## LONGITUDINAL DYNAMICS



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## Scope and Summary of the 2 lectures:

The goal of an accelerator is to provide a stable particle beam.

The particles nevertheless perform transverse betatron oscillations.
We will see that they also perform oscillations in the longitudinal plane and in energy.

We will look at the stability of these oscillations, and their dynamics.

More related lectures:

- Linacs
- RF Systems
- Acceleration methods
- Accelerating structures
- Linac: Phase Stability + Energy-Phase oscillations
- Circular accelerators: Cyclotron / Synchrotron
- Stability in a Synchrotron
- Longitudinal Phase Space Motion
- Bunch and Bucket
- Injection Matching + Filamentation
- RF manipulations in the PS
- Alessandra Lombardi
- Heiko Damerau


## The CERN Accelerator Complex



- Linear accelerators scale in size and cost(!) ~linearly with the energy.
- Circular accelerators can each turn reuse
- the accelerating system
- the vacuum chamber
- the bending/focusing magnets
- beam instrumentation, ...
-> economic solution to reach higher particle energies
But each accelerator has a limited energy range.


## Particle types and acceleration

The accelerating system will depend upon the evolution of the particle velocity:

- electrons reach a constant velocity ( $\sim$ speed of light) at relatively low energy
- heavy particles reach a constant velocity only at very high energy
$\rightarrow$ we need different types of resonators, optimized for different velocities
$\rightarrow$ the revolution frequency will vary, so the RF frequency will be changing
-> magnetic field needs to follow the momentum increase
Particle rest mass $m_{0}$ : electron 0.511 MeV proton 938 MeV $239 \mathrm{U} \sim 220000 \mathrm{MeV}$

Total Energy: $E=m_{0} c^{2}$ Relativistic gamma factor:

$$
=\frac{E}{E_{0}}=\frac{m}{m_{0}}=\frac{1}{\sqrt{1 r^{2}}}
$$

Momentum:

$$
p=m v=\frac{E}{c^{2}} \quad c=\quad \frac{E}{c}=
$$



Particle energy (MeV)

## Revolution frequency variation

The revolution and RF frequency will be changing during acceleration Much more important for lower energies (values are kinetic energy - protons).

PS Booster: $\quad 50 \mathrm{MeV}(\beta=0.314)$-> $1.4 \mathrm{GeV}(\beta=0.915)$
(pre LS2)
(post LS2):
$602 \mathrm{kHz} \rightarrow 1746 \mathrm{kHz}=>190 \%$ frequency increase
$160 \mathrm{MeV}(\beta=0.520) \rightarrow 2 \mathrm{GeV}(\beta=0.948) \Rightarrow 95 \%$ increase
PS: $\quad 1.4 \mathrm{GeV}(\beta=0.915) \rightarrow 25.4 \mathrm{GeV}(\beta=0.9994)$
$437 \mathrm{KHz} \rightarrow 477 \mathrm{kHz}=>9 \%$ increase
(post LS2): $\quad 2 \mathrm{GeV}(\beta=0.948)$-> $25.4 \mathrm{GeV}(\beta=0.9994)$ ) $5 \%$ increase
SPS: $\quad 25.4 \mathrm{GeV}$-> $450 \mathrm{GeV}(\beta=0.999998)$
=> 0.06\% frequency increase
LHC: $\quad 450 \mathrm{GeV} \rightarrow 7 \mathrm{TeV}(\beta=0.999999991)$
=> only $210^{-6}$ increase
RF system needs more flexibility in lower energy accelerators.

## Acceleration: May the force be with you

To accelerate, we need a force in the direction of motion!


Newton-Lorentz Force on a charged particle with charge $q$

$$
\vec{F}=\frac{\mathrm{d} \vec{p}}{\mathrm{~d} t}=q(\vec{E}+\vec{v} \times \vec{B})
$$ to motion => no acceleration

Hence, it is necessary to have an electric field E (preferably) along the direction of the initial momentum (z), which changes the momentum of the particle.

$$
\frac{\mathrm{d} p}{\mathrm{~d} t}=q E_{z}
$$

The $2^{\text {nd }}$ term - larger at high velocities - is used for:

- BENDING: generated by a magnetic field perpendicular to the plane of the particle trajectory. The bending radius $\rho$ obeys the relation :

$$
\frac{p}{q}=B \rho \quad \text { in practical units for } q=e: \quad B \quad[\mathrm{Tm}] \quad \frac{p[\mathrm{GeV} / \mathrm{c}]}{0.3}
$$

- FOCUSING: the bending effect is used to bring the particles trajectory closer to the axis, hence to increase the beam density.


## Energy Gain

The acceleration increases the momentum, providing kinetic energy to the charged particles.

In relativistic dynamics, total energy $E$ and momentum $p$ are linked by

$$
E^{2}=E_{0}^{2}+p^{2} c^{2}
$$

$$
\left(E=E_{0}+W\right) \quad W \text { kinetic energy }
$$

$$
E_{0} \text { rest energy }
$$

Hence: $\quad d E=v d p$
$\left(2 E d E=2 c^{2} p d p \Leftrightarrow d E=c^{2} m v / E d p=v d p\right)$
The rate of energy gain per unit length of acceleration (along $z$ ) is then:

$$
\frac{d E}{d z}=v \frac{d p}{d z}=\frac{d p}{d t}=q E_{z}
$$

and the kinetic energy gained from the field along the $z$ path is:

$$
d W=d E=q E_{z} d z \quad \rightarrow \quad W=q \int E_{z} d z=q V
$$

where $V$ is just a potential.

## Unit of Energy

Today's accelerators and future projects work/aim at the TeV energy range.
LHC: $7 \mathrm{TeV} \rightarrow 14 \mathrm{TeV}$
CLIC: 3 TeV
FCC-hh: ~100 TeV
In fact, this energy unit comes from acceleration:
1 eV (electron Volt ) is the energy that 1 elementary charge e (like one electron or proton) gains when it is accelerated in a potential (voltage) difference of 1 Volt.

Basic Unit: eV (electron Volt)

$$
\begin{aligned}
& \mathrm{keV}=1000 \mathrm{eV}=10^{3} \mathrm{eV} \\
& \mathrm{MeV}=10^{6} \mathrm{eV} \\
& \mathrm{GeV}=10^{9} \mathrm{eV} \\
& \mathrm{TeV}=10^{12} \mathrm{eV}
\end{aligned}
$$

LHC $=\sim 450$ Million km of batteries!!! $3 \times$ distance Earth-Sun


## Electrostatic Acceleration



## Electrostatic Field:

Force: $\vec{F}=\frac{\mathrm{d} \vec{p}}{\mathrm{~d} t}=q \vec{E}$
Energy gain: $\mathrm{W}=\mathrm{q} \Delta \mathrm{V}$
used for first stage of acceleration: particle sources, electron guns, $x$-ray tubes

Limitation: insulation problems maximum high voltage ( $\sim 10 \mathrm{MV}$ )


Van-de-Graaf generator at MIT

## Radio-Frequency (RF) Acceleration


G.Ising

Electrostatic acceleration limited by insulation possibilities => use time-varying fields

1924: Ising suggests drift-tubes with time-varying fields

1928: Widerge builds first demonstration linac

Prinzip einer Methode zin Iterstellung von Kanalstrahlen hoher Yoltzahl.

Von
GUSTAF ISING.

Sit 2 Fignren in Texte.
Sitgeteilt am 12. März 1924 durch C. W. Osegn und M. Sizadagn.

P.Lebrun

## Radio-Frequency (RF) Acceleration



Cylindrical electrodes (drift tubes) separated by gaps and fed by a RF generator, as shown above, lead to an alternating electric field polarity
Synchronism condition
$\longrightarrow$
$L=v T / 2$
$v=$ particle velocity $T=R F$ period

Consequence: We can only accelerate bunched beam!
$\square$
Similar for standing wave cavity as shown (with $v \approx c$ )


## RF acceleration: Alvarez Structure

9 Used for protons, ions (50-200 MeV, f~200 MHz)


Synchronism condition $(g \ll L)$

$$
\Rightarrow \quad L=v_{s} T_{R F}=\beta_{s} \lambda_{R F}
$$

$$
\omega_{R F}=2 \pi f_{R F}=2 \pi \frac{v_{s}}{L}
$$



## Resonant RF Cavities

- Considering RF acceleration, it is obvious that when particles get high velocities the drift spaces get longer and one looses on the efficiency. => The solution consists of using a higher operating frequency.
- The power lost by radiation, due to circulating currents on the electrodes, is proportional to the RF frequency.
=> The solution consists of enclosing the system in a cavity which resonant frequency matches the RF generator frequency.

- The electromagnetic power is now constrained in the resonant volume
- Each such cavity can be independently powered from the RF generator
- Note however that joule losses will occur in the cavity walls (unless made of superconducting materials)


## Some RF Cavity Examples



## Common Phase Conventions

1. For circular accelerators, the origin of time is taken at the zero crossing of the RF voltage with positive slope
2. For linear accelerators, the origin of time is taken at the positive crest of the RF voltage

Time $t=0$ chosen such that:


$$
E_{1}(t)=E_{0} \sin \left({ }_{R F} t\right)
$$


$E_{2}(t)=E_{0} \cos \left({ }_{R F} t\right)$
3. I will stick to convention 1 in the following to avoid confusion

## Principle of Phase Stability (Linac)

Let's consider a succession of accelerating gaps, operating in the $2 \pi$ mode, for which the synchronism condition is fulfilled for a phase $\Phi_{s}$.
$e V_{S}=e \hat{V} \sin$ is the energy gain in one gap for the particle to reach the $S$ next gap with the same RF phase: $\mathbb{P}_{1}, P_{2}, \ldots . .$. are fixed points.


If an energy increase is transferred into a velocity increase =>

$$
\begin{array}{ll}
M_{1} \& N_{1} \text { will move towards } P_{1} & \Rightarrow \text { stable } \\
M_{2} \& N_{2} \text { will go away from } P_{2} & \Rightarrow \text { unstable }
\end{array}
$$

(Highly relativistic particles have no significant velocity change)

## A Consequence of Phase Stability




The divergence of the field is zero according to Maxwell :

$$
\nabla \vec{E}=0 \Rightarrow \frac{\partial E_{x}}{\partial x}+\frac{\partial E_{z}}{\partial z}=0 \Rightarrow \frac{\partial E_{x}}{\partial x}=\frac{\partial E_{z}}{\partial z}
$$

Transverse fields

- focusing at the entrance and
- defocusing at the exit of the cavity.

Electrostatic case: Energy gain inside the cavity leads to focusing RF case:

Field increases during passage => transverse defocusing!

## External focusing (solenoid, quadrupole) is then necessary

## Energy-phase Oscillations (Small Amplitude) (1)

- Rate of energy gain for the synchronous particle ( $q=e$ ):

$$
\frac{d E_{s}}{d z}=\frac{d p_{s}}{d t}=e E_{0} \sin
$$

- Use reduced variables with respect to synchronous particle

$$
w=W-W_{s}=E-E_{s} \quad \varphi=\phi-\phi_{s}
$$

Energy gain: $\frac{d w}{d z}=e E_{0}\left[\sin \left(\phi_{s}+\varphi\right)-\sin \phi_{s}\right] \approx e E_{0} \cos \phi_{s} \cdot \varphi \quad(\operatorname{small} \varphi)$

- Rate of phase change with respect to the synchronous one:

$$
\frac{d \varphi}{d z}=\omega_{R F}\left(\frac{d t}{d z}-\left(\frac{d t}{d z}\right)_{s}\right)=\omega_{R F}\left(\frac{1}{v}-\frac{1}{v_{s}}\right) \cong-\frac{\omega_{R F}}{v_{s}^{2}}\left(v-v_{s}\right)
$$

Leads finally to:

$$
\frac{d \varphi}{d z}=-\frac{\omega_{R F}}{m_{0} v_{s}^{3} \gamma_{s}^{3}} w
$$

## Energy-phase Oscillations (Small Amplitude) (2)

Combining the two $1^{\text {st }}$ order equations into a $2^{\text {nd }}$ order equation gives the equation of a harmonic oscillator:

$$
\frac{d^{2} \varphi}{d z^{2}}+\Omega_{s}^{2} \varphi=0 \quad \text { with }
$$

Stable harmonic oscillations imply:

$$
\Omega_{s}^{2}=\frac{e E_{0} \omega_{R F} \cos \phi_{s}}{m_{0} v_{s}^{3} \gamma_{s}^{3}}
$$

Slower for higher energy! hence: $\quad \cos \phi_{s}>0$

And since acceleration also means:

$$
\sin \phi_{s}>0
$$

You finally get the result for the stable phase range:

$$
\begin{gathered}
0<\phi_{s}<\frac{\pi}{2} \quad \begin{array}{c}
\text { Positive rising } \\
\text { RF slope! }
\end{array}
\end{gathered}
$$



## Summary up to here...

- Acceleration by electric fields, static fields limited => time-varying fields
- Synchronous condition needs to be fulfilled for acceleration
- Particles perform oscillation around synchronous phase
- Stable acceleration on the rising slope in a linac.
- Electrons are quickly relativistic, speed does not change
- Protons and ions need changing structure geometry and certain RF frequency range


## Circular accelerators

## Betatron

Cyclotron

## Synchrotron

## Acceleration by Induction: The Betatron

## A ramping magnetic field

- Guides particles on a circular trajectory and
- Creates a tangential electric field that accelerates the particles

Limited by saturation in iron ( $\sim 300 \mathrm{MeV}$ e-)
Used in industry and medicine, as they are compact accelerators for electrons


Donald Kerst with the first betatron, invented at the University of Illinois in 1940


## Circular accelerators: Cyclotron

Courtesy: EdukiteLearning, https://youtu.be/cNnNM2ZqIsc

## Circular accelerators: Cyclotron



Used for protons, ions

$$
\begin{aligned}
& \mathrm{B}=\text { constant } \\
& \omega_{\mathrm{RF}}=\text { constant }
\end{aligned}
$$



Synchronism condition

$$
\Rightarrow \begin{aligned}
\omega_{s} & =\omega_{R F} \\
2 \pi \rho & =v_{s} T_{R F}
\end{aligned}
$$



Cyclotron frequency

$$
\omega=\frac{q B}{m_{0} \gamma}
$$

1. $\quad \gamma$ increases with the energy $\Rightarrow$ no exact synchronism
2. if $v \ll c \Rightarrow \gamma \cong 1$
[^0]
## Cyclotron / Synchrocyclotron



Synchrocyclotron: Same as cyclotron, except a modulation of $\omega_{\mathrm{RF}}$

$$
\text { B } \quad=\text { constant }
$$

$$
\gamma \omega_{\mathrm{RF}} \quad=\text { constant } \quad \omega_{\mathrm{RF}} \text { decreases with time }
$$

The condition:

$$
\omega_{s}(t)=\omega_{R F}(t)=\frac{q B}{m_{0} \gamma(t)}
$$

Allows to go beyond the non-relativistic energies

## Circular accelerators: Cyclotron



Courtesy Berkeley Lab, https://www.youtube.com/watch?v=cutKuFxeXmQ

## Circular accelerators: The Synchrotron



Synchronism condition

1. Constant orbit during acceleration
2. To keep particles on the closed orbit, B should increase with time
3. $\omega$ and $\omega_{R F}$ increase with energy

RF frequency can be multiple of revolution frequency

$$
\omega_{R F}=h \omega
$$

$$
T_{s}=h T_{R F}
$$

$\frac{2 \pi R}{v_{s}}=h T_{R F}$
$h$ integer, harmonic number: number of RF cycles per revolution
$h$ is the maximum number of bunches in the synchrotron.
Normally less bunches due to gaps for kickers, collision constraints,...

## Circular accelerators: The Synchrotron



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## The Synchrotron

The synchrotron is a synchronous accelerator since there is a synchronous RF phase for which the energy gain fits the increase of the magnetic field at each turn. That implies the following operating conditions:


If $v \approx c, \omega$ hence $\omega_{\text {RF }}$ remain constant (ultra-relativistic $e^{-}$)

## The Synchrotron - LHC Operation Cycle

The magnetic field (dipole current) is increased during the acceleration.


## The Synchrotron - Energy ramping

Energy ramping by increasing the $B$ field (frequency has to follow v):

$$
p=e B \underset{\rho \text { const. }}{\Rightarrow} \frac{d p}{d t}=e \quad \dot{B} \Rightarrow(p)_{t u r n}=e \quad \dot{B} T_{r}=\frac{2 \quad R \dot{B}}{v}
$$

With
$E^{2}=E_{0}^{2}+p^{2} c^{2} \Rightarrow E=v p \quad(E)_{\text {turn }}=(W)_{s}=2$ e $R \dot{B}=e \hat{V} \sin$
Synchronous phase $\varphi_{s}$ changes during energy ramping

$$
\begin{aligned}
& \qquad \sin \phi_{s}=2 \pi \rho R \frac{\dot{B}}{\hat{V}_{R F}} \Rightarrow \phi_{s}=\arcsin \left(2 \pi \rho R \frac{\dot{B}}{\hat{V}_{R F}}\right) \\
& \text { he synchronous phase depends on } \\
& \text { - the change of the magnetic field } \\
& \text { and the RF voltage } \\
& \text { CAS Basic, } 9-13 \text { May } 2022
\end{aligned}
$$

## The Synchrotron - Frequency change

During the energy ramping, the RF frequency increases to follow the increase of the revolution frequency:

$$
\omega=\frac{\omega_{R F}}{h}=\omega\left(B, R_{S}\right)
$$

Hence: $\frac{f_{R F}(t)}{h}=\frac{v(t)}{2 R_{s}}=\frac{1}{2} \frac{e c^{2}}{E_{s}(t)} \frac{R_{s}}{R_{s}} B(t) \quad\left(\right.$ using $\left.p(t)=e B(t), \quad E=m c^{2}\right)$
Since $E^{2}=\left(m_{0} c^{2}\right)^{2}+p^{2} c^{2}$ the RF frequency must follow the variation of the $B$ field with the law

$$
\frac{f_{R F}(t)}{h}=\frac{c}{2 R_{s}} \frac{B(t)^{2}}{\left(m_{0} c^{2} / e c\right)^{2}+B(t)^{2}}{ }^{1 / 2}
$$

RF frequency program during acceleration determined by B-field!

## Example: PS - Field / Frequency change

During the energy ramping, the B-field and the revolution frequency increase


## Overtaking in a roundabout

Finally a real-life problem: what is the fastest way through a roundabout?
Most CERN people encounter this near the French entrance to CERN.


## Optimize the roundabout!



The magic roundabout in Swindon, UK!
Video: https://www.youtube.com/watch?v=60Gvj7GZSIo

## Overtaking in a Formula 1 Race




## Overtaking in a Formula 1 Race

## Overtaking in a Formula 1 Race



A F1 car wants to overtake another car! It will have a

- a different track length due to a 'dispersion orbit'
- and a different velocity.

$$
\begin{aligned}
& T=\frac{L}{v}=\frac{2 \pi R}{v} \text { and } f_{r}=\frac{1}{T}=\frac{v}{2 \pi R} \\
& =>\frac{\Delta T}{T}=\frac{\Delta R}{R}-\frac{\Delta v}{v}
\end{aligned}
$$

The winner depends on the relative change in speed compared to the relative change in track length!

> If the relative change in speed is larger than the relative change in track length $=>$ the red car will win!

## Overtaking in a Synchrotron


$\mathrm{p}=$ particle momentum
$\mathrm{R}=$ synchrotron physical radius
$f_{r}=$ revolution frequency

A particle slightly shifted in momentum will have a

- dispersion orbit and a different orbit length (higher momentum $=>$ less bent in magnet)
- a different velocity.

As a result of both effects the revolution frequency changes with a "slip factor $n$ ":

$$
\eta=\frac{d T / T}{d p / p}
$$

Note: you also find $n$ defined with a minus sign!

Effect from orbit defined by Momentum compaction factor:

$$
\alpha_{c}=\frac{d L / L}{d p / p}
$$

$$
\alpha_{c}=\frac{1}{L} \int_{C} \frac{D_{x}(s)}{\rho(s)} d s_{0}
$$

## Dispersion Effects - Revolution Period

The two effects of the orbit length and the particle velocity change the revolution period as:

$$
\begin{gathered}
T=\frac{L}{\beta c} \quad \Rightarrow \quad \frac{d T}{T}=\frac{d L}{L}-\frac{d \beta}{\beta}=\alpha_{\uparrow} \frac{d p}{p}-\frac{d \beta}{\beta} \\
\frac{d T}{T}=\left(\alpha_{c}-\frac{1}{\gamma^{2}}\right) \frac{d p}{p} \quad p=m v=\frac{E_{0}}{c} \quad \frac{d p}{p}=\frac{d}{\text { definition of momentum }} \text { compaction factor }
\end{gathered}+\frac{d\left(\begin{array}{ll}
1 & 2^{2}
\end{array}\right)^{1 / 2}}{\left(\begin{array}{ll}
2
\end{array}\right)^{1 / 2}}=\underbrace{\left(\begin{array}{ll}
2
\end{array}\right)^{1}}_{2} \frac{d}{}-1 .
$$

$$
\operatorname{Slip}_{\text {factor: }} \eta=\alpha_{c}-\frac{1}{\gamma^{2}} \quad \text { or } \quad \eta=\frac{1}{\gamma_{t}^{2}}-\frac{1}{\gamma^{2}} \quad \text { with } \quad \gamma_{t}=\frac{1}{\sqrt{\alpha_{c}}}
$$

Note: you also find $n$ defined with a minus sign!
At transition energy, $\eta=0$, the velocity change and the path length change with momentum compensate each other. So the revolution frequency there is independent from the momentum deviation.

## Phase Stability in a Synchrotron

From the definition of $\eta$ it is clear that an increase in momentum gives

- below transition ( $\eta<0$ ) a higher revolution frequency (increase in velocity dominates) while
- above transition ( $\eta>0$ ) a lower revolution frequency ( $v \approx c$ and longer path) where the momentum compaction (generally $>0$ ) dominates.



## Crossing Transition

At transition, the velocity change and the path length change with momentum compensate each other. So the revolution frequency there is independent from the momentum deviation.
Crossing transition during acceleration makes the previous stable synchronous phase unstable. The RF system needs to make a rapid change of the RF phase, a 'phase jump'.


In the PS: $\gamma_{+}$is at $\sim 6 \mathrm{GeV}$
In the SPS: $\gamma_{\dagger}=22.8$, injection at $\gamma=27.7$
$\Rightarrow$ no transition crossing!
In the $\mathrm{LHC}: \gamma_{+}$is at $\sim 55 \mathrm{GeV}$, also far below injection energy
Transition crossing not needed in leptons machines, why?

## Dynamics: Synchrotron oscillations

Simple case (no accel.): $B=$ const., below transition $\quad \gamma<\gamma_{t}$
The phase of the synchronous particle must therefore be $\phi_{0}=0$.
$\Phi_{1} \quad$ - The particle $B$ is accelerated

- Below transition, an energy increase means an increase in revolution frequency
- The particle arrives earlier - tends toward $\phi_{0}$

- The particle is decelerated
- decrease in energy - decrease in revolution frequency
- The particle arrives later - tends toward $\phi_{0}$


## Synchrotron oscillations



Particle B performs Synchrotron Oscillations around synchronous particle $A$.

The amplitude depends on the initial phase and energy.
The oscillation frequency is much slower than in the transverse plane.
It takes a large number of revolutions for one complete oscillation.
The restoring electric force is smaller than the magnetic force.

- proton synchrotrons of the order of 1000 turns
- electron storage rings of the order of $\sim 10$ turns


## The Potential Well



Cavity voltage

## Potential well

## Longitudinal phase space

The energy - phase oscillations can be drawn in phase space. Similar to transverse, but here it's TIME and ENERGY!


The particle trajectory in the phase space ( $\Delta p / p, \phi$ ) describes its longitudinal motion.


Emittance: phase space area including all the particles

NB: if the emittance contour correspond to a possible orbit in phase space, its shape does no $\dagger$ change with time (matched beam)

## Longitudinal Phase Space Motion

Particle B oscillates around particle A in a synchrotron oscillation.
Plotting this motion in longitudinal phase space (time, energy) gives:


## Synchrotron oscillations - No acceleration



## Synchrotron motion in phase space

$\Delta \mathrm{E}-\phi$ phase space of a stationary bucke $\dagger$ (when there is no acceleration)

Dynamics of a particle
Non-linear, conservative oscillator $\rightarrow$ e.g. pendulum

Particle inside the separatrix:

Particle at the unstable fix-point

Bucket area: area enclosed by the separatrix The area covered by particles is the longitudinal emittance

Particle outside the separatrix:

## (Stationary) Bunch \& Bucket

The bunches of the beam fill usually a part of the bucket area.


Bucket area = longitudinal Acceptance [eVs]
Bunch area $=$ longitudinal beam emittance $=4 \pi \sigma_{E} \sigma_{\dagger}[\mathrm{eVs}]$
Attention: Different definitions are used!

## Synchrotron motion in phase space

The restoring force is non-linear.
$\Rightarrow$ speed of motion depends on position in phase-space
(here shown for a stationary bucket)


## Synchrotron oscillations (with acceleration)

Case with acceleration B increasing

$$
\gamma<\gamma_{t}
$$



Phase space picture

$$
\phi_{s}<\phi<\pi-\phi_{s}
$$



## RF Acceptance versus Synchronous Phase



The areas of stable motion (closed trajectories) are called "BUCKET". The number of circulating buckets is equal to " $h$ ".

The phase extension of the bucket is maximum for $\phi_{s}=180^{\circ}$ (or $0^{\circ}$ ) which means no acceleration.

During acceleration, the buckets get smaller, both in length and energy acceptance.
=> Injection preferably without acceleration.

## Longitudinal Dynamics in Synchrotrons

Now we will look more quantitatively at the "synchrotron motion".
The RF acceleration process clearly emphasizes two coupled variables, the energy gained by the particle and the RF phase experienced by the same particle.
Since there is a well defined synchronous particle which has always the same phase $\phi_{s}$, and the nominal energy $E_{s}$, it is sufficient to follow other particles with respect to that particle.
So let's introduce the following reduced variables:
revolution frequency:
particle RF phase :
particle momentum
particle energy
azimuth angle

$$
\begin{aligned}
& \Delta f_{r}=f_{r}-f_{r s} \\
& \Delta \phi=\phi-\phi_{s} \\
& \Delta p=p-p_{s} \\
& \Delta E=E-E_{s} \\
& \Delta \theta=\theta-\theta_{s}
\end{aligned}
$$

Look at difference from synchronous particle

## Equations of Longitudinal Motion

In these reduced variables, the equations of motion are (see Appendix):

$$
\begin{gathered}
\frac{\Delta E}{\omega_{0}}=\frac{p_{0} R}{h \eta \omega_{0}} \dot{\phi} \quad 2 \pi \frac{d}{d t}\left(\frac{\Delta E}{\omega_{0}}\right)=e \hat{V}\left(\sin \phi-\sin \phi_{s}\right) \\
\frac{d}{d t}\left[\frac{-p_{0} R}{h \eta \omega_{0}} \frac{d \phi}{d t}\right]+\frac{e \hat{V}}{2 \pi}\left(\sin \phi-\sin \phi_{s}\right)=0
\end{gathered}
$$

This second order equation is non linear. Moreover the parameters within the bracket are in general slowly varying with time.
We will study some cases in the following...

## Small Amplitude Oscillations

Let's assume constant parameters $R, p_{0}, \omega_{0}$ and $\eta$ :

$$
\ddot{\phi}+\frac{\Omega_{s}^{2}}{\cos \phi_{s}}\left(\sin \phi-\sin \phi_{s}\right)=0 \quad \text { with } \quad \Omega_{s}^{2}=\frac{-q \hat{V}_{R F} \eta h \omega_{0}}{2 \pi R p_{0}} \cos \phi_{s}
$$

Consider now small phase deviations from the reference particle:

$$
\sin \phi-\sin \phi_{s}=\sin \left(\phi_{s}+\Delta \phi\right)-\sin \phi_{s} \cong \cos \phi_{s} \Delta \phi
$$

and the corresponding linearized motion reduces to a harmonic oscillation:
${ }^{\bullet}+{ }_{s}^{2}=0$ where $\Omega_{s}$ is the synchrotron angular frequency.
The synchrotron tune $v_{s}$ is the number of synchrotron oscillations per revolution:

$$
v_{s}=\Omega_{s} / \omega_{0}
$$

Typical values are <<1, as it takes several 10-1000 turns per oscillation.

- proton synchrotrons of the order $10^{-3}$
- electron storage rings of the order $10^{-1}$


## Stability condition for $\phi_{s}$

$$
\Omega_{s}^{2}=\frac{-q \widehat{V}_{R F} \eta h \omega_{0}}{2 \pi R p_{0}} \cos \phi_{s} \Leftrightarrow \Omega_{s}^{2}=\omega_{0}^{2} \frac{-q \widehat{V}_{R F} \eta h}{2 \pi \beta^{2} E} \cos \phi_{s} \quad \begin{aligned}
& \text { with } \\
& R p=\frac{\beta^{2} E}{\omega}
\end{aligned}
$$

Stability is obtained when $\Omega_{s}$ is real and so $\Omega_{s}{ }^{2}$ positive:

$$
\begin{gathered}
\Omega_{s}^{2}>0 \\
\mathbb{\Downarrow} \\
\eta \cos \phi_{s}<0
\end{gathered}
$$

Stable in the region if


## Energy Acceptance

From the equation of the separatrix, we can calculate (see appendix) the acceptance in energy:

$$
\begin{aligned}
& \left(\frac{\Delta E}{E_{S}}\right)_{\max }= \pm \beta \sqrt{\frac{-q \widehat{V}}{\pi h \eta E_{s}} G\left(\phi_{s}\right)} \\
& G\left({ }_{s}\right)=2 \cos s_{s}+\left(2_{s}\right) \sin { }_{s}
\end{aligned}
$$



This "RF acceptance" depends strongly on $\phi_{s}$ and plays an important role for the capture at injection, and the stored beam lifetime.
It's largest for $\phi_{s}=0$ and $\phi_{s}=\pi$ (no acceleration, depending on $\eta$ ).
It becomes smaller during acceleration, when $\phi_{s}$ is changing
Need a higher RF voltage for higher acceptance => need more $\$ €:$

## RF Acceptance versus Synchronous Phase



The areas of stable motion (closed trajectories) are called "BUCKET". The number of circulating buckets is equal to " $h$ ".

The phase extension of the bucket is maximum for $\phi_{s}=180^{\circ}$ (or $0^{\circ}$ ) which means no acceleration.

During acceleration, the buckets get smaller, both in length and energy acceptance.
=> Injection preferably without acceleration.

## Injection: Bunch-to-bucket transfer

- Bunch from sending accelerator into the bucket of receiving


Advantages:

$\rightarrow$ Particles always subject to longitudinal focusing
$\rightarrow$ No need for RF capture of de-bunched beam in receiving accelerator
$\rightarrow$ No particles at unstable fixed point
$\rightarrow$ Time structure of beam preserved during transfer

## Effect of a Mismatch

Injected bunch: short length and large energy spread after $1 / 4$ synchrotron period: longer bunch with a smaller energy spread.


For larger amplitudes, the angular phase space motion is slower
( $1 / 8$ period shown below) $\Rightarrow$ can lead to filamentation and emittance growth

restoring force is non-linear

stationary bucket
CAS Basic, 9-13 May 2022

accelerating bucket

## Effect of a Mismatch (2)

- Long. emittance is only preserved for correct RF voltage

$\rightarrow$ Bunch is fine, longitudinal emittance remains constant


## Effect of a Mismatch (3)

Evolution of an injected beam for the first 100 turns.
For a mismatched transfer, the emittance increases (right).

matched beam

mismatched beam - phase error

## Bunch Rotation

Phase space motion can be used to make short bunches.
Start with a long bunch and extract or recapture when it's short.


initial beam

## Capture of a Debunched Beam with Fast Turn-On



CAS Basic, 9-13 May 2022

## Capture of a Debunched Beam with Adiabatic Turn-On





## Generating a 25 ns Bunch Train in the PS

- Longitudinal bunch splitting (basic principle)
- Reduce voltage on principal RF harmonic and simultaneously rise voltage on multiple harmonics (adiabatically with correct phase, etc.)


Use double splitting at 25 GeV to generate 50ns bunch trains instead CAS Basic, 9-13 May 2022

## Production of the LHC 25 ns beam

## 1. Inject four bunches $\sim 180 \mathrm{~ns}, 1.3 \mathrm{eVs}$



Wait 1.2 s for second injection
2. Inject two bunches


$$
\sim 0.7 \text { eVs }
$$

4. Accelerate from $1.4 \mathrm{GeV}\left(\mathrm{E}_{\text {kin }}\right)$ to 26 GeV

## Production of the LHC 25 ns beam

5. During acceleration: longitudinal emittance blow-up: 0.7 - 1.3 eVs

6. Fine synchronization, bunch rotation $\rightarrow$ Extraction!

## The LHC25 (ns) cycle in the PS




$\rightarrow$ Each bunch from the Booster divided by $12 \rightarrow 6 \times 3 \times 2 \times 2=72$

## Triple splitting in the PS




## Two times double splitting in the PS

Two times double splitting and bunch rotation:



- Bunch is divided twice using RF systems at $h=21 / 42(10 / 20 \mathrm{MHz})$ and $h=42 / 84(20 / 40 \mathrm{MHz})$
- Rotation: first part h84 only +h168 (80 MHz) for final part


## Summary

- Cyclotrons/Synchrocylotrons for low energy
- Synchrotrons for high energies, constant orbit, rising field and frequency synchronously
- Particles with higher energy have a longer orbit (normally) but a higher velocity
- at low energies (below transition) velocity increase dominates
- at high energies (above transition) velocity almost constant
- Particles perform oscillations around synchronous phase
- synchronous phase depending on acceleration
- below or above transition
- Bucket is the stable region in phase space inside the separatrix
- Bunch is the area filled with beam
- Matching the shape of the bunch to the bucket is essential


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## Appendix

- Summary Relativity and Energy Gain
- Velocity, Energy, and Momentum
- Momentum compaction factor
- Synchrotron energy-phase oscillations
- Stability condition
- Separatrix stationary bucket
- Large amplitude oscillations
- Bunch matching into stationary bucket


## Appendix: Relativity + Energy Gain

Newton-Lorentz Force $\vec{F}=\frac{\mathrm{d} \vec{p}}{\mathrm{dt}}=e\left(\begin{array}{ll}\vec{E}+\vec{v} & \vec{B}\end{array}\right)$
$2^{\text {nd }}$ term always perpendicular to motion $=>$ no acceleration

## Relativistics Dynamics

$\beta=\frac{v}{c}=\sqrt{1-\frac{1}{\gamma^{2}}} \quad=\frac{E}{E_{0}}=\frac{m}{m_{0}}=\frac{1}{\sqrt{12^{2}}}$
$p=m v=\frac{E}{c^{2}} \quad c=\frac{E}{c}=\quad m_{0} c$
$E^{2}=E_{0}^{2}+p^{2} c^{2} \longrightarrow \quad d E=v d p$
$\frac{d E}{d z}=v \frac{d p}{d z}=\frac{d p}{d t}=e E_{z}$
$d E=d W=e E_{z} d z \quad \rightarrow W=e \quad E_{z} d z$

## RF Acceleration

$$
E_{z}=\hat{E}_{z} \sin { }_{R F} t=\hat{E}_{z} \sin (t)
$$

$$
\hat{E}_{z} d z=\hat{V}
$$

$$
W=e \hat{V} \sin \phi
$$

(neglecting transit time factor)
The field will change during the passage of the particle through the cavity
=> effective energy gain is lower

## Appendix: Velocity, Energy and Momentum

normalized velocity $\beta=\frac{v}{c}=\sqrt{1-\frac{1}{\gamma^{2}}}$
=> electrons almost reach the speed of light very quickly (few MeV range)
total energy

$$
E=m_{0} c^{2}
$$

rest energy

$$
\gamma=\frac{E}{E_{0}}=\frac{m}{m_{0}}=\frac{1}{\sqrt{1-v^{2} / c^{2}}}=\frac{1}{\sqrt{1-\beta^{2}}}
$$

Momentum $\quad p=m v=\frac{E}{c^{2}} \quad c=\frac{E}{c}=\quad m_{0} c$
=> Magnetic field needs to follow the momentum increase



## Appendix: Momentum Compaction Factor

$$
\begin{array}{ll}
\alpha_{c}=\frac{p}{L} \frac{d L}{d p} & d s_{0}=d \\
d s=(+x) d
\end{array}
$$

The elementary path difference from the two orbits is: definition of dispersion $D_{x}$

$$
\frac{d l}{d s_{0}}=\frac{d s \quad d s_{0}}{d s_{0}}=\frac{x}{\square} \frac{D_{x}}{=} \frac{d p}{p}
$$

leading to the total change in the circumference:

$$
\begin{aligned}
& d L=d l=\frac{x}{c} d s_{0}=\frac{D_{x}}{} \frac{d p}{p} d s_{0} \\
& \alpha_{c}=\frac{1}{L} \int_{C} \frac{D_{x}(s)}{\rho(s)} d s_{0} \begin{array}{l}
\text { With p=o in } \\
\text { straight sections } \\
\text { we get: }
\end{array} \alpha_{c}=\frac{\left\langle D_{x}\right\rangle_{m}}{R}
\end{aligned}
$$

$\left\langle>_{m}\right.$ means that the average is considered over the bending magnet only

## Appendix: First Energy-Phase Equation



For a given particle with respect to the reference one:

$$
\Delta \omega .=\frac{d}{d t}(\Delta \theta)=-\frac{1}{h} \frac{d}{d t}(\Delta \phi)=-\frac{1}{h} \frac{d \phi}{d t}
$$

Since: $\quad \eta=-\frac{p_{0}}{\omega_{0}}\left(\frac{d \omega}{d p}\right)_{s} \quad$ and $\quad \begin{gathered}E^{2}=E_{0}^{2}+p^{2} c^{2} \\ \Delta E=v_{s} \Delta p=\omega_{0} R \Delta p\end{gathered}$
one gets: $\quad \frac{\Delta E}{\omega_{0}}=\frac{p_{0} R}{h \eta \omega_{0}} \frac{d(\Delta \phi)}{d t}=\frac{p_{0} R}{h \eta \omega_{0}} \dot{\phi}$

## Appendix: Second Energy-Phase Equation

The rate of energy gained by a particle is: $\quad \frac{d E}{d t}=e \hat{V} \sin \phi \frac{\omega_{r}}{2 \pi}$
The rate of relative energy gain with respect to the reference particle is then:

$$
2 \pi \Delta\left(\frac{\dot{E}}{\omega_{0}}\right)=e \hat{V}\left(\sin \phi-\sin \phi_{s}\right)
$$

Expanding the left-hand side to first order:

$$
\left(\dot{E} T_{r}\right) \quad \dot{E} \quad T_{r}+T_{r s} \quad \dot{E}=E \dot{T}_{r}+T_{r s} \quad \dot{E}=\frac{d}{d t}\left(T_{r s} \quad E\right)
$$

leads to the second energy-phase equation:

$$
2 \pi \frac{d}{d t}\left(\frac{\Delta E}{\omega_{0}}\right)=e \hat{V}\left(\sin \phi-\sin \phi_{s}\right)
$$

## Appendix: Stability condition for $\phi_{s}$

$$
\Omega_{s}^{2}=\frac{-q \widehat{V}_{R F} \eta h \omega_{0}}{2 \pi R p_{0}} \cos \phi_{s} \Leftrightarrow \Omega_{s}^{2}=\omega_{0}^{2} \frac{-q \widehat{V}_{R F} \eta h}{2 \pi \beta^{2} E} \cos \phi_{s} \quad \begin{aligned}
& \text { with } \\
& R p=\frac{\beta^{2} E}{\omega}
\end{aligned}
$$

Stability is obtained when $\Omega_{s}$ is real and so $\Omega_{s}{ }^{2}$ positive:

$$
\begin{gathered}
\Omega_{s}^{2}>0 \\
\mathbb{I} \\
\eta \cos \phi_{s}<0
\end{gathered}
$$

Stable in the region if


## Appendix: Stationary Bucket - Separatrix

This is the case $\sin \phi_{s}=0$ (no acceleration) which means $\phi_{s}=0$ or $\pi$. The equation of the separatrix for $\phi_{s}=\pi$ (above transition) becomes:

$$
\frac{\dot{\phi}^{2}}{2}+\Omega_{s}^{2} \cos \phi=\Omega_{s}^{2}
$$

$$
\frac{\dot{\phi}^{2}}{2}=2 \Omega_{s}^{2} \sin ^{2} \frac{\phi}{2}
$$

Replacing the phase derivative by the (canonical) variable W:


## Stationary Bucket (2)

Setting $\phi=\pi$ in the previous equation gives the height of the bucket:

$$
W_{b k}=\frac{C}{h c} \sqrt{\frac{e \hat{V} E_{s}}{2 h}}
$$

This results in the maximum energy acceptance:

$$
E_{\max }={ }_{r f} W_{b k}=s \sqrt{2 \frac{e \hat{V}_{R F} E_{s}}{h}}
$$

The area of the bucket is: $\quad A_{b k}=2 \int_{0}^{2 \pi} W d \phi$
Since: $\quad \int_{0}^{2 \pi} \sin \frac{\phi}{2} d \phi=4$
one gets: $\quad A_{b k}=8 W_{b k}=8 \frac{C}{h c} \sqrt{\frac{e \hat{V} E_{s}}{2 h}} \quad \longrightarrow \quad W_{b k}=\frac{A_{b k}}{8}$

## Appendix: Large Amplitude Oscillations

For larger phase (or energy) deviations from the reference the second order differential equation is non-linear:

$$
\ddot{\phi}+\frac{\Omega_{s}^{2}}{\cos \phi_{s}}\left(\sin \phi-\sin \phi_{s}\right)=0 \quad\left(\Omega_{s} \text { as previously defined }\right)
$$

Multiplying by $\dot{\phi}$ and integrating gives an invariant of the motion:

$$
\frac{\dot{\phi}^{2}}{2}-\frac{\Omega_{s}^{2}}{\cos \phi_{s}}\left(\cos \phi+\phi \sin \phi_{s}\right)=I
$$

which for small amplitudes reduces to:

(the variable is $\Delta \phi$, and $\phi_{s}$ is constant)

Similar equations exist for the second variable : $\Delta \mathrm{E} \propto \mathrm{d} \phi / \mathrm{d} \dagger$

## Large Amplitude Oscillations (2)

When $\phi$ reaches $\pi-\phi_{s}$ the force goes to zero and beyond it becomes non restoring.
Hence $\pi-\phi_{s}$ is an extreme amplitude for a stable motion which in the phase space ( -, ) is shown as closed trajectories.

Equation of the separatrix:


$$
\frac{\dot{\phi}^{2}}{2}-\frac{\Omega_{s}^{2}}{\cos \phi_{s}}\left(\cos \phi+\phi \sin \phi_{s}\right)=-\frac{\Omega_{s}^{2}}{\cos \phi_{s}}\left(\cos \left(\pi-\phi_{s}\right)+\left(\pi-\phi_{s}\right) \sin \phi_{s}\right)
$$

Second value $\phi_{m}$ where the separatrix crosses the horizontal axis:

$$
\cos \phi_{m}+\phi_{m} \sin \phi_{s}=\cos \left(\pi-\phi_{s}\right)+\left(\pi-\phi_{s}\right) \sin \phi_{s}
$$

## Energy Acceptance

From the equation of motion it is seen that $\dot{\phi}$ reaches an extreme when $\ddot{\phi}=0$, hence corresponding to $\phi=\phi_{s}$.
Introducing this value into the equation of the separatrix gives:

$$
\cdot_{\max }^{2}=2{\underset{s}{2}\left\{2+\left(2_{s}\right) \tan { }_{s}\right\}, ~}_{\text {a }}
$$

That translates into an acceptance in energy:

$$
\begin{gathered}
\left(\frac{\Delta E}{E_{s}}\right)_{\max }= \pm \beta \sqrt{\frac{e \widehat{V}}{\pi h \eta E_{S}} G\left(\phi_{s}\right)} \\
G\left({ }_{s}\right)=2 \cos { }_{s}+\left(2_{s}\right) \sin { }_{s}
\end{gathered}
$$

This "RF acceptance" depends strongly on $\phi_{s}$ and plays an important role for the capture at injection, and the stored beam lifetime.
It's largest for $\phi_{s}=0$ and $\phi_{s}=\pi$ (no acceleration, depending on $\eta$ ).
Need a higher RF voltage for higher acceptance.

## Bunch Matching into a Stationary Bucket

A particle trajectory inside the separatrix is described by the equation:

$$
\frac{\dot{\phi}^{2}}{2}-\frac{\Omega_{s}^{2}}{\cos \phi_{s}}\left(\cos \phi+\phi \sin \phi_{s}\right)=I \quad \xrightarrow{\phi_{s}=\pi} \quad \frac{\dot{\phi}^{2}}{2}+\Omega_{s}^{2} \cos \phi=I
$$



The points where the trajectory crosses the axis are symmetric with respect to $\phi_{s}=\pi$

$$
\begin{array}{r}
\frac{\dot{\phi}^{2}}{2}+\Omega_{s}^{2} \cos \phi=\Omega_{s}^{2} \cos \phi_{m} \\
\dot{\phi}= \pm \Omega_{s} \sqrt{2\left(\cos \phi_{m}-\cos \phi\right)} \\
W= \pm W_{b k} \sqrt{\cos ^{2} \frac{m}{2} \cos ^{2} \frac{2}{2}} \\
\cos ()=2 \cos ^{2} \frac{1}{2}
\end{array}
$$

## Bunch Matching into a Stationary Bucket (2)

Setting $\phi=\pi$ in the previous formula allows to calculate the bunch height:

$$
\begin{gathered}
W_{b}=W_{b k} \cos \frac{m}{2}=W_{b k} \sin \frac{\hat{2}}{2} \quad \text { or: } \quad W_{b}=\frac{A_{b k}}{8} \cos \frac{\phi_{m}}{2} \\
\longrightarrow\left(\frac{E}{E_{s}}\right)_{b}=\left(\frac{E}{E_{s}}\right)_{R F} \cos \frac{m}{2}=\left(\frac{E}{E_{s}}\right)_{R F} \sin \frac{\hat{2}}{2}
\end{gathered}
$$

This formula shows that for a given bunch energy spread the proper matching of a shorter bunch ( $\phi_{m}$ close to $\pi$, " small) will require a bigger RF acceptance, hence a higher voltage

For small oscillation amplitudes the equation of the ellipse reduces to:

$$
W=\frac{A_{b k}}{16} \sqrt{\wedge^{2}(\quad)^{2}} \longrightarrow\left(\frac{16 W}{A_{b k}}\right)^{2}+\left(\overline{{ }^{2}}\right)^{2}=1
$$

Ellipse area is called longitudinal emittance

$$
A_{b}=\frac{-}{16} A_{b k}{ }^{\wedge}
$$


[^0]:    Animation: https://phyanim.sciences.univ-nantes.fr/Meca/Charges/cyclotron.php

