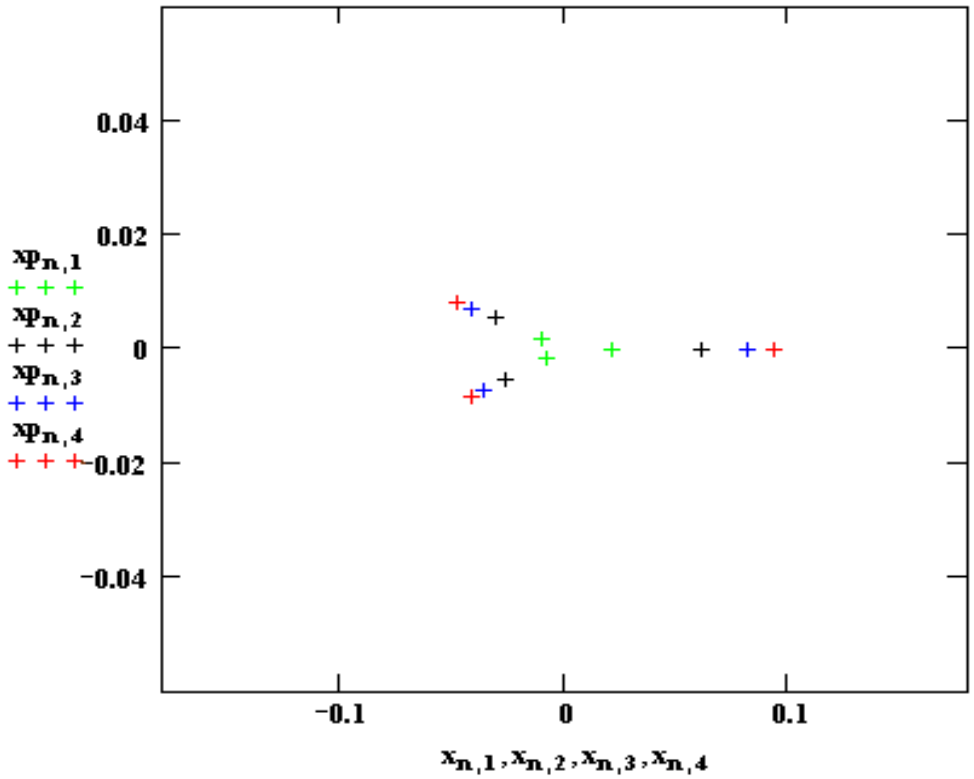


… to be very clear:

—> *as long as our particle is running on an ellipse in x, x' space ...*

everything is alright, the beam is stable and we can sleep well at nights.

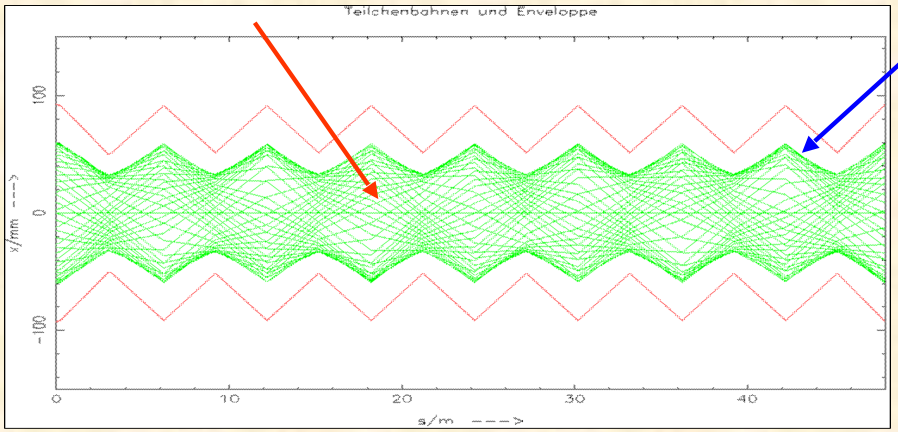
—> *if however we have scattering at the rest gas, or non-linear fields, or beam collisions (!) the particle will jump around in x-x' and ε will increase*



Emittance of the Particle Ensemble:

$$x(s) = \sqrt{\varepsilon}\,\sqrt{\beta(s)} \cdot \cos(\Psi(s) + \phi)$$

$$\hat{x}(s) = \sqrt{\varepsilon}\,\sqrt{\beta(s)}$$



single particle trajectories, N ≈ 10 ¹¹ per bunch

Gauß Particle Distribution:

$$\rho(x) = \frac{N \cdot e}{\sqrt{2\pi}\,\sigma_x} \cdot e^{-\frac{1}{2}\frac{x^2}{\sigma_x^2}}$$

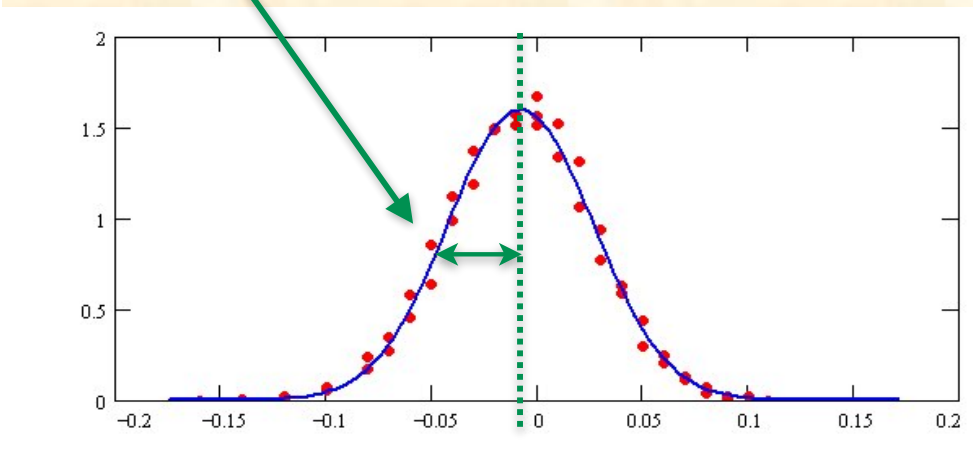
particle at distance 1 σ from centre
↔ 68.3 % of all beam particles

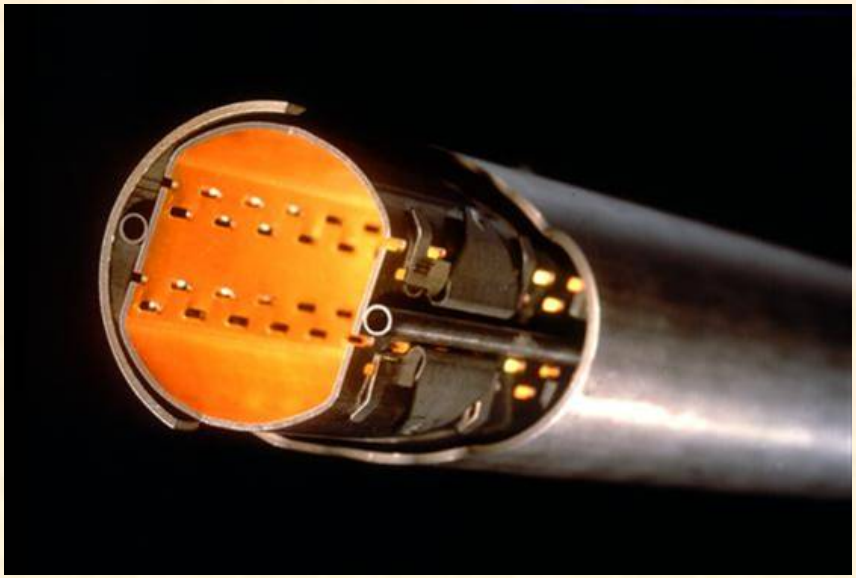
LHC:

$$\beta = 180\,m$$

$$\varepsilon = 5 * 10^{-10}\,mrad$$

$$\sigma = \sqrt{\varepsilon * \beta} = \sqrt{5 * 10^{-10}\,m * 180\,m} = 0.3\,mm$$

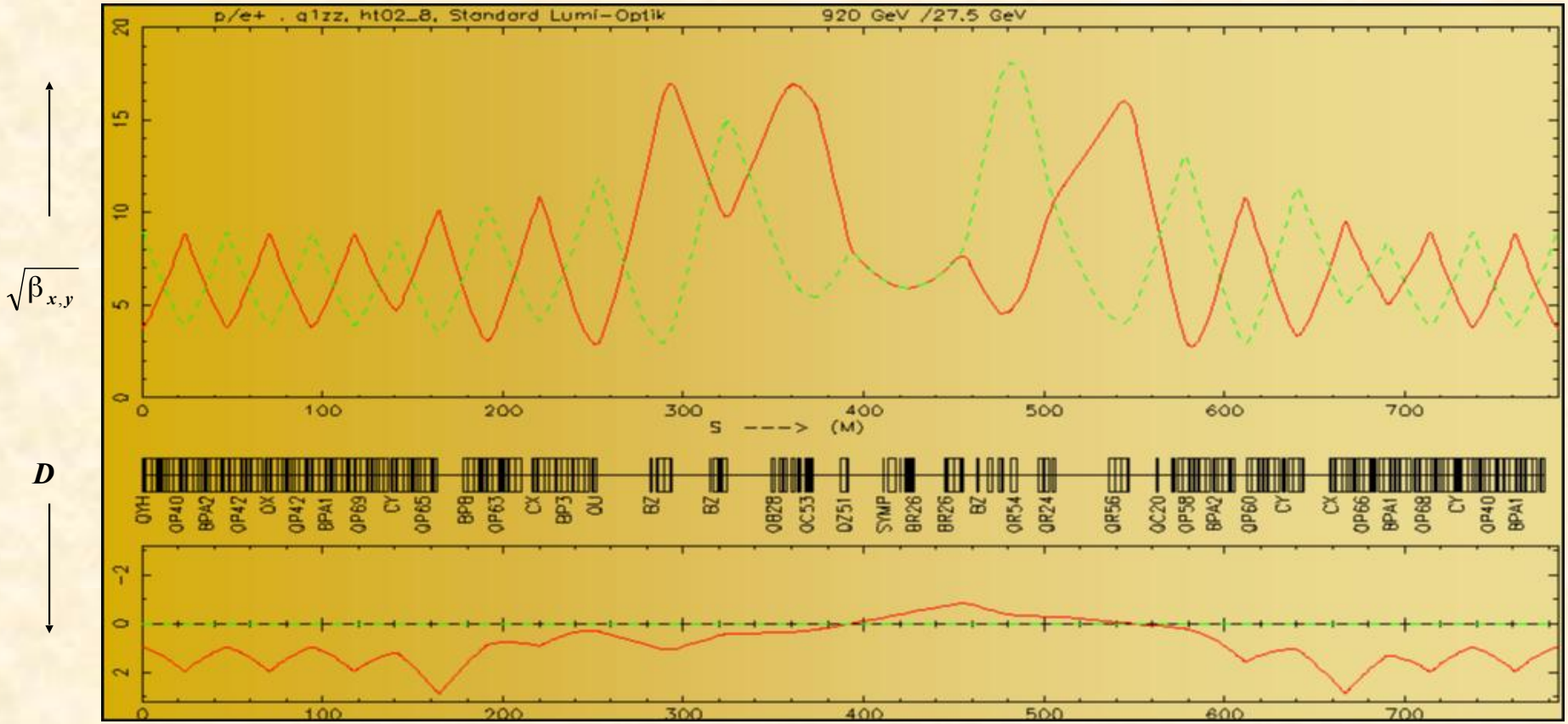




aperture requirements: $r_0 = 18 * \sigma$

The „not so ideal“ World

Lattice Design in Particle Accelerators



1952: Courant, Livingston, Snyder:

Theory of strong focusing in particle beams

Recapitulation: ...the story with the matrices !!!

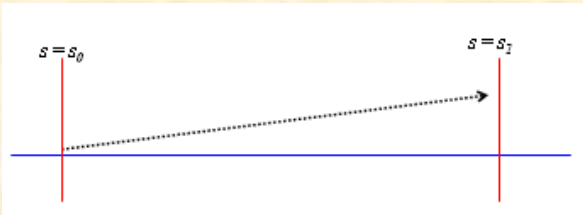
Equation of Motion:

$$x'' + K\,x = 0 \qquad K = 1/\rho^2 - k \;\ldots\; \textit{hor. plane:}$$

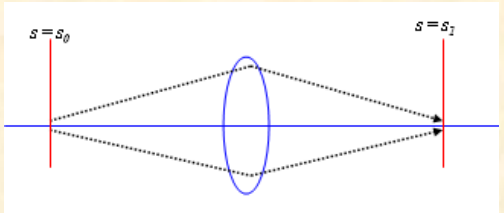
$$K = k \qquad \ldots\; \textit{vert. Plane:}$$

Solution of Trajectory Equations

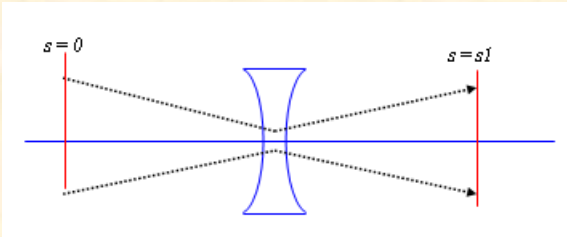
$$\begin{pmatrix} x \\ x' \end{pmatrix}_{s1} = M * \begin{pmatrix} x \\ x' \end{pmatrix}_{s0}$$



$$M_{drift} = \begin{pmatrix} 1 & l \\ 0 & 1 \end{pmatrix}$$



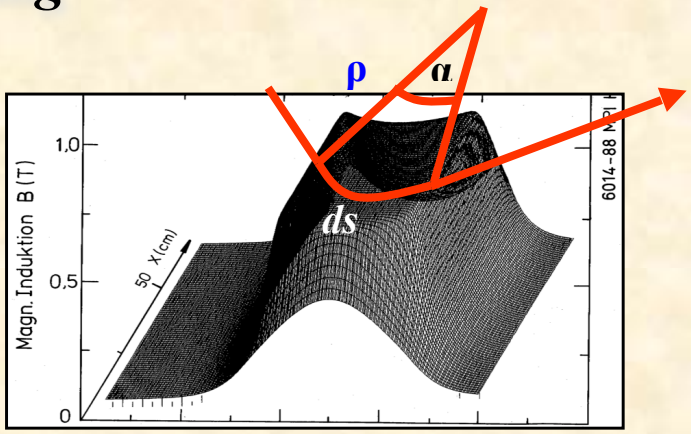
$$M_{foc} = \begin{pmatrix} \cos(\sqrt{|K|}\,l) & \frac{1}{\sqrt{|K|}}\sin(\sqrt{|K|}\,l) \\ -\sqrt{|K|}\sin(\sqrt{|K|}\,l) & \cos(\sqrt{|K|}\,l) \end{pmatrix}$$



$$M_{defoc} = \begin{pmatrix} \cosh(\sqrt{|K|}\,l) & \frac{1}{\sqrt{|K|}}\sinh(\sqrt{|K|}\,l) \\ \sqrt{|K|}\sinh(\sqrt{|K|}\,l) & \cosh(\sqrt{|K|}\,l) \end{pmatrix}$$

$$M_{total} = M_{QF} * M_D * M_B * M_D * M_{QD} * M_D * \quad \ldots$$

9.) Lattice Design: „… how to build a storage ring“



Geometry of the ring:

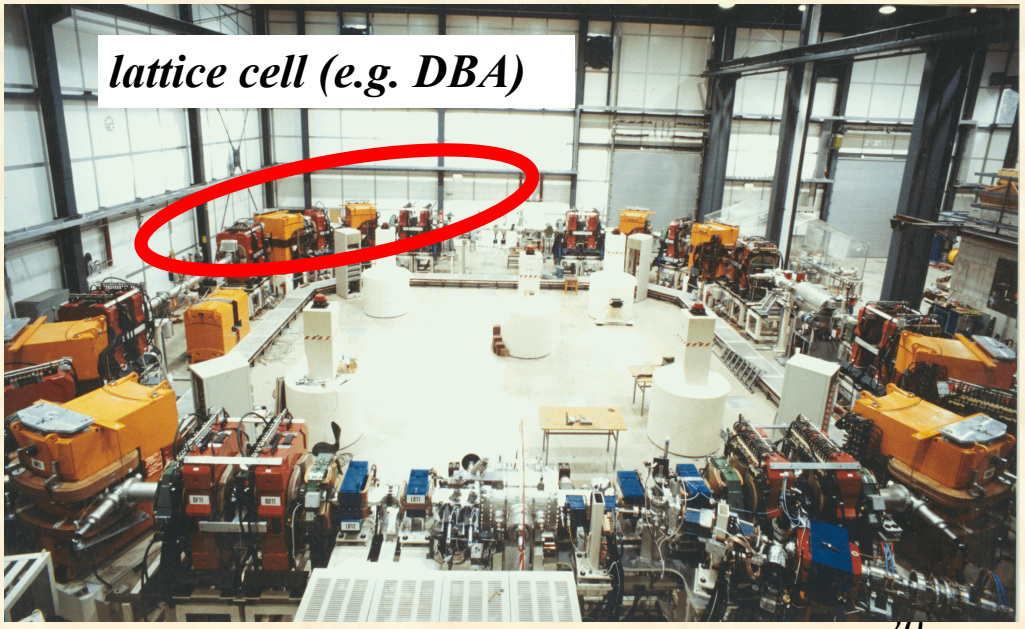
Circular Orbit: ***bending angle of one dipole***

$$\alpha = \frac{ds}{\rho} = \frac{\int Bdl}{B\rho}$$

The angle of all dipoles, swept out in one revolution must be 2π, so

$$\alpha = \frac{\int Bdl}{B\rho} = 2\pi \quad \longrightarrow \quad \int Bdl = 2\pi \cdot \frac{p}{q}$$

… defines the integrated dipole field around the machine.

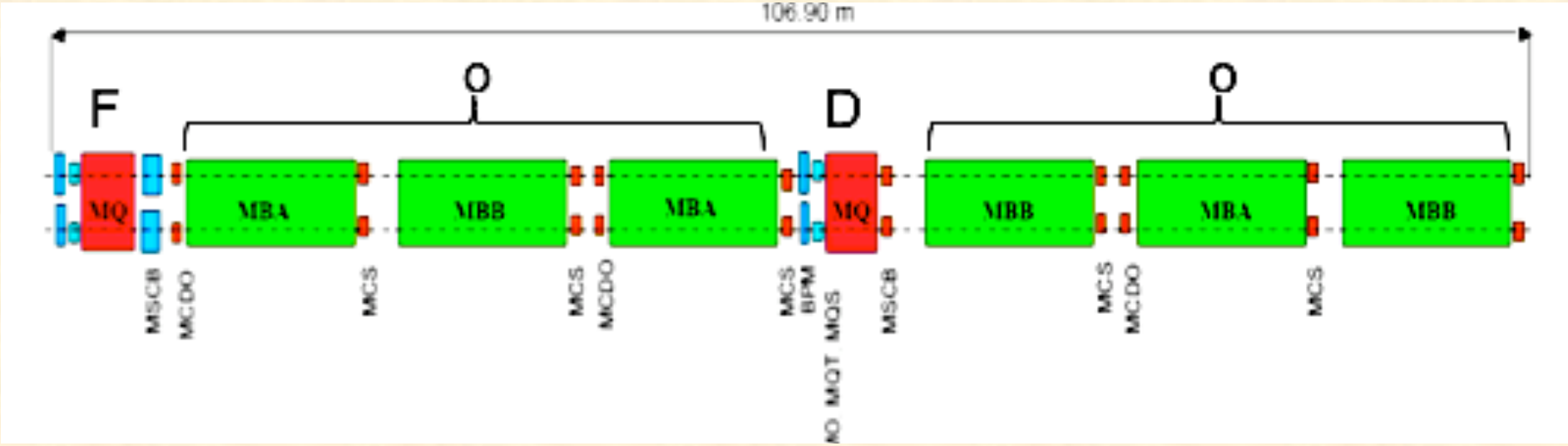


7000 GeV Proton storage ring
dipole magnets N = 1232
l = 15 m
q = +1 e

$$\int B\, dl \approx N\, l\, B = 2\pi\ p/e$$

$$B \approx \frac{2\pi\ \ 7000\ \ 10^9\, eV}{1232\ \ 15\, m\ \ 3\ 10^8\, \frac{m}{s}\ \ e} = 8.3\ Tesla$$

The Basic Cell of LHC:
… a 90° FoDo lattice



equipped with additional corrector coils

MB: main dipole
MQ: main quadrupole
MQT: Trim quadrupole
MQS: Skew trim quadrupole
MO: Lattice octupole (Landau damping)
MSCB: Skew sextupole
Orbit corrector dipoles
MCS: Spool piece sextupole
MCDO: Spool piece 8 / 10 pole
BPM: Beam position monitor + diagnostics

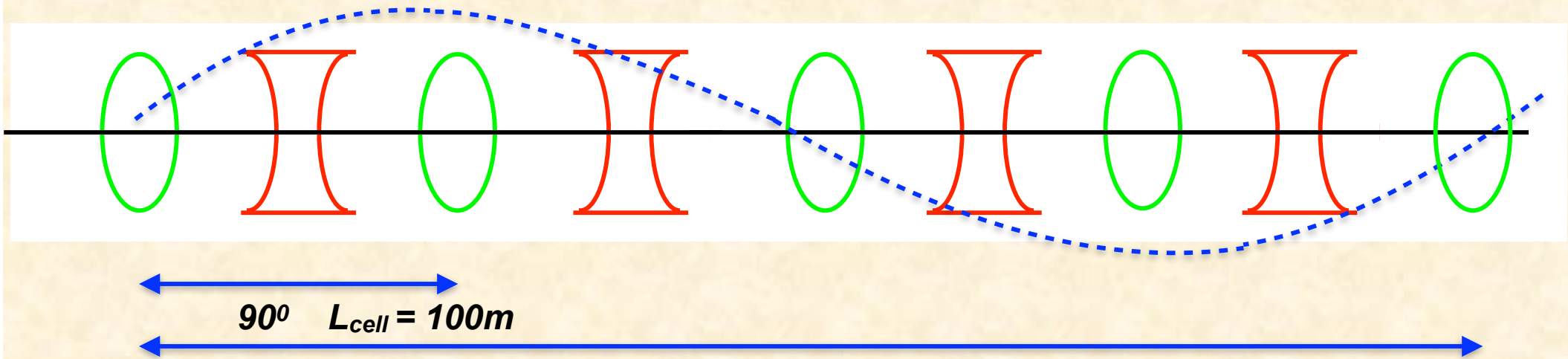
Dipoles

... The sum of the dipole fileds (in Tesla) multiplied by their length defines the particle momentum that we store in the ring.

Quadrupoles (gradient * length) define the transverse oscillation frequency. In LHC we need 4 cells (100m long each) for a full 360^0 oscillation, which is called a FoDo lattice with 90^0 phase advance.

And just like in playing a guitar, the higher the restoring force (quad gradient) the higher is the frequency (i.e. the phase advance per cell or for the complete ring the tune) … and we could even hear it !!!

"phase advance"



The Tune ...

...is the number of transverse oscillations per turn and corresponds to the „Eigenfrequency" or sound of the particle oscillations. As in any oscillating system (e.g. pendulum) we have to avoid resonance conditions between the eigenfrequemcy of the system (= partcicle) and any external frequency that might act on the beam. Most prominent external frequency is the revolution frequemcy itself !! -> avoid integer tunes.

The Beta function

shows the overall effect of all focusing fields; it has a certain value (m) that depends on the actual position in the ring, and is a measure of the transverse beam size.

The beam emittance

describes - independent of the focusing fields - the quality of the particle ensemble. It measures the area in phase space and can be considered like the temperature of a gas.

Small emittance —> high beam quality.

Together with the beta function it defines the beam dimension.

"Once more unto the breach, dear friends, once more"
(W. Shakespeare, Henry 5)

"Do they actually drop ?
do our particles follow gravity and fall down ?"

Answer: No

And in between the arcs ???

What about ...

Short Straight Sections

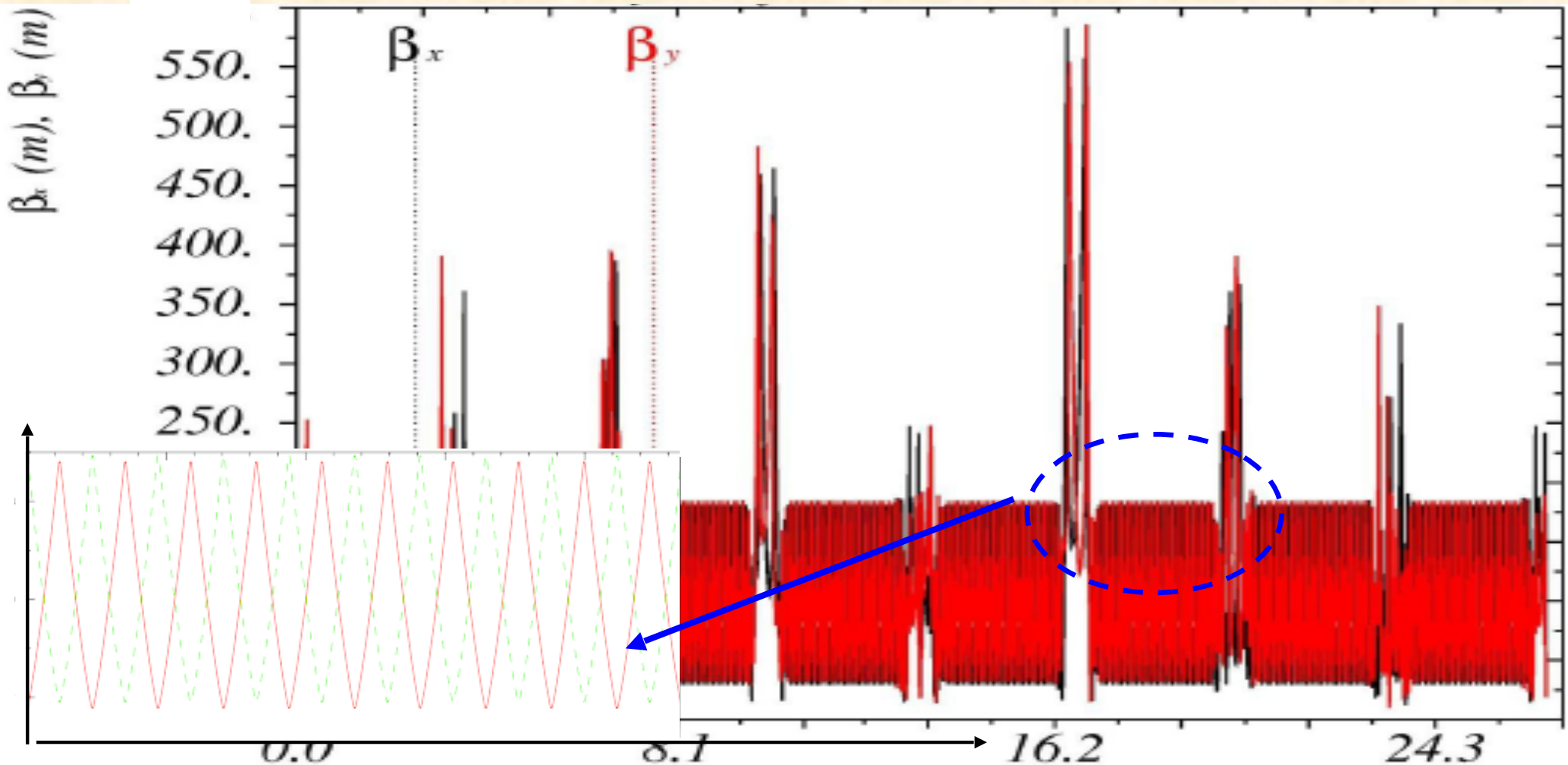
Long Straight Sections

Mini-Beta Insertions

etc etc

FoDo-Lattice

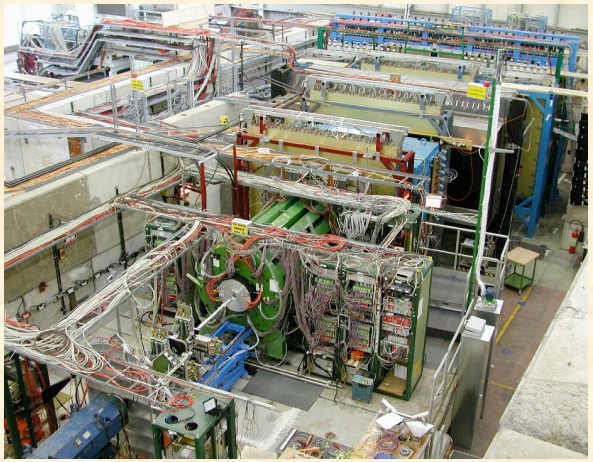
A magnet structure consisting of focusing and defocusing quadrupole lenses in alternating order with nothing in .
(Nothing = elements that can be neglected on first sight: drift, bending magnets, RF structures ... and especially experiments...)



Starting point for the calculation: in the middle of a focusing quadrupole
Phase advance per cell μ = 45°,
→ calculate the twiss parameters for a periodic solution

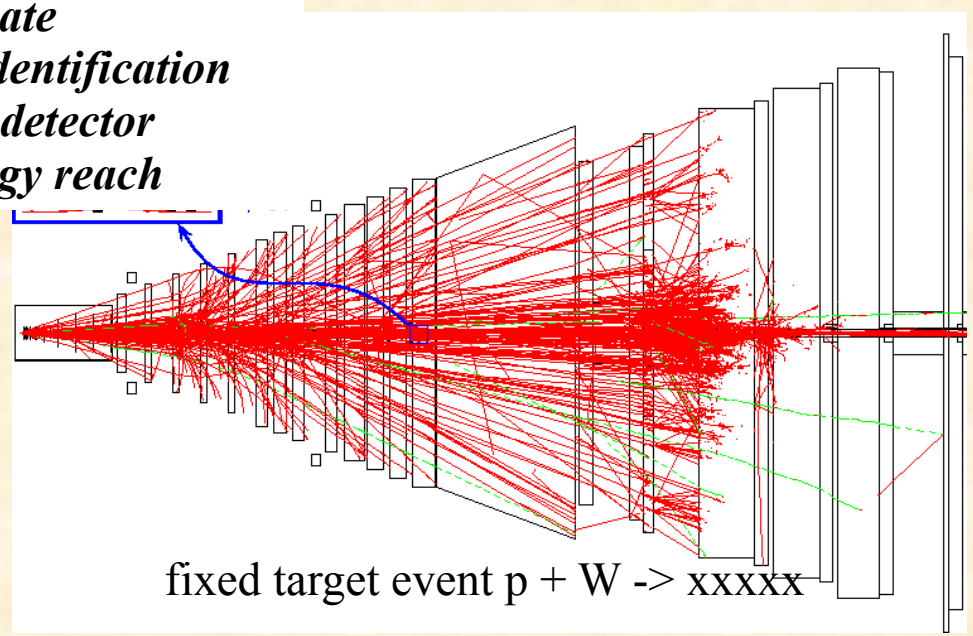
11.) The structure of matter:

Fixed target experiments:



HARP Detector, CERN

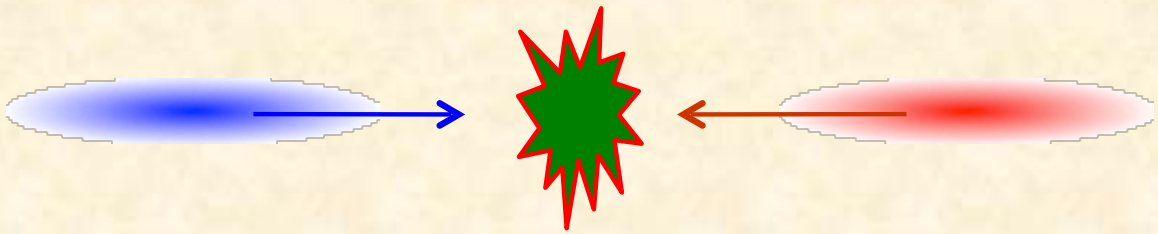
high event rate
easy track identification
asymmetric detector
limited energy reach

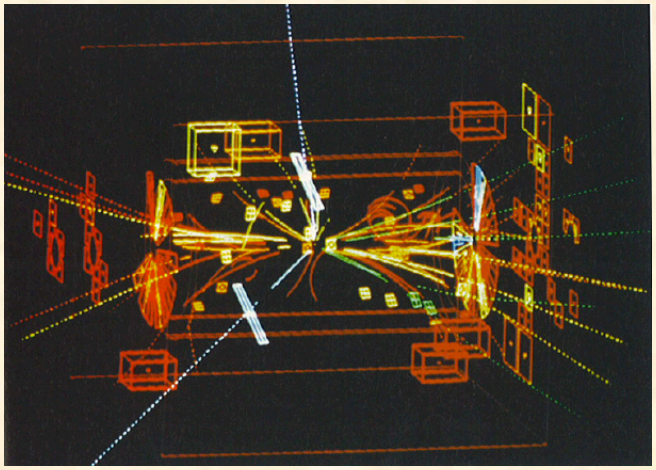


fixed target event p + W -> xxxxx

Collider experiments:

$E=mc^2$



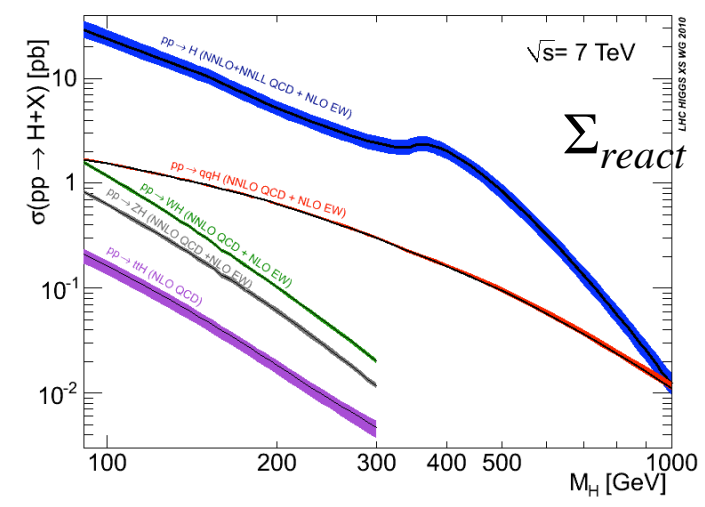


low event rate (luminosity)
but higher energy at IP

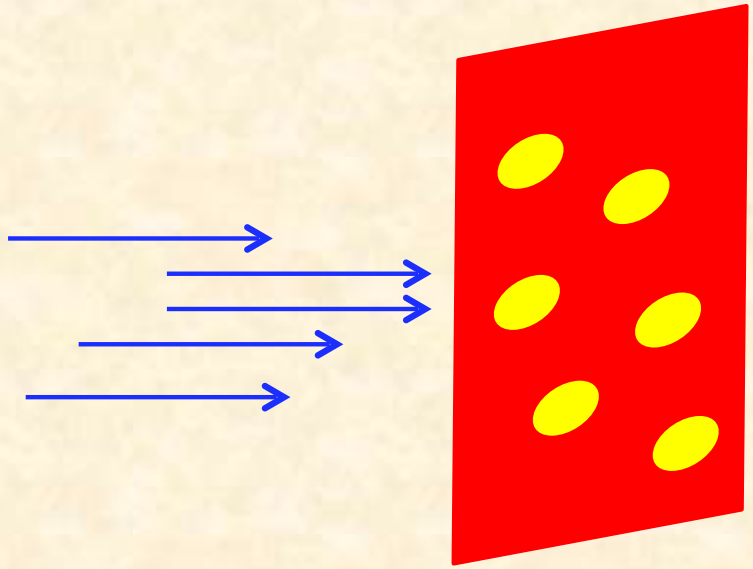
$E_{lab} = E_{cm}$

Problem: Our particles are VERY small !!

Overall cross section of the Higgs:



$$\Sigma_{react} \approx 1pb$$



The particles are "very small"

$$1b = 10^{-24} cm^2$$

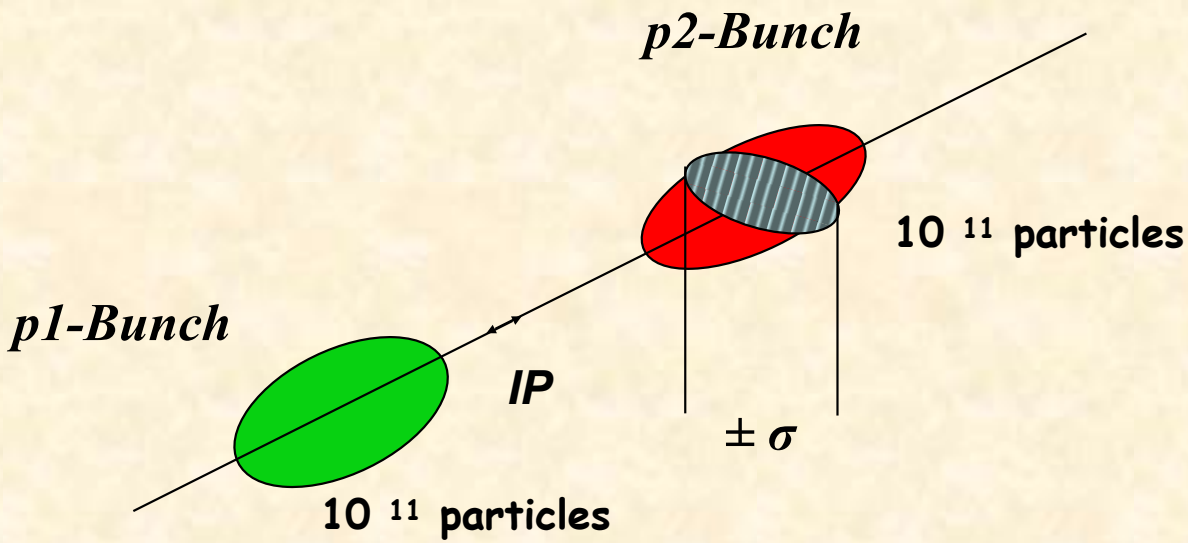
$$1\ pb = 10^{-12} \cdot 10^{-24} cm^2 = \frac{1}{mio} \cdot \frac{1}{mio} \cdot \frac{1}{mio} \cdot \frac{1}{mio} \cdot \frac{1}{mio} \cdot \frac{1}{10000}\ mm^2$$

The only chance we have:
compress the transverse beam size … at the IP

LHC typical:

σ = 0.1 mm → 16 μm

Luminosity



Example: Luminosity at LHC

$$f_0 = 11.245\ kHz$$
$$n_b = 2808$$
$$\beta_{x,y} = 0.55\ m$$
$$\varepsilon_{x,y} = 5 * 10^{-10}\ rad\ m$$
$$\sigma_{x,y} = 16\ \mu m$$
$$I_p = 584\ mA$$

$$L = \frac{1}{4\pi e^2 f_0 n_b} * \frac{I_{p1} I_{p2}}{\sigma_x \sigma_y}$$

Number of particles
in the target
in the moving beam
per second
per cm²

→ *how dense is the moscito cloud*

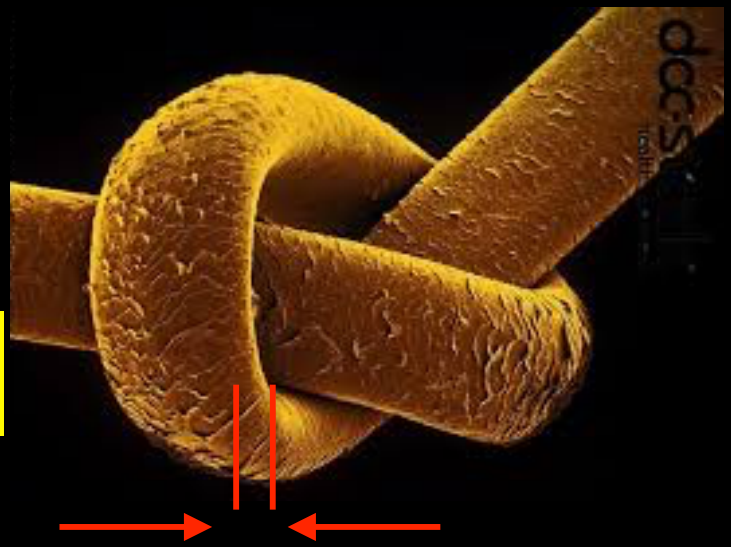
$$L = 1.0 * 10^{34}\ 1/cm^2 s$$

The Luminosity defines the number of "hits". It depends on the particle density at the collision point.

The Beta function at the IP "β*" should be made as small as possible to increase the particle density. In a drift β is growing quadratically and proportional to 1/β*, which sets the ultimate limit to the achievable luminosity.

The distance L* of the focusing magnets from the IP should be as small as possible.
… try to avoid detectors like ATLAS or CMS whenever possible. LOL.

The beam dimensions at the IP are typically a few µm.



Human hair:
d ≈ 70 µm