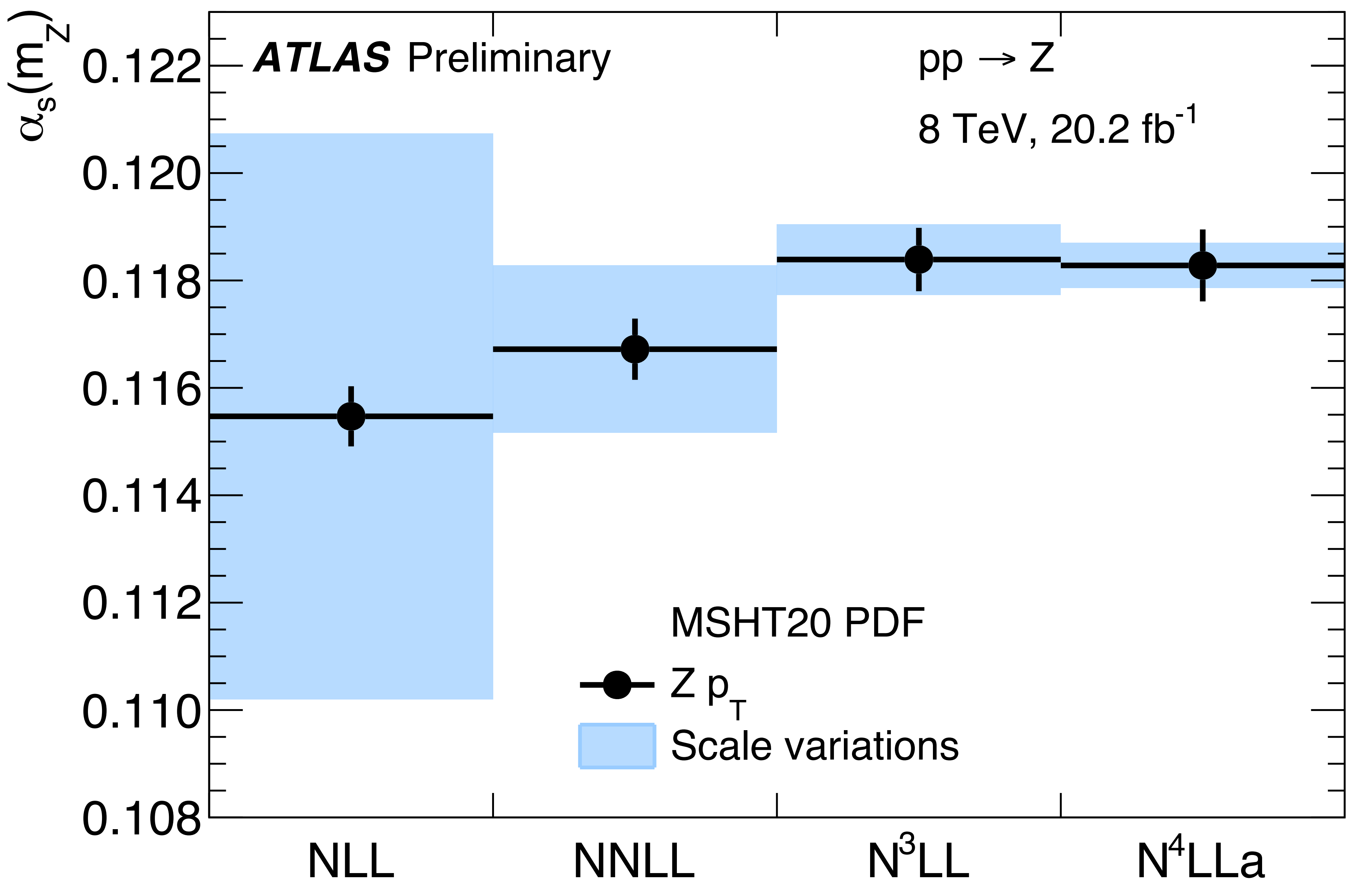


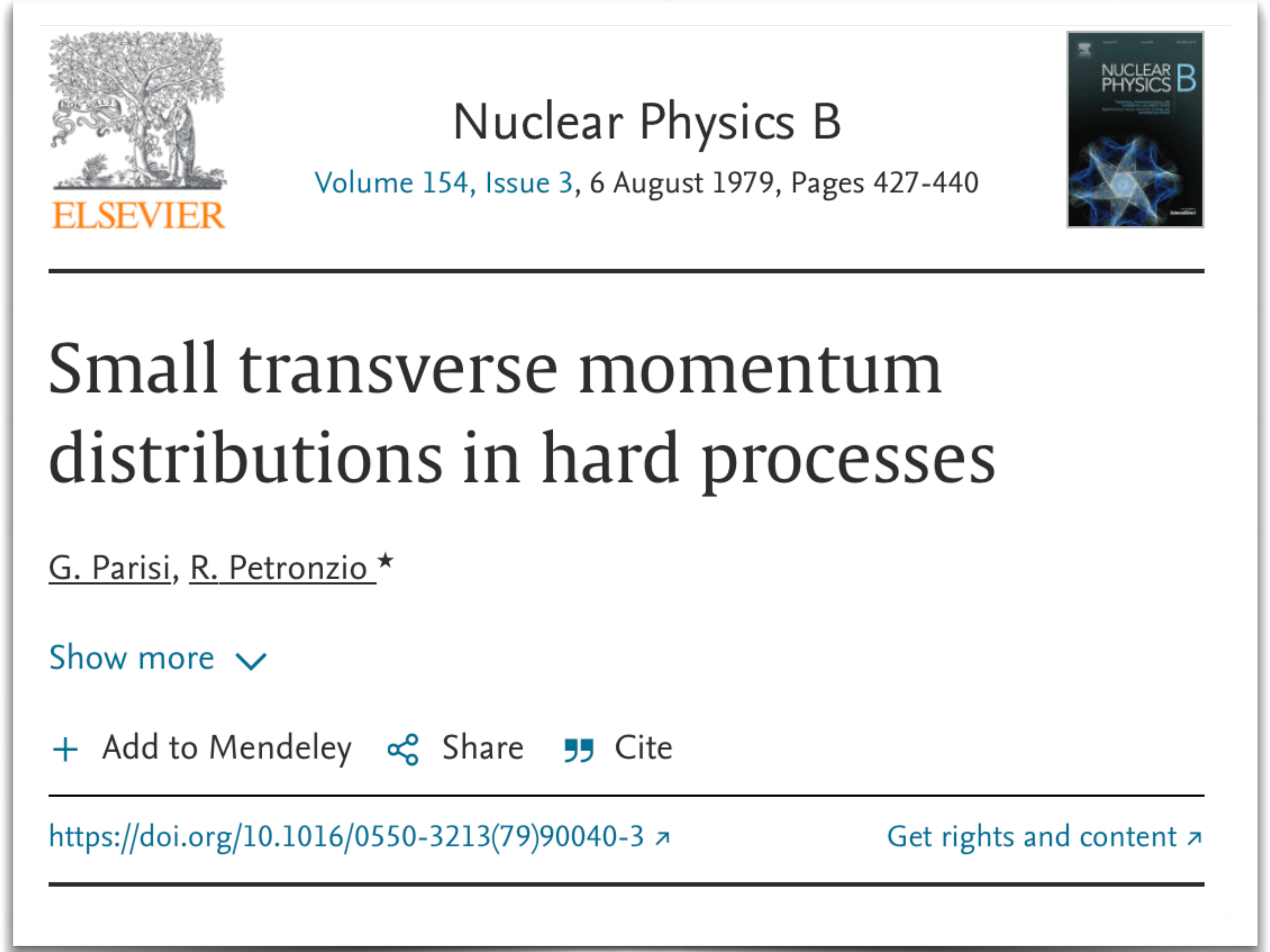
UNIVERSITÄT BERN

AEC
ALBERT EINSTEIN CENTER FOR FUNDAMENTAL PHYSICS

Progress in Resummed Calculations

Thomas Becher
University of Bern

Parton Showers and Resummation (PSR 23),
June 6-8 2023, University of Milano-Bicocca



Nuclear Physics B

Volume 154, Issue 3, 6 August 1979, Pages 427-440

Small transverse momentum distributions in hard processes

G. Parisi, R. Petronzio *

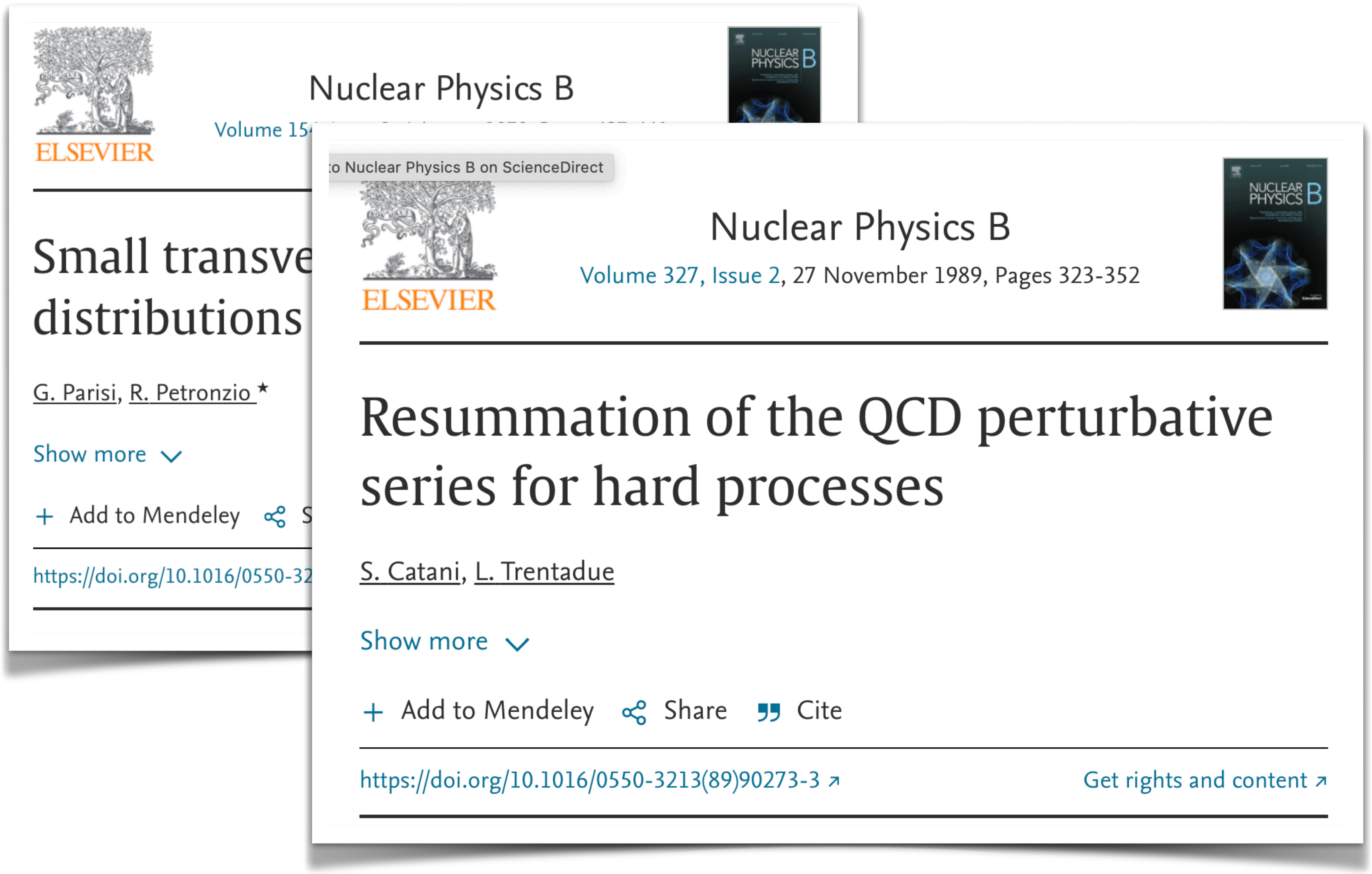
Show more

+ Add to Mendeley Share Cite

https://doi.org/10.1016/0550-3213(79)90040-3 Get rights and content

I will not present a historical introduction to the field (please see Andrea Banfi's Mount Rushmor of resummation from last year!) …

… but let me note that Italy is the motherland of resummation (and Milano its hometown)!


Nuclear Physics B

Volume 15…

Small transve…
distributions

G. Parisi, R. Petronzio *

Show more

+ Add to Mendeley S…

https://doi.org/10.1016/0550-32…

…o Nuclear Physics B on ScienceDirect

Nuclear Physics B

Volume 327, Issue 2, 27 November 1989, Pages 323-352

Resummation of the QCD perturbative series for hard processes

S. Catani, L. Trentadue

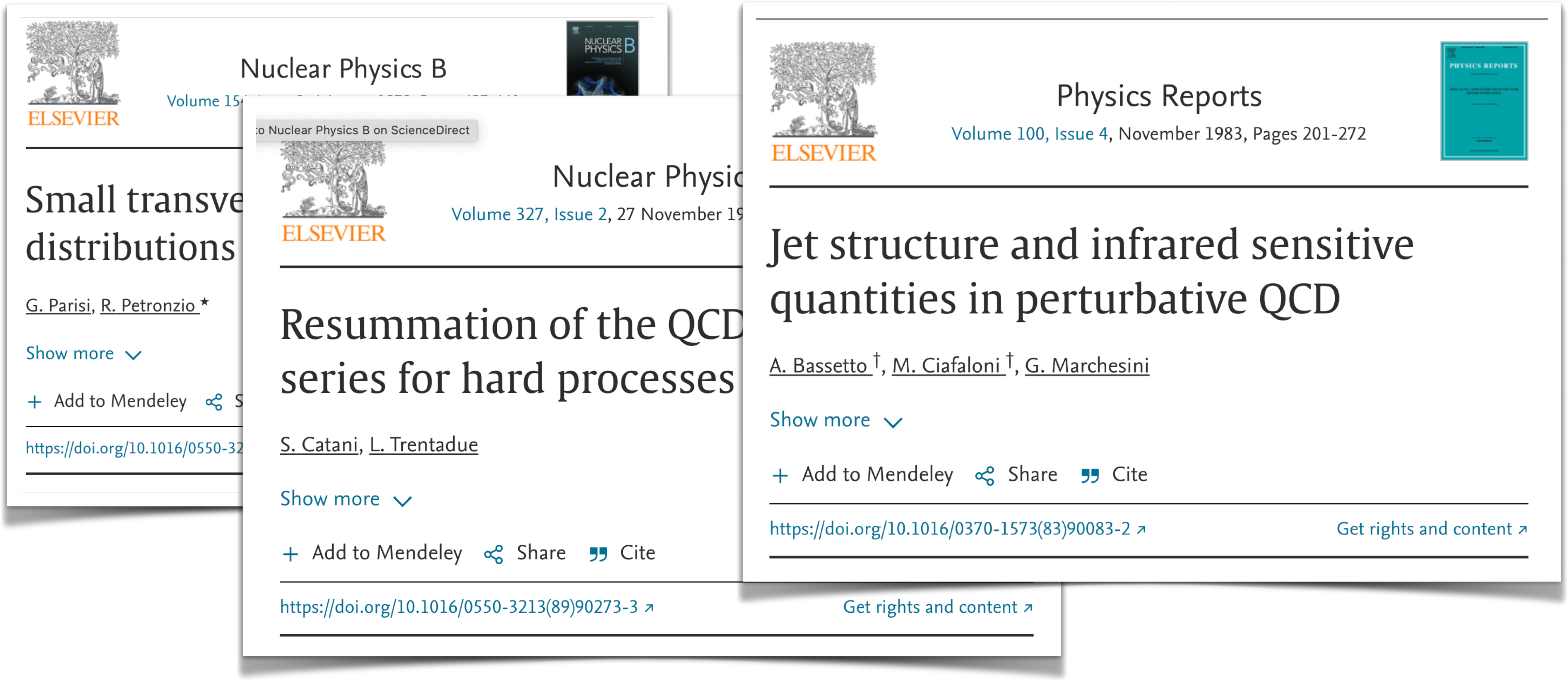
Show more

+ Add to Mendeley Share Cite

https://doi.org/10.1016/0550-3213(89)90273-3 Get rights and content

I will not present a historical introduction to the field (please see Andrea Banfi's Mount Rushmor of resummation from last year!) …

… but let me note that Italy is the motherland of resummation (and Milano its hometown)!



Nuclear Physics B
Volume 15...

Small transve...
distributions

G. Parisi, R. Petronzio *

Show more

+ Add to Mendeley

https://doi.org/10.1016/0550-32...

...o Nuclear Physics B on ScienceDirect

Nuclear Physic...
Volume 327, Issue 2, 27 November 19...

Resummation of the QCD
series for hard processes

S. Catani, L. Trentadue

Show more

+ Add to Mendeley   Share   Cite

https://doi.org/10.1016/0550-3213(89)90273-3   Get rights and content

Physics Reports
Volume 100, Issue 4, November 1983, Pages 201-272

Jet structure and infrared sensitive quantities in perturbative QCD

A. Bassetto †, M. Ciafaloni †, G. Marchesini

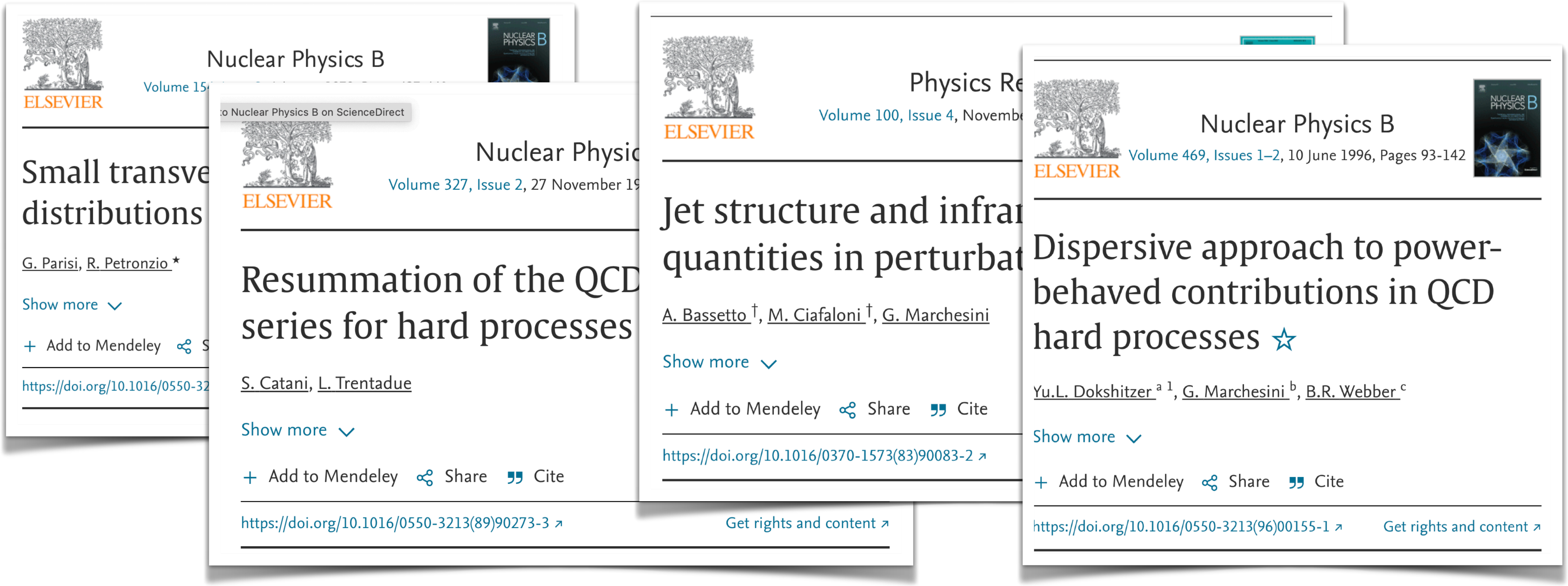
Show more

+ Add to Mendeley   Share   Cite

https://doi.org/10.1016/0370-1573(83)90083-2   Get rights and content

I will not present a historical introduction to the field (please see Andrea Banfi's Mount Rushmor of resummation from last year!) …

… but let me note that Italy is the motherland of resummation (and Milano its hometown)!



ELSEVIER
Nuclear Physics B
Volume 15...
Small transve...
distributions
G. Parisi, R. Petronzio *
Show more
+ Add to Mendeley
https://doi.org/10.1016/0550-32...

...o Nuclear Physics B on ScienceDirect
ELSEVIER
Nuclear Physic...
Volume 327, Issue 2, 27 November 19...
Resummation of the QCD...
series for hard processes
S. Catani, L. Trentadue
Show more
+ Add to Mendeley Share Cite
https://doi.org/10.1016/0550-3213(89)90273-3 ↗ Get rights and content ↗

ELSEVIER
Physics Re...
Volume 100, Issue 4, Novemb...
Jet structure and infra...
quantities in perturbat...
A. Bassetto †, M. Ciafaloni †, G. Marchesini
Show more
+ Add to Mendeley Share Cite
https://doi.org/10.1016/0370-1573(83)90083-2 ↗

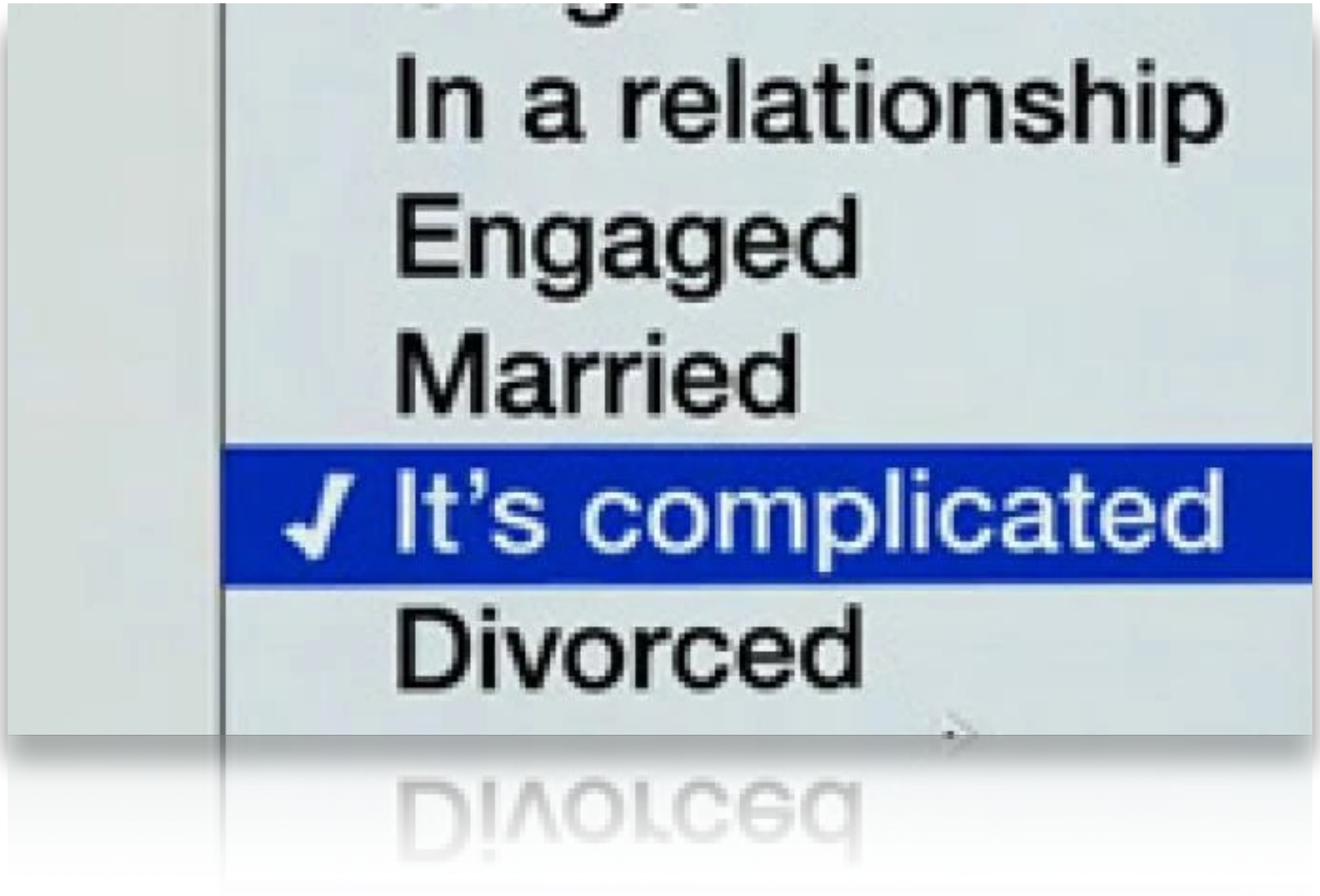
ELSEVIER
Nuclear Physics B
Volume 469, Issues 1–2, 10 June 1996, Pages 93-142
Dispersive approach to power-behaved contributions in QCD hard processes ☆
Yu.L. Dokshitzer [a] [1], G. Marchesini [b], B.R. Webber [c]
Show more
+ Add to Mendeley Share Cite
https://doi.org/10.1016/0550-3213(96)00155-1 ↗ Get rights and content ↗

I will not present a historical introduction to the field (please see Andrea Banfi's Mount Rushmor of resummation from last year!) …

… but let me note that Italy is the motherland of resummation (and Milano its hometown)!

Outline

1) α_s from event shapes in e^+e^-
   - hadronization effects in the three-jet region
   - Sudakov shoulders
2) energy-energy correlators
3) α_s from transverse momentum spectrum in $pp \to Z+X$
4) non-global observables
   - subleading non-global logarithms (NGLs)
   - superleading logarithms (SLLs)

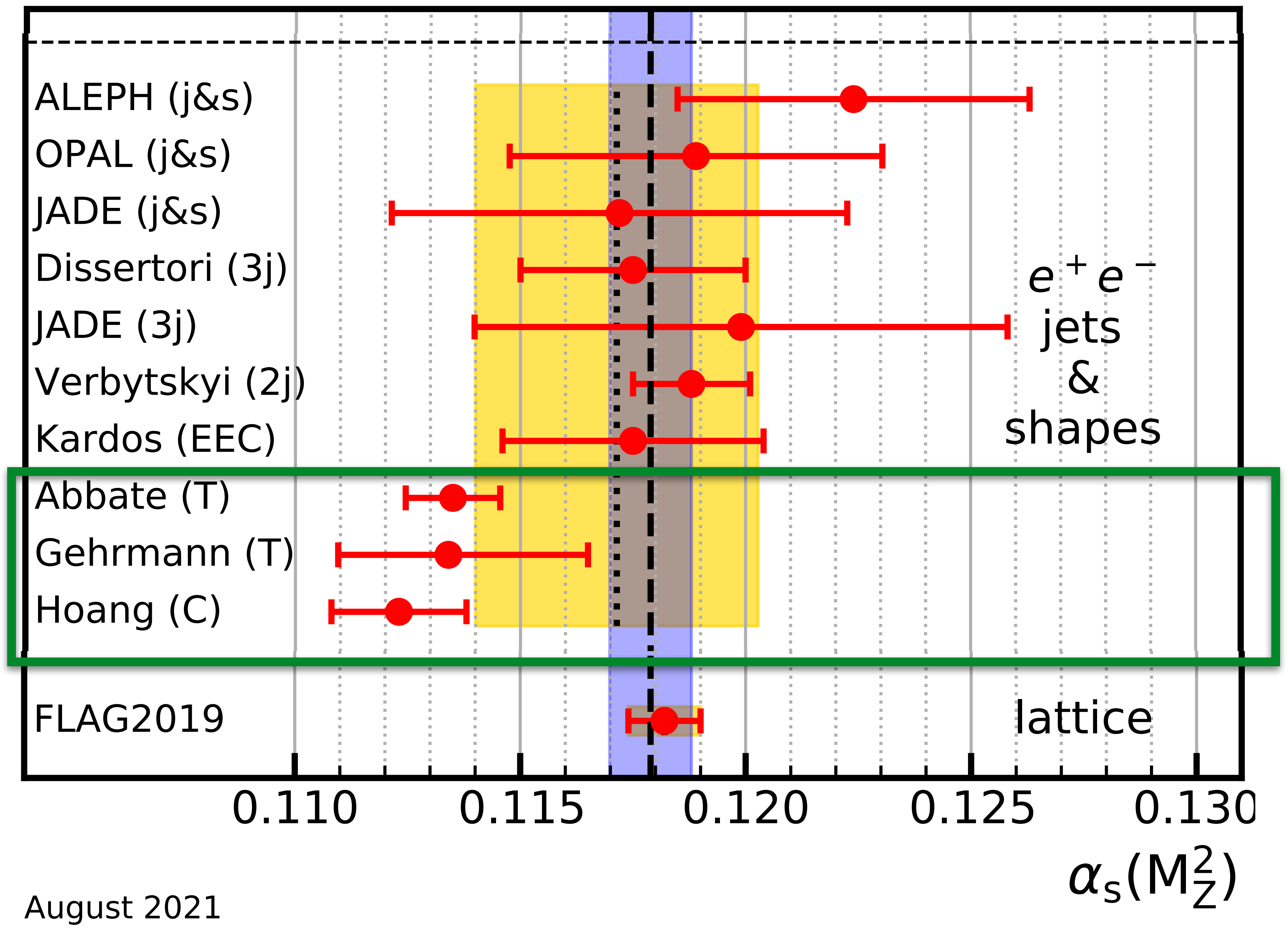


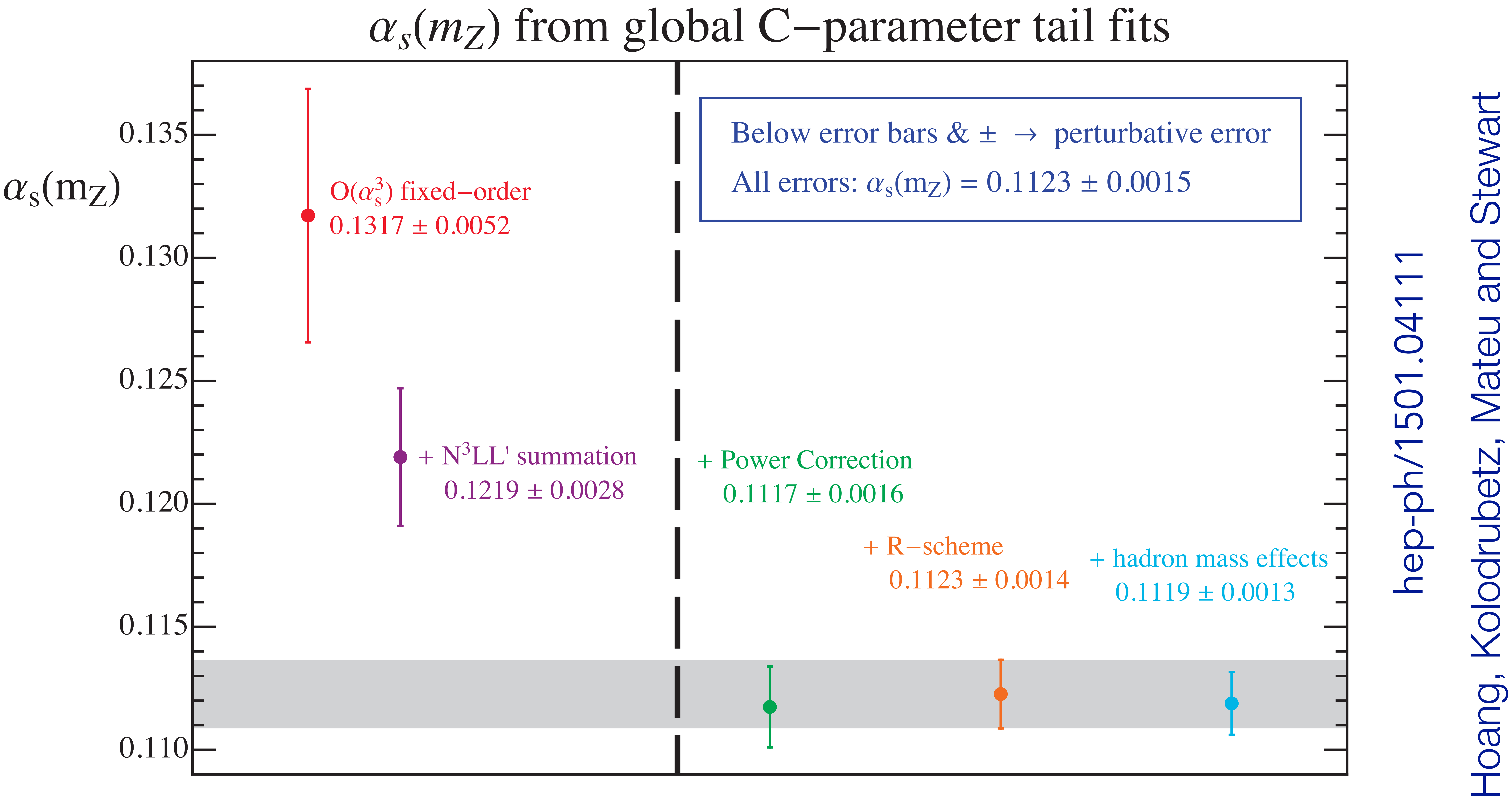
α_s from event shapes in e^+e^-

A long-standing discrepancy

N^{3}LL + NNLO computations with **hadronization corrections determined from fit to data** a precise values of α_s ~4σ lower than world average.

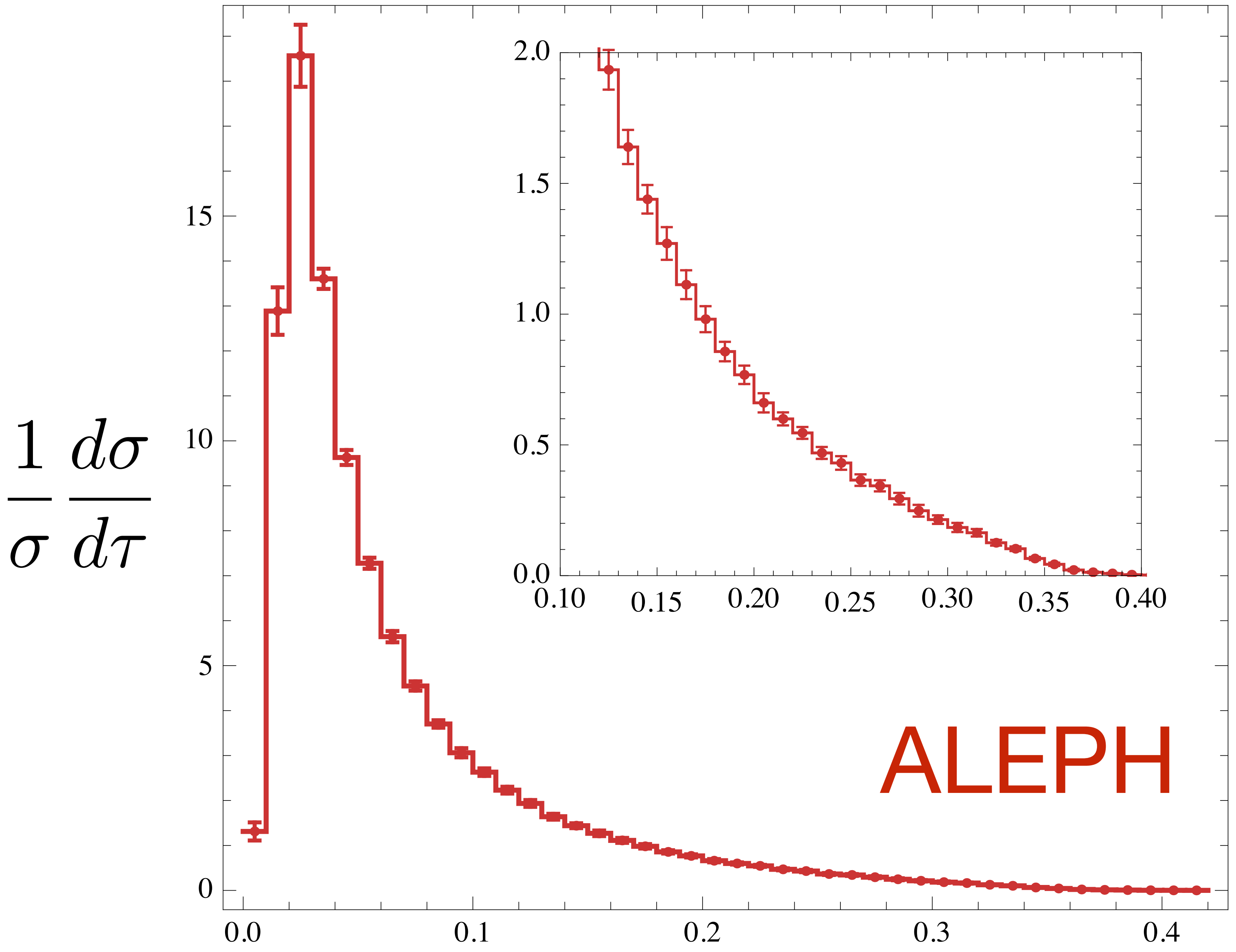
Fitted hadronization corrections are sizable and larger than hadronization models of parton shower MCs.



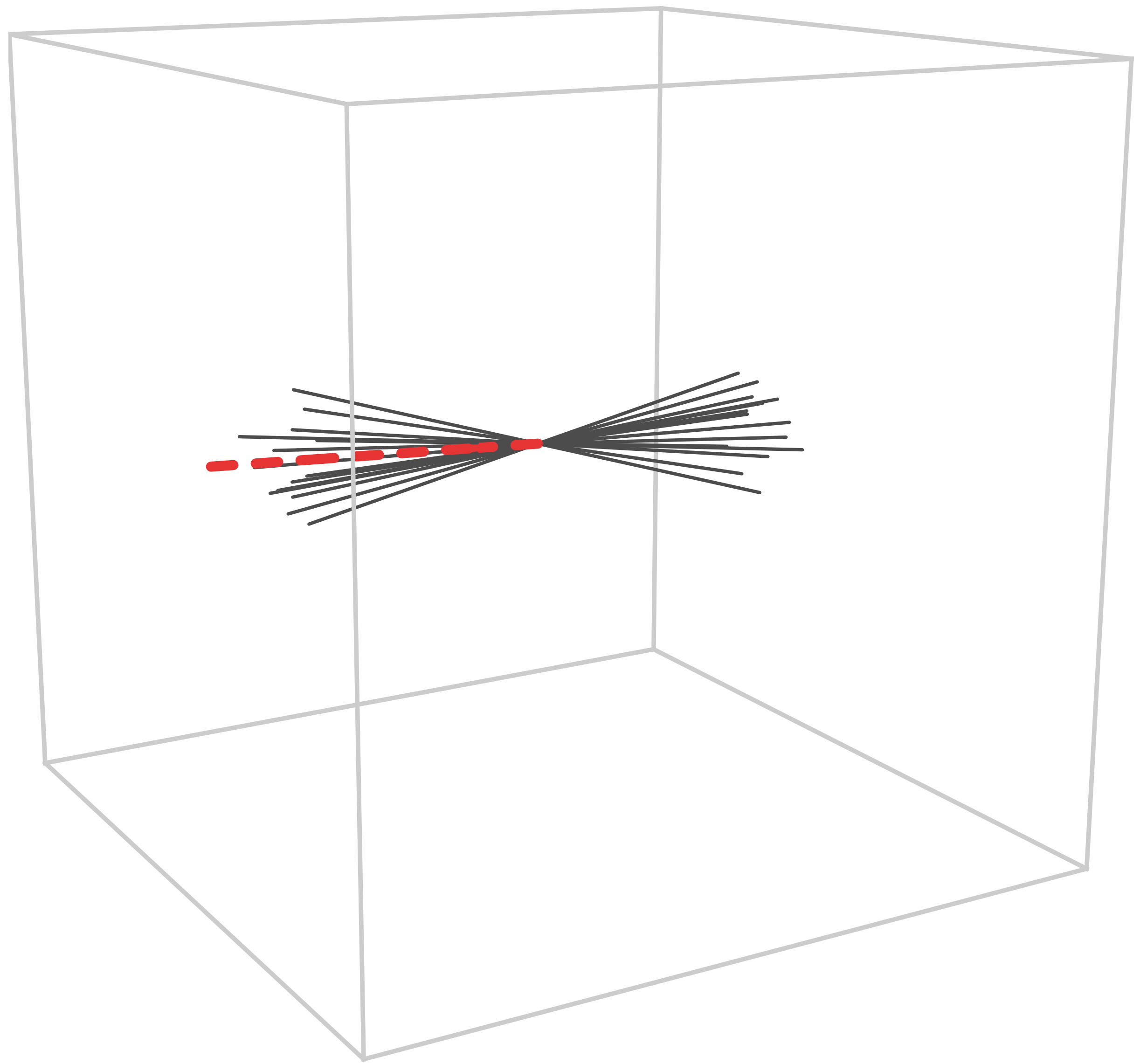


Hadronization effects are large ~ 9% !

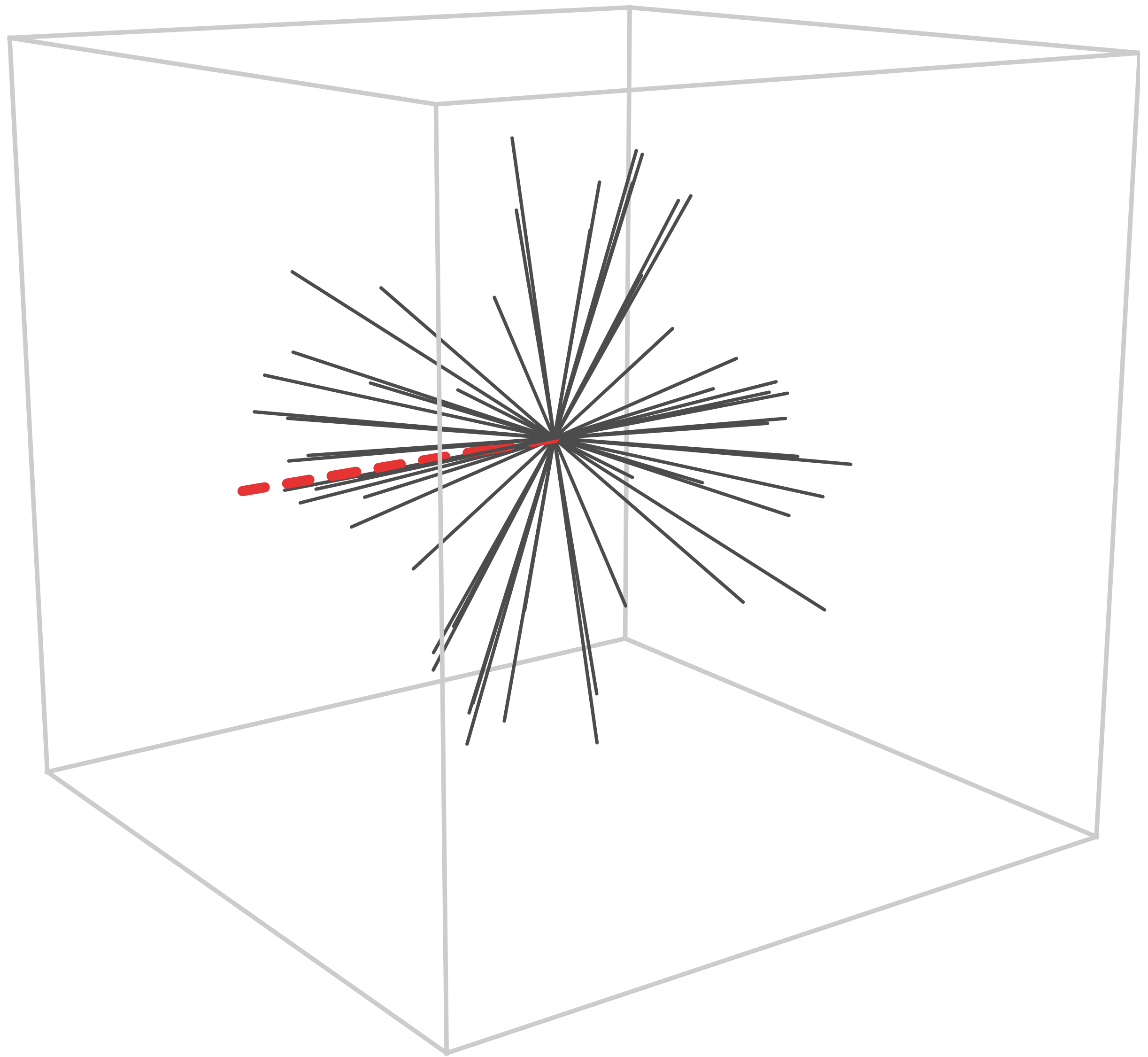
Event-shape distributions



τ

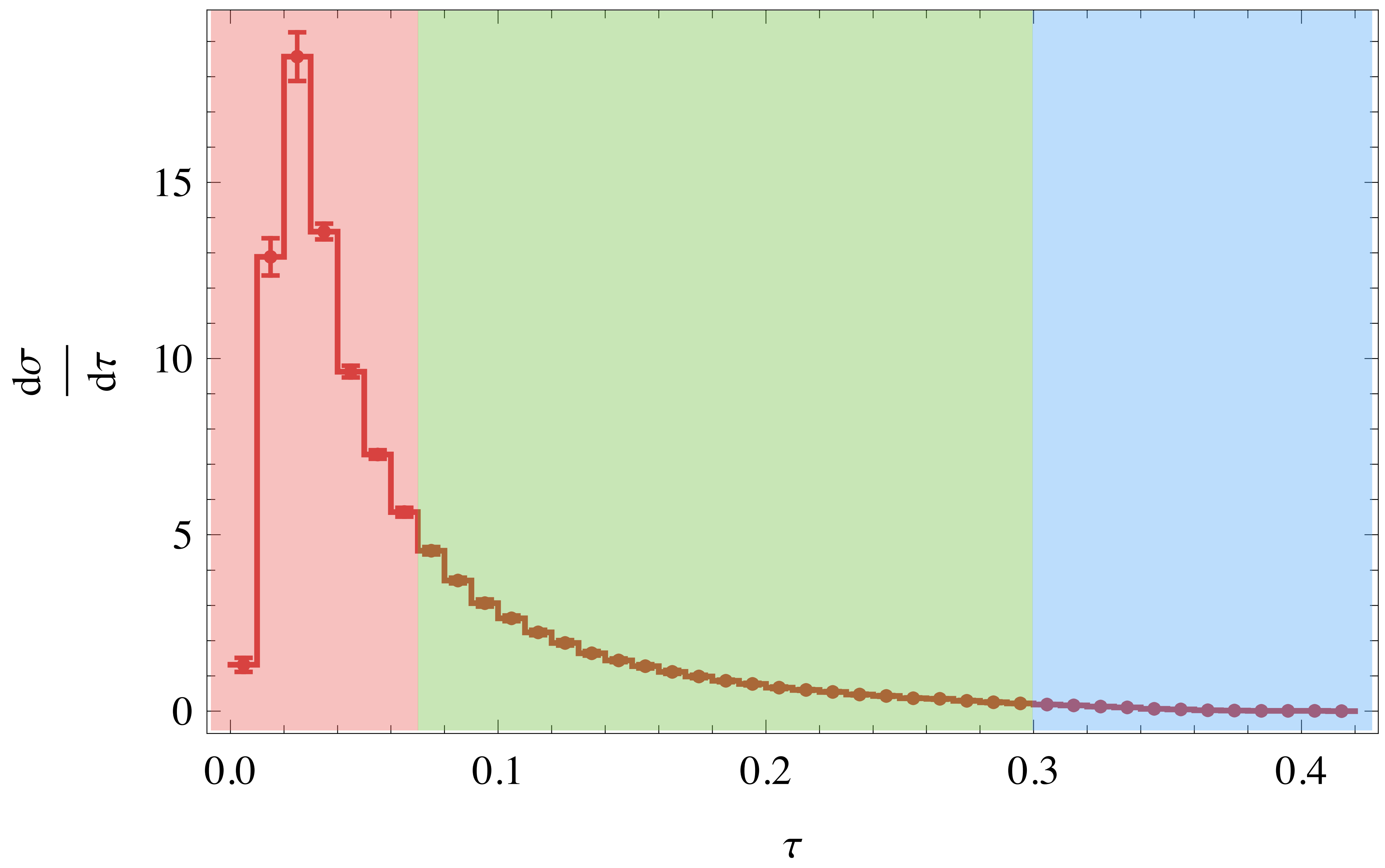


$\tau \approx 0$



$\tau \approx 1/2$

Thrust $T = 1 - \tau$ measures momentum along thrust axis, broadening B transverse momentum, jet masses M_H, M_D measure invariant mass in hemispheres.



- Peak region: strongly affected by hadronisation
- Tail region: used in fit for α_s, resummation + matching + fitted hadronisation
- Far-tail region: strongly affected by higher-order QCD

Factorization

Resummation and treatment of hadronization are based on the factorization theorem → Gherardo's talk

$$\frac{1}{\tau}\frac{d\sigma}{d\tau} = \mathcal{H} \cdot \mathcal{J} \otimes \mathcal{J} \otimes \mathcal{S} \otimes \mathcal{S}_{\rm np}$$

Scale: $\tau\, Q$ $\Lambda_{\rm QCD}$

Shape-function
Korchemsky, Sterman '99

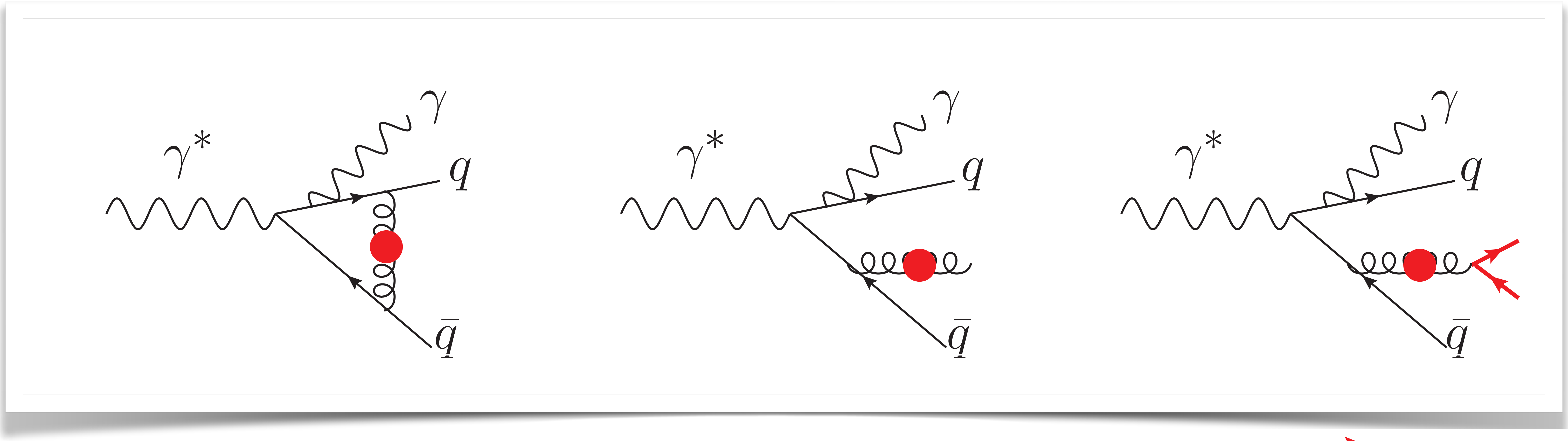
obtained in the limit $\tau \to 0$.

For values $\tau \gg \Lambda_{\rm QCD}/Q$ one can multipole expand $\mathcal{S}_{\rm np}$. In this limit hadronisation correponds to a shift of the perturbative distribution.

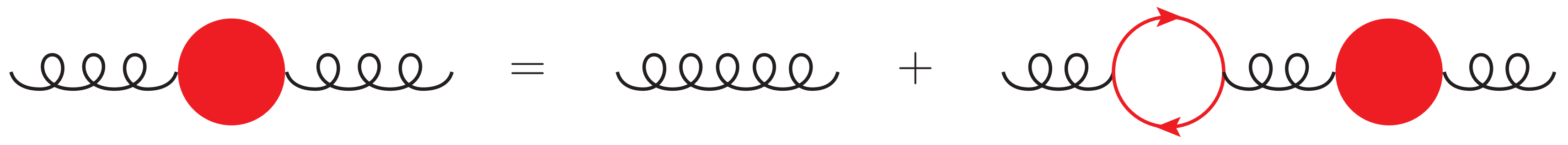
Two or three jets?

- The α_s fit region extends over the full three-jet region and it has been questioned (e.g. by Salam ‘17) whether it is appropriate to use hadronization based on an analysis of the two-jet limit at higher values.
  - Perhaps the ``shift” depends on the values of τ and C.
- New: analysis of hadronisation in 3-jet region
  - C-parameter in the symmetric 3-jet limit Luisoni, Monni, Salam ’20
  - General formula + analytic results for C and τ Caola, Ferrario Ravasio, Limatola, Melnikov, Nason ‘21+ Ozcelik ’22
  - Numerical evaluation for other event shapes + α_s fit Nason, Zanderighi ‘23

Hadronisation in the three-jet region



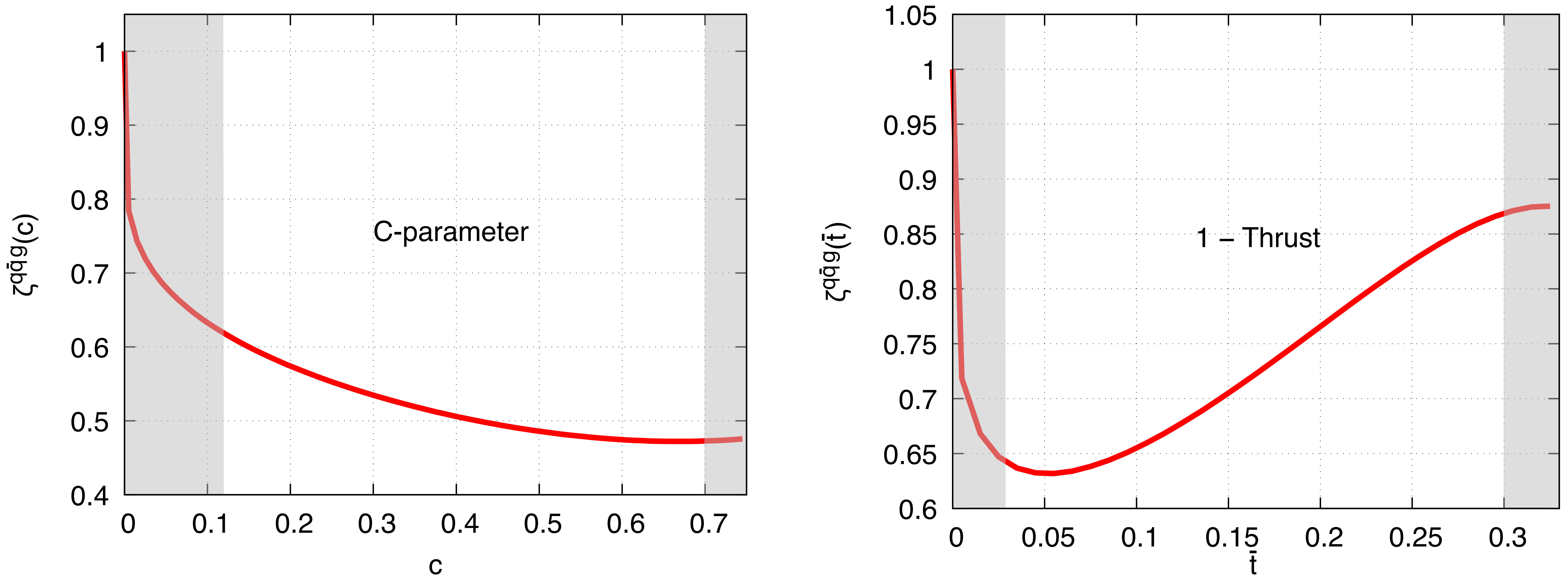
Caola et al. ’21 ‘22



- Renormalon-type computation. A model, but based on QFT:
  - Power corrections identified will be present, but there could be other sources.
  - True hadronisation will be less universal than predicted by the model.
- Caola et al. compute with photon instead of gluon in the final state
  - For hadronization associated purely with soft function (no recoil, no collinear contributions), one can reconstruct one-emission QCD result as sum of dipoles.

Hadronization results

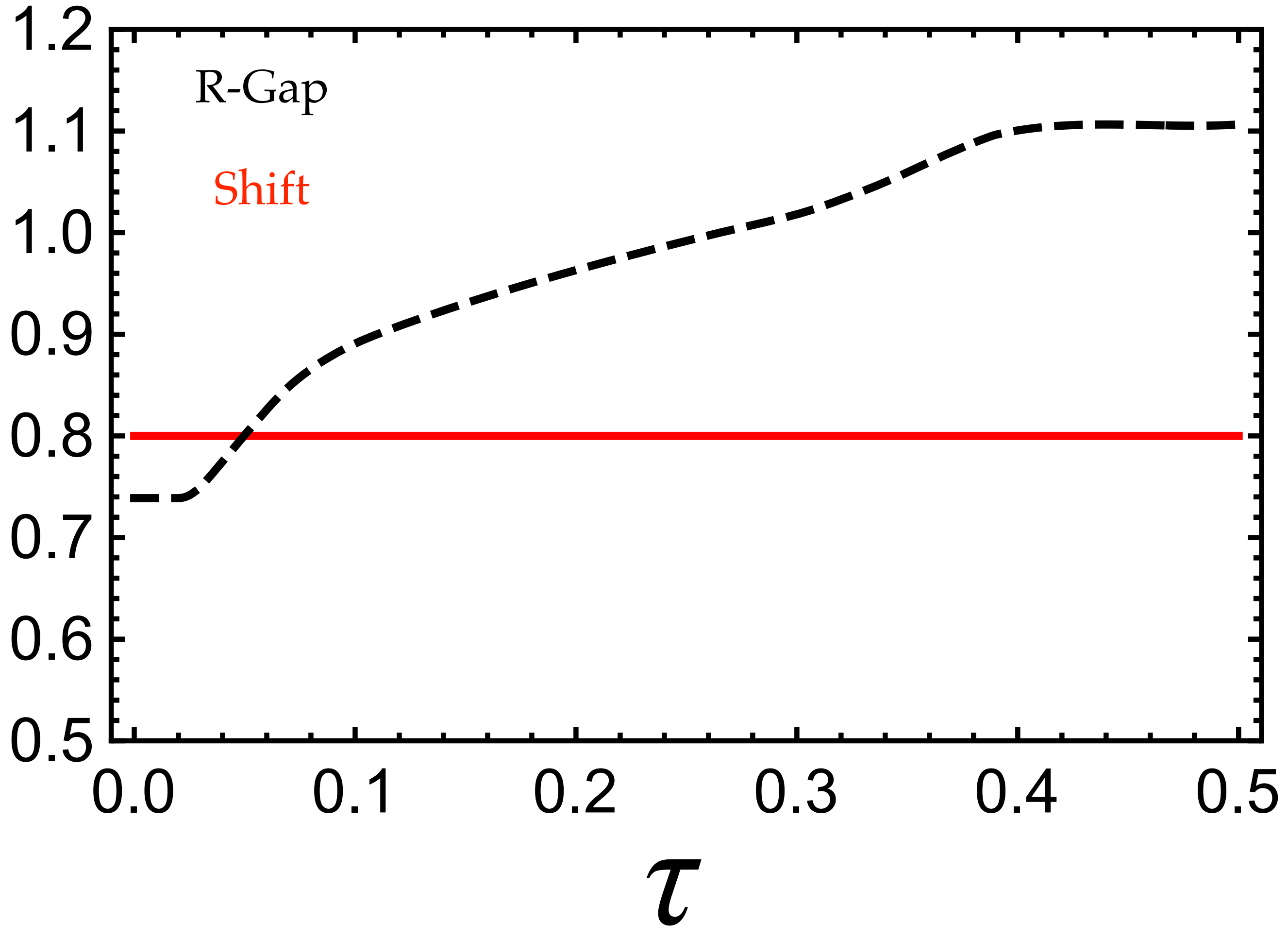
Caola et al. '22; Nason, Zanderighi '23



- Hadronisation is implemented as an observable-dependent shift ζ
- Naively: smaller shifts ζ than in the two-jet limit $\longrightarrow$ larger α_s .
- Sidenote: jet masses show very abrupt transition at very low values

Remarks

The implementation of hadronisation with shape function S_{np} is not the same as a simple shift. For the scheme used in the fit of Abbate et al. '10 one finds



from talk of Jim Talbert

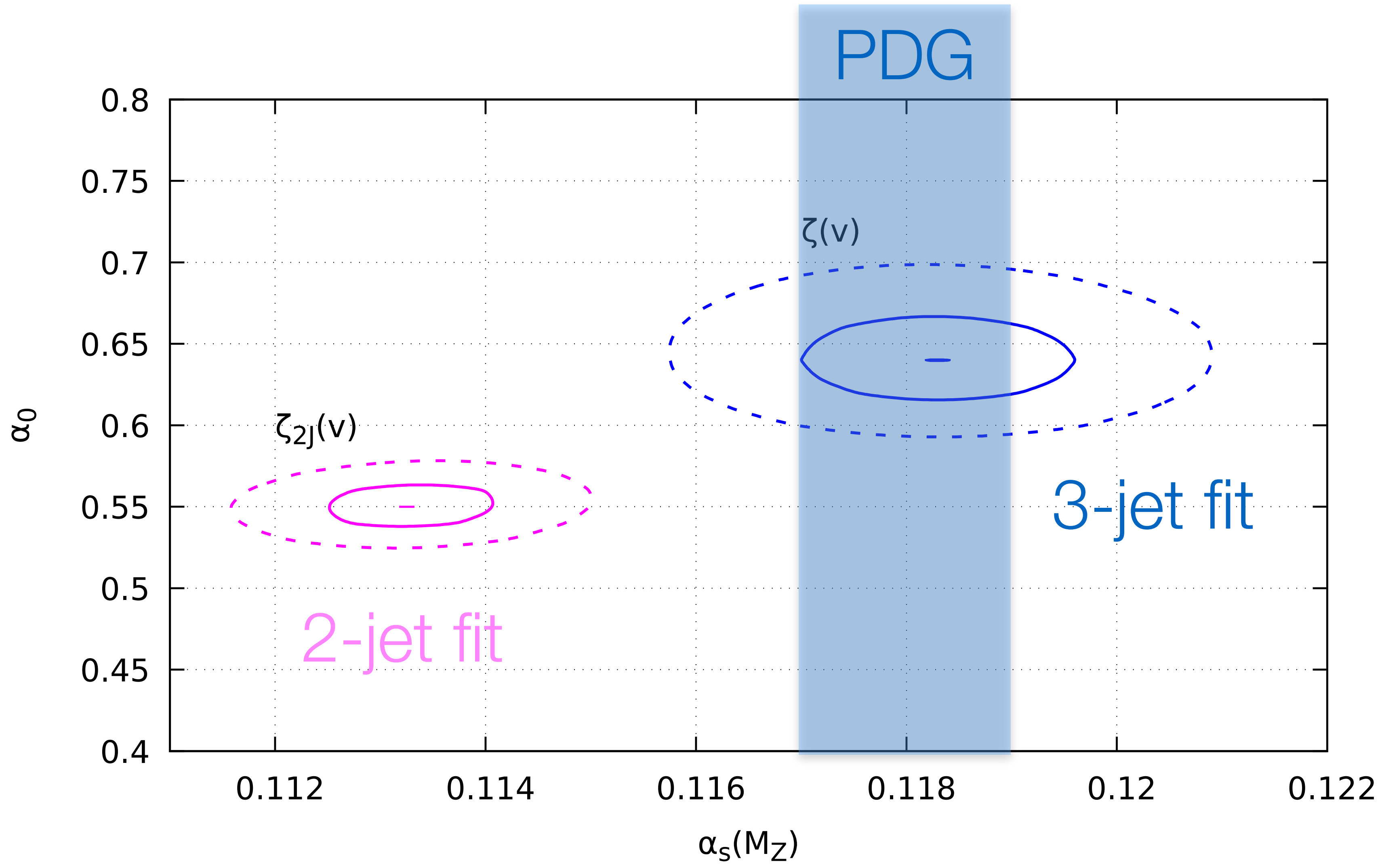
Bell, Lee, Makris, Prager, Talbert, Yan, in preparation

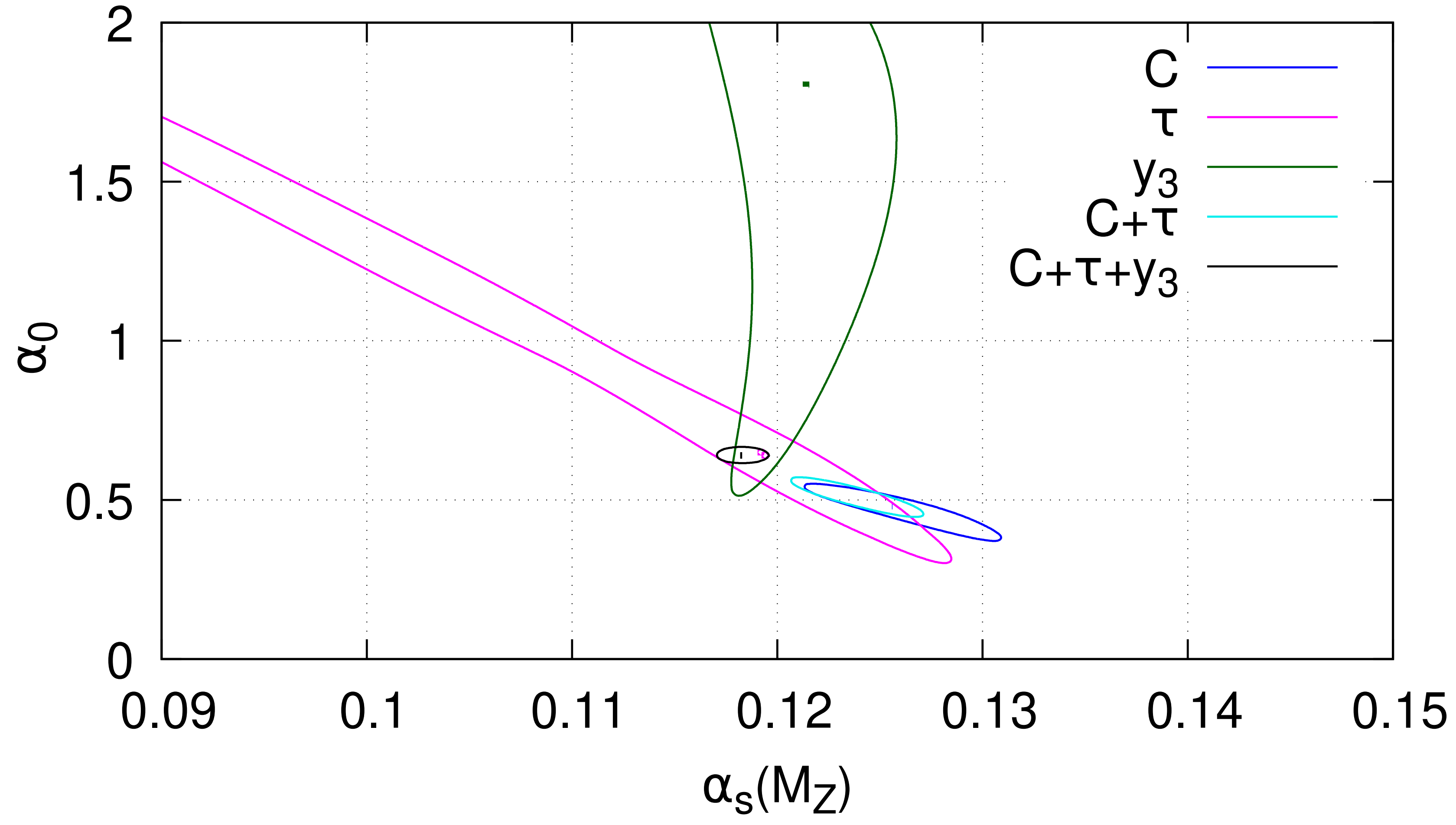
Shape is similar to new three-jet result.

Universality of shift between τ and C parameter in two-jet limit is model independent Lee and Sterman '06. Universality in three-jet limit?

α_s fit with 3-jet hadronisation

Nason, Zanderighi '23

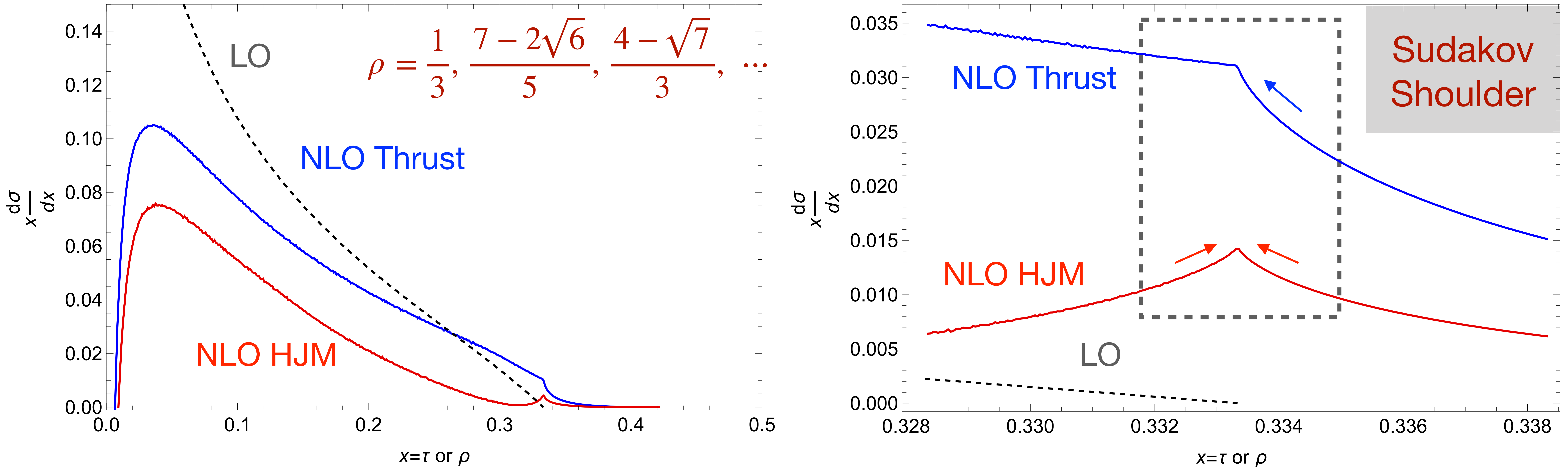




- Fit does not include resummation
  - would lead to smaller α_s
- Strictly speaking, hadronisation computation does not apply to 3-jet resolution y_3
  - additional model assumptions
- Find few per-cent differences among hadron mass schemes
- Fit with other observables? B_W, M_H, M_D?

Sudakov shoulders

Bhattacharya, Schwartz, Zhang '22 + Michel, Stewart in progress, see Matt's talk



At the 3-jet end-point, the thrust and heavy-jet-mass (HJM) distributions suffer from enhanced higher-order corrections, which can be resummed. Effect on α_s extraction?

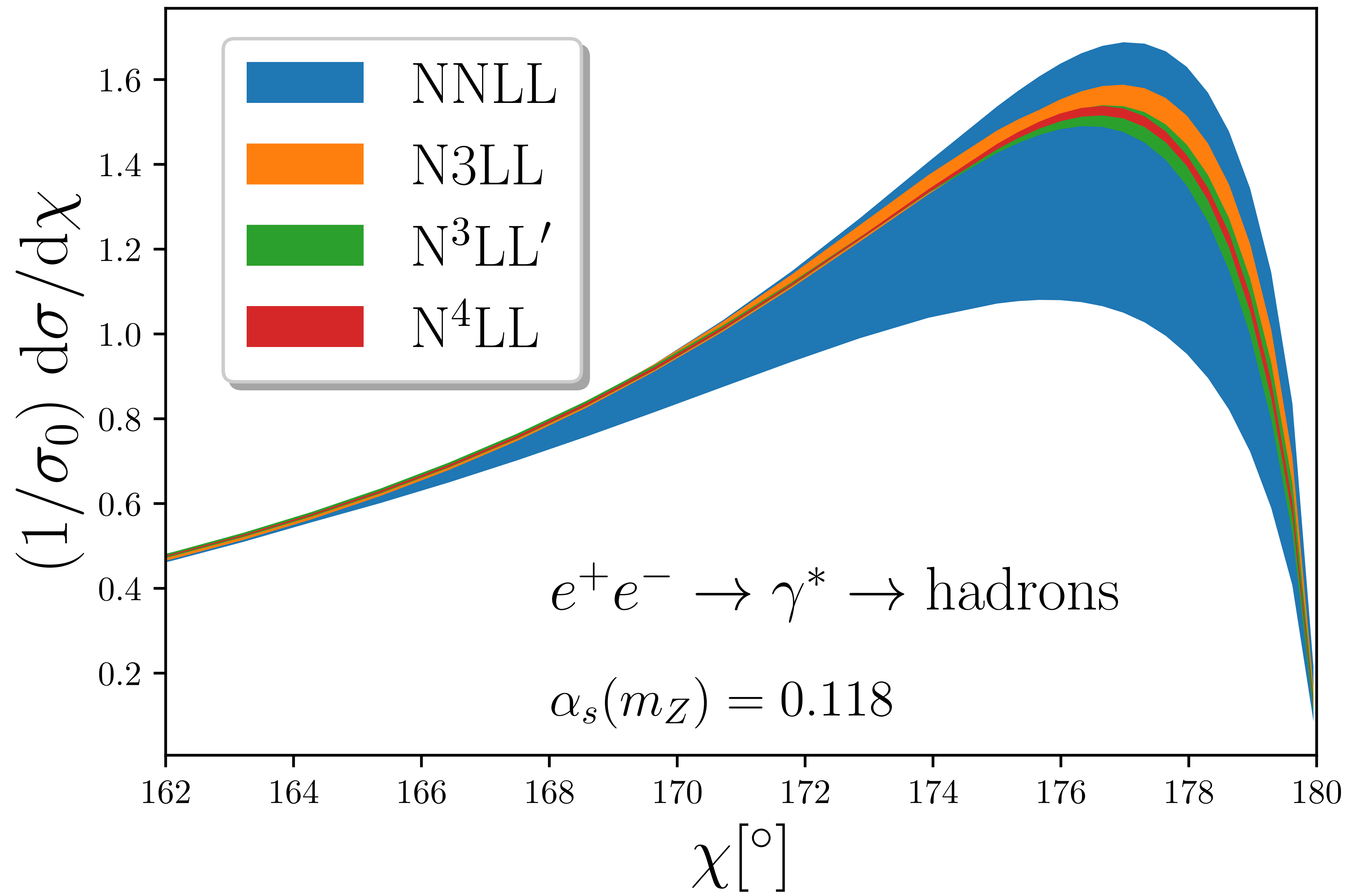
Soft-drop and hadron colliders

Due to their sensitivity to soft radiation, it is difficult to use traditional event shapes at hadron colliders

- Huge corrections from underlying event, pile-up, hadronisation

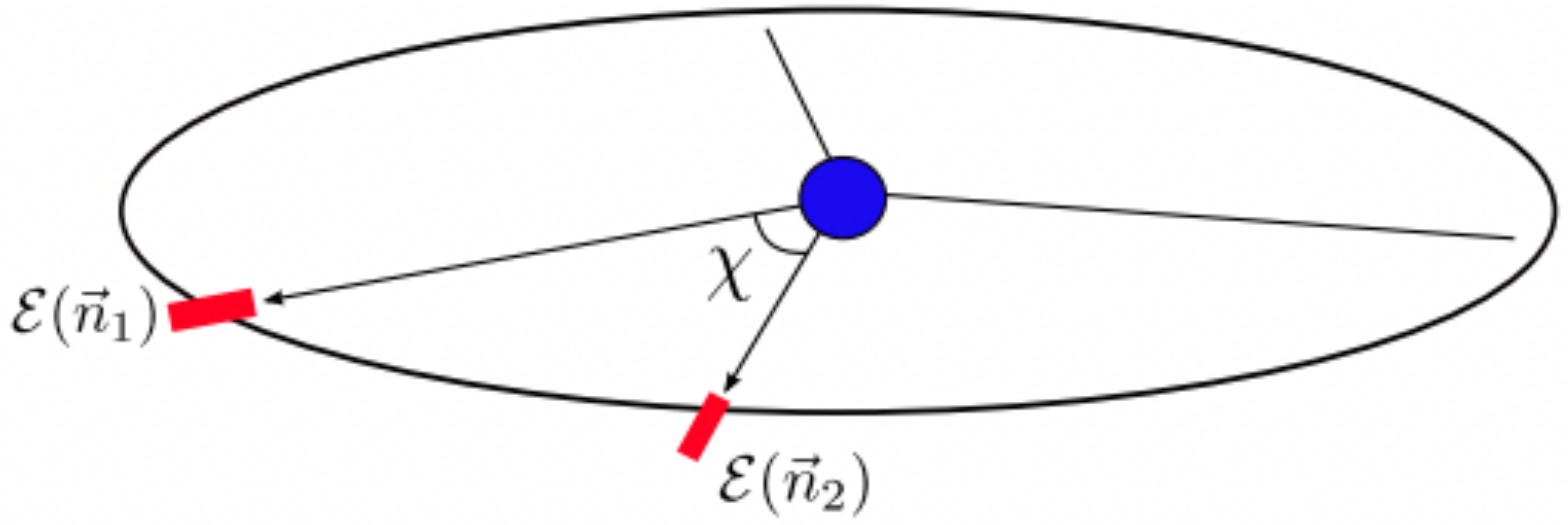
Can try to mitigate these problems by removing soft emissions from observables using soft-drop Larkoski, Marzani, Soyez, Thaler '14

- Resummed results for jet mass at N^{2}LL Frye, Larkoski, Schwartz,Yan '16 and N^{3}LL at e^+e^- Kardos, Larkoski, Trócsányi '20
- Could allow for α_s extractions at hadron colliders at the 10% level, perhaps 5% in the future Hannesdottir, Pathak, Schwartz, Stewart '22



Energy-energy correlators

Matrix elements



Energy-flow operator

$$\langle \Psi | \mathcal{E}(\hat{n}_1) \cdots \mathcal{E}(\hat{n}_k) | \Psi \rangle \quad \text{with} \quad \mathcal{E}(\hat{n}) = \int_0^\infty dt \lim_{r \to \infty} r^2 n^i T_{0i}(t, r\hat{n})$$

characterize energy flow into the detector

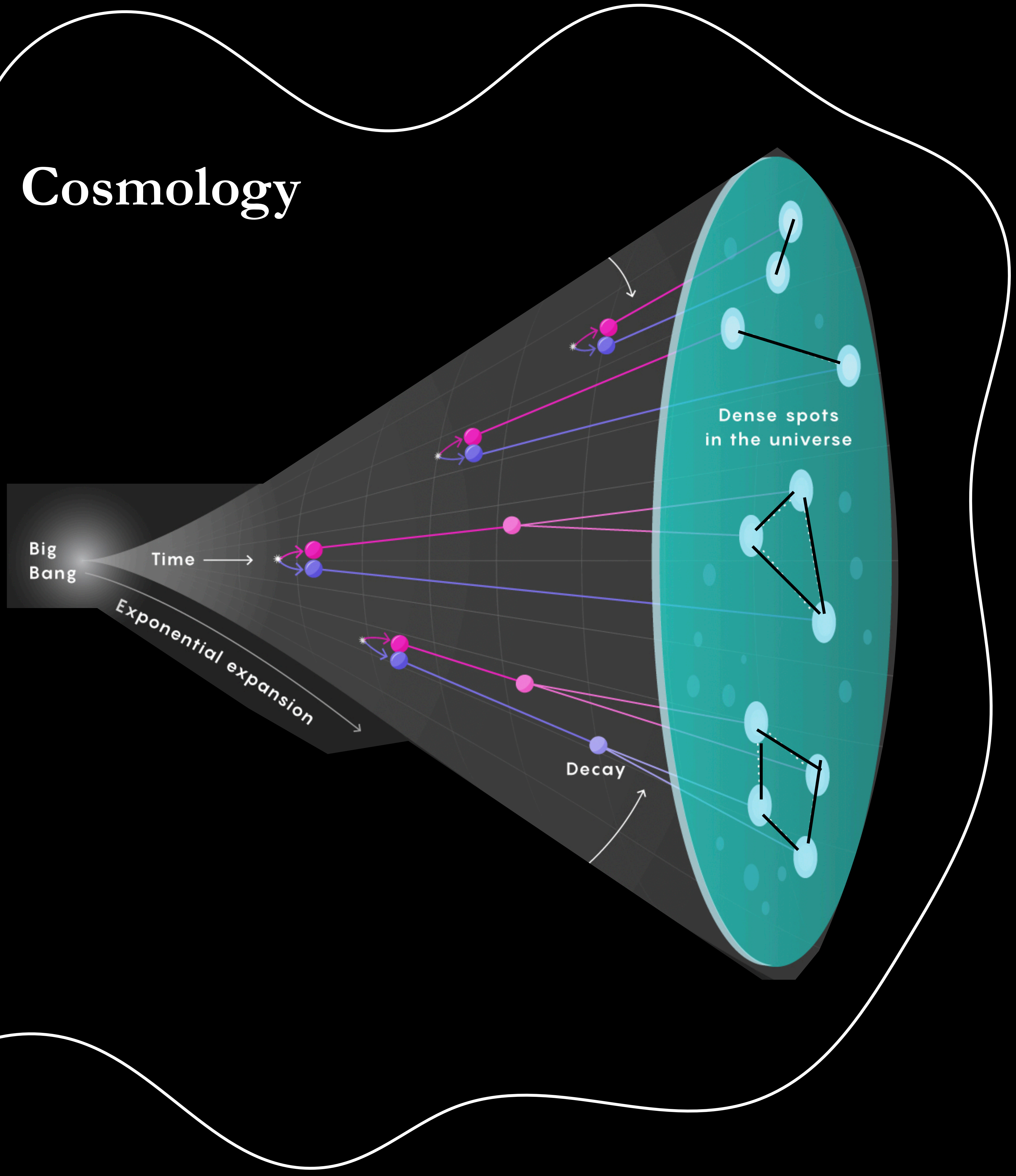
Sveshnikov, Tkachov '95

A lot of new interesting developments in using these energy-energy correlators to study jet subtructure, determine α_s and m_t , …

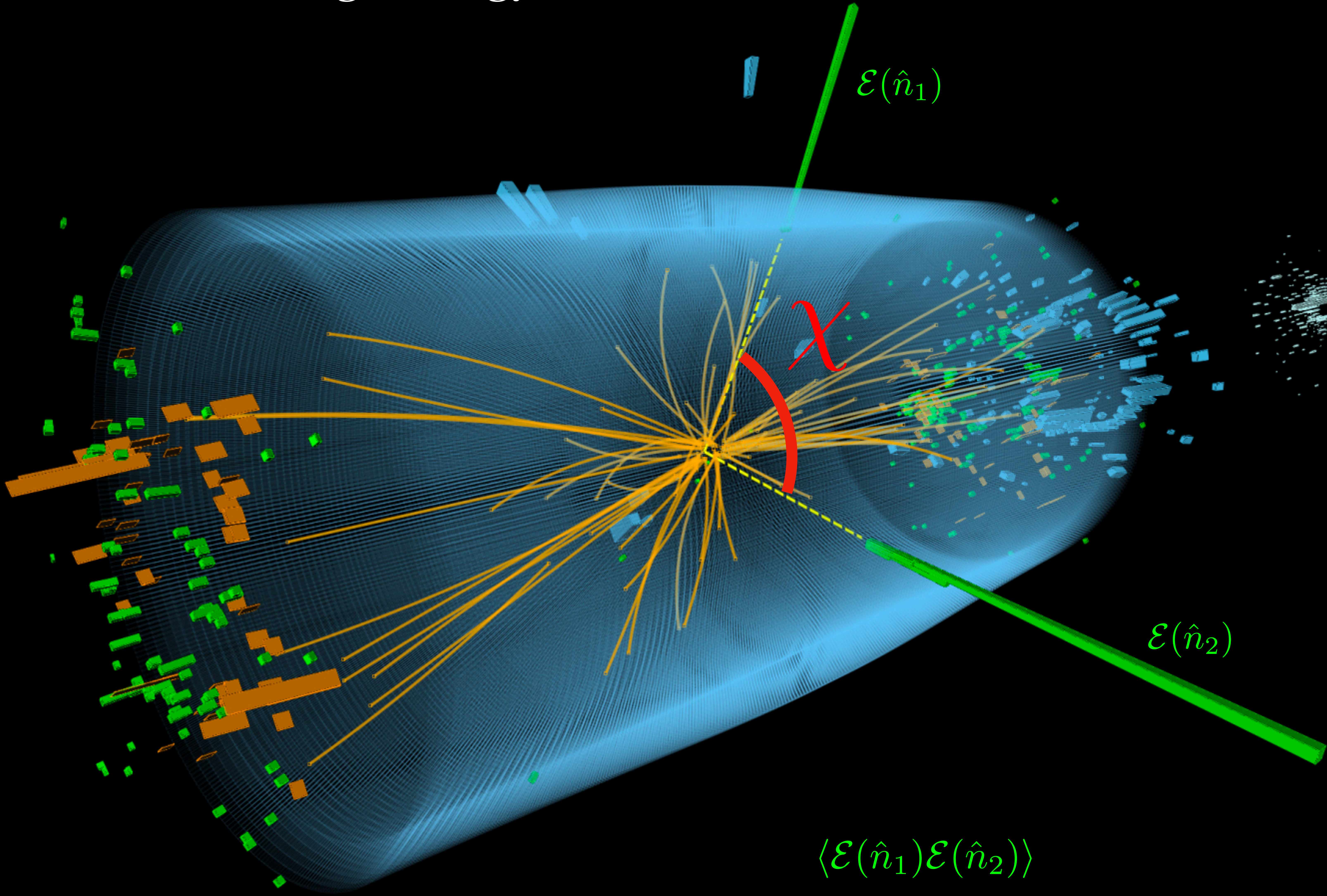
Correlators have many good properties

- Insensitive to soft radiation
- Factorization, Light-ray OPE, CFT techniques Hofman, Maldacena '08

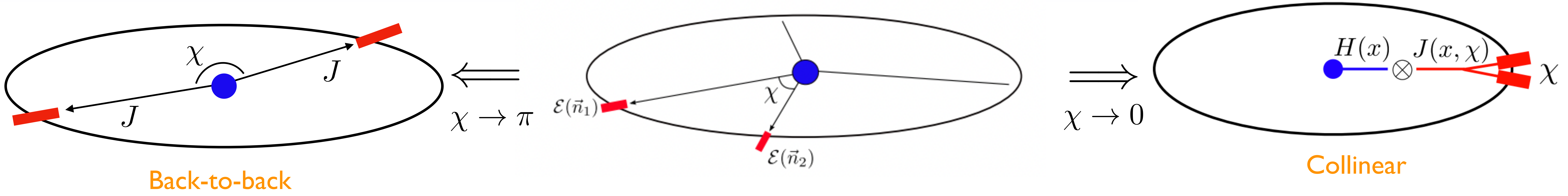
Cosmology



High Energy Collider



from Kyle Lee's talk at SCET23



$$\frac{d\Sigma}{d\chi} = \sum_{i,j} \int d\sigma \frac{E_i E_j}{Q^2} \delta\left(\chi - \chi_{ij}\right) \quad \text{and} \quad \sum_{i,j} E_i E_j = Q^2 \quad \Longrightarrow \quad \int_0^{\pi} d\chi \frac{d\Sigma}{d\chi} = \sigma_{\text{tot}}$$

Sum rule

Dixon, Moult, Zhu, `19

Simplest correlator is familiar EEC Basham, Brown, Ellis, Love '78.

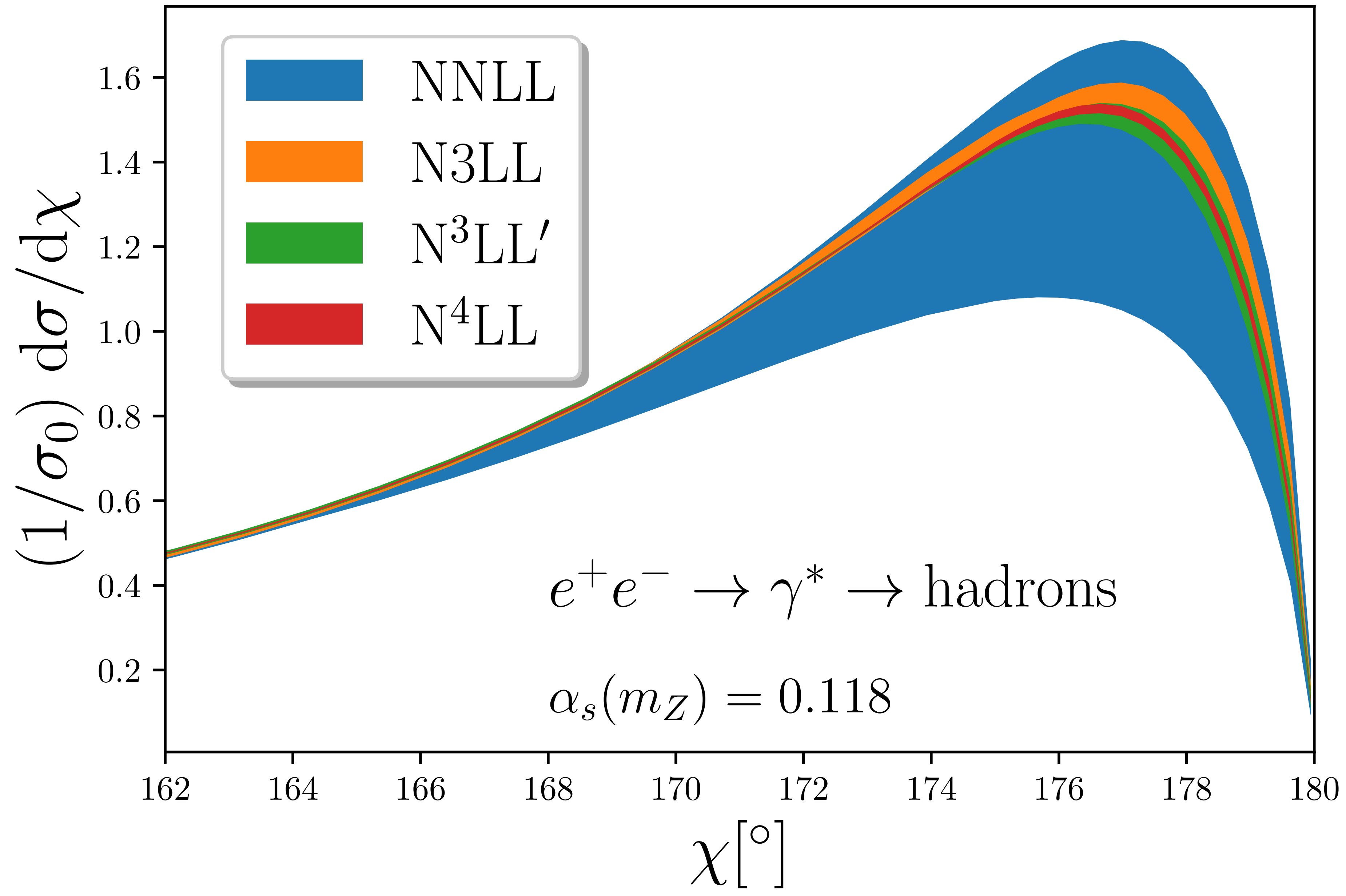
Factorization theorems in collinear and in back-to-back limit. Second case

$$\frac{\mathrm{d}\sigma}{\mathrm{d}z} = \frac{\hat{\sigma}_0}{8} H_{q\bar{q}}(Q,\mu) \int_0^{\infty} \mathrm{d}(b_T Q)^2 \, J_0\left(b_T Q\sqrt{1-z}\right) \; \mathcal{J}_q\left(b_T, \mu, \frac{Q b_T}{\upsilon}\right) \mathcal{J}_{\bar{q}}\left(b_T, \mu, Q b_T \upsilon\right)$$

Gao, Li, Moult, Zhu '19

$$z \equiv \tfrac{1}{2}(1-\cos\chi)$$

First N⁴LL resummation!

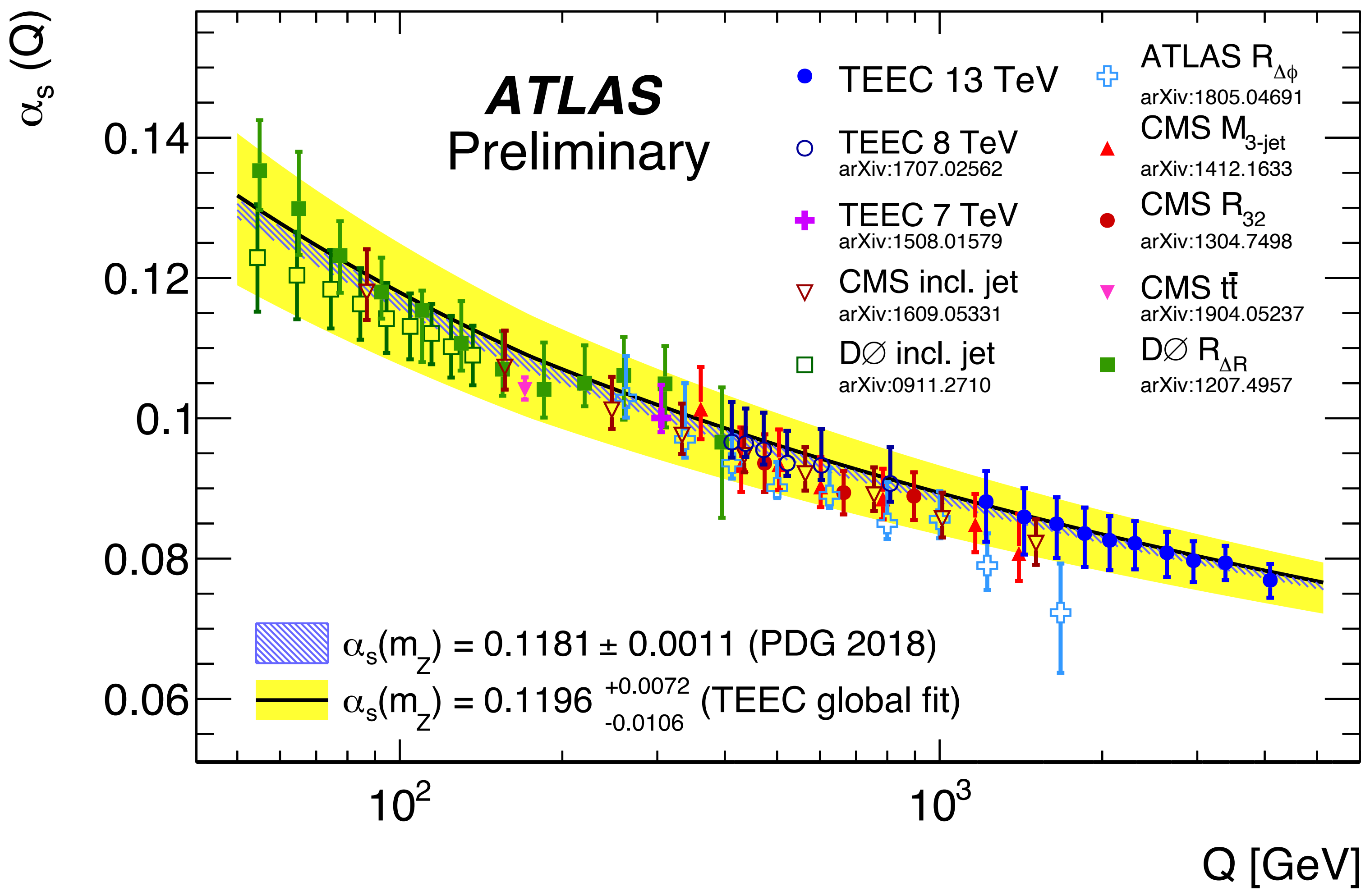


Duhr, Mistlberger, Vita ’22

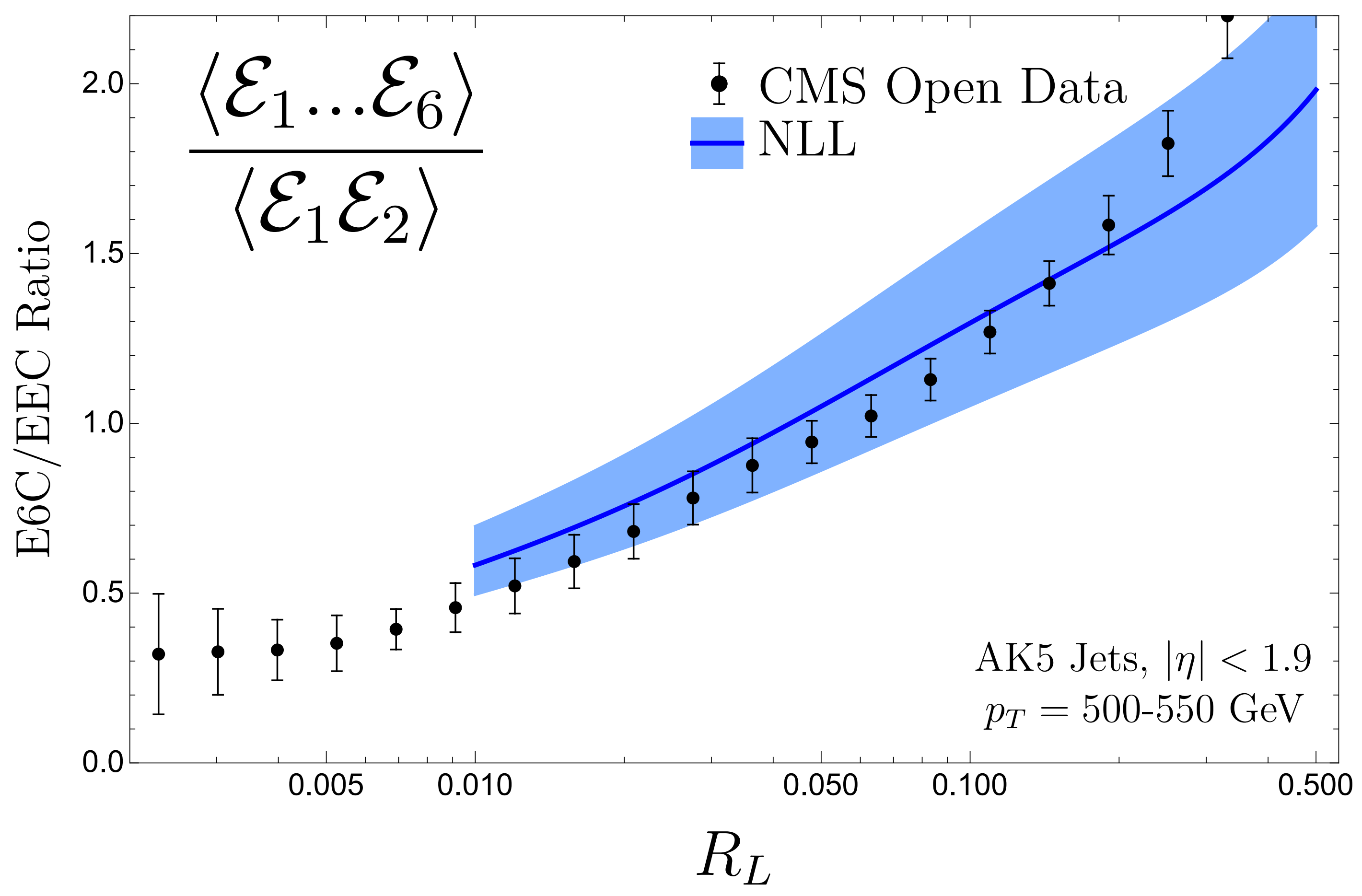
Ingredients:

- 3-loop jet functions Ebert, Mistlberger, Vita ’20
- 4-loop rapidity anomalous dimension Duhr, Mistlberger, Vita ’22, Moult, Zhu, Zhu ‘22.
- four-loop hard anomalous dimensions Manteuffel, Panzer, and Schabinger ’20; Lee, Manteuffel, Schabinger, Smirnov, Smirnov, and M. Steinhauser ’22.
- four-loop cusp Henn, Korchemsky, Mistlberger ’19; Manteuffel, Panzer, and Schabinger ’20 + … 5-loop cusp is missing, estimated to have very small effect.

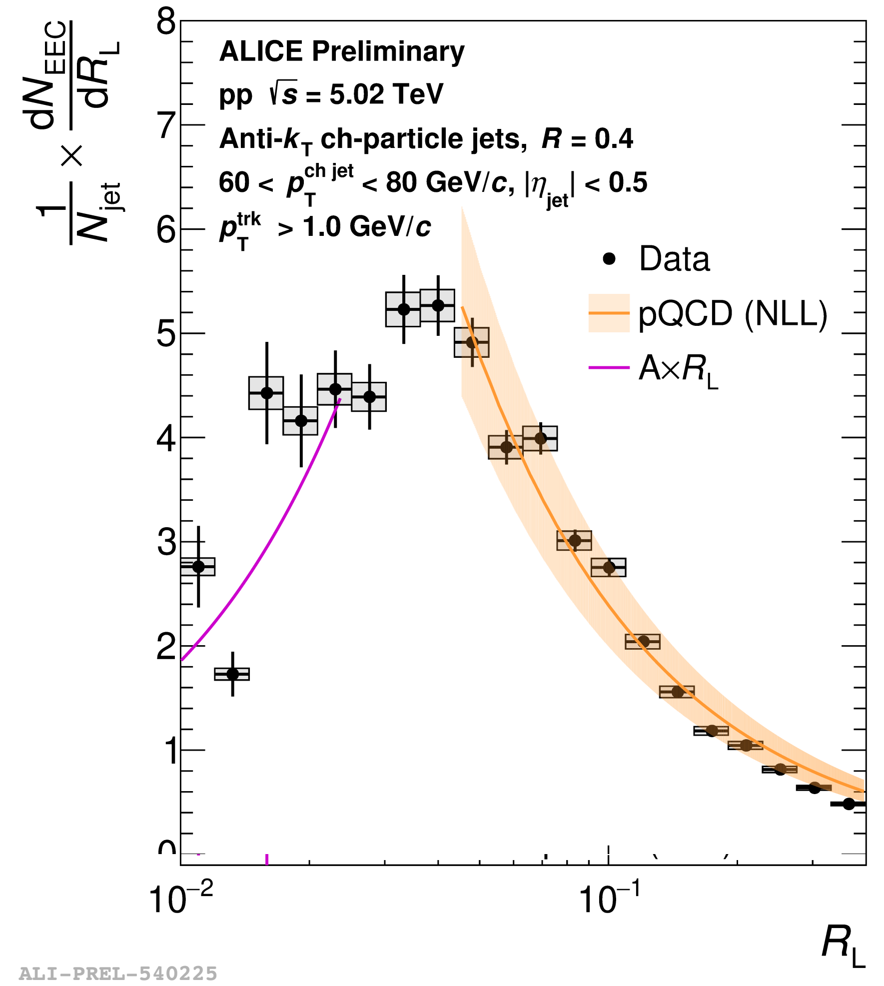
"Conformal Colliders Meet the LHC"



ATLAS-CONF-2020-025



Lee, Mecaj, Moult '22

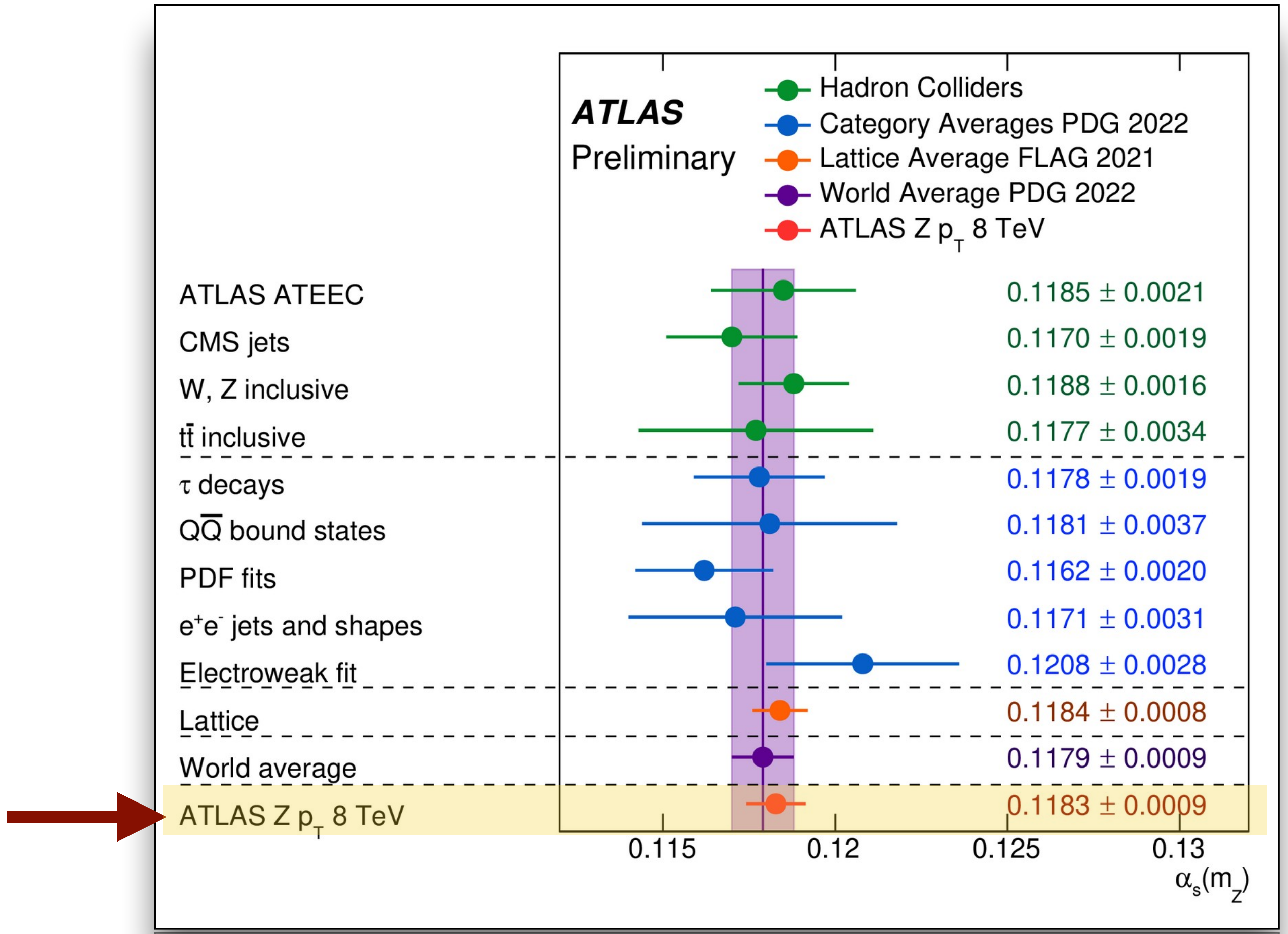


ALI-PREL-539525

- At hadron collider use transverse EECs Ali, Pietarinen, Stirling '84. ATLAS α_s determination is based on NLO + MC but NNLL resummation is available Gao, Li, Moult, Zhu '19
- Factorization theorems for multi-point correlators in jets in collinear limit; results for higher-point correlators Lee, Mecaj, Moult '22

Many new ideas and results

- EECs to measure the top-quark mass Holguin, Moult, Pathak, Procura ’22
- EECs for *b*- and *c*-quarks Lee, Mecaj Moult ’22
- Non-Gaussianities in collider energy flux Chen, Moult, Thaler, Zhu ‘22
- EECs for nuclear matter at the electron-ion collider (EIC) Devereaux, Fan, Ke, Lee, Moult ’23
- Nucleon energy correlators Liu, Zhu ’22, Cao, Liu, Zhu ‘23
- EECs for studying the quark-gluon plasma
  Andres, Dominguez, Holguin, Marquet, Moult ’23; Liu, Liu, JPan, Yuan and Zhu ’23
- Renormalons in the EEC Schindler, Stewart, Sun ’23
- …



α_s from from q_T spectrum of Z-bosons

$pp \to$ "EW bosons" $+ X$ at low q_T

hard function: Born + virtual

$$\mathrm{d}\sigma_{ij}(p_1,p_2,\{\underline{q}\}) = \int_0^1 \mathrm{d}\xi_1 \int_0^1 \mathrm{d}\xi_2\, \mathrm{d}\sigma^0_{ij}(\xi_1 p_1, \xi_2 p_2, \{\underline{q}\})\, \mathcal{H}_{ij}(\xi_1 p_1, \xi_2 p_2, \{\underline{q}\}, \mu) \cdot$$

$$\frac{1}{4\pi}\int \mathrm{d}^2 x_\perp\, e^{-iq_\perp x_\perp} \left(\frac{x_T^2 Q^2}{b_0^2}\right)^{-F_{ij}(x_\perp,\mu)} B_i(\xi_1, x_\perp, \mu)\cdot B_j(\xi_2, x_\perp, \mu)$$

collinear anomaly

beam functions

- Ingredients known to high accuracy
  - three-loop beam functions Ebert, Mistlberger, Vita '20
  - three-loop hard functions for Z/W/γ (new: singlet contributions Gehrmann, Primo '21 with top mass Chen, Czakon, Niggetiedt '21), two-loop for diboson processes
  - new: four-loop anomalous dimensions and anomaly exponent

4-loop anomalous dimensions

- Anomaly exponent aka rapidity anomalous dimension can be extracted from regular 4-loop soft anomalous dimension obtained in Das, Moch, Vogt '19, Duhr, Mistlberger Vita, '22 through conformal mapping at $\beta(\epsilon^*) = 0$ Vladimirov '16.
  - Independent extractions by Duhr, Mistlberger, Vita '22 and Moult, Zhu, Zhu '22
- four-loop hard anomalous dimensions Manteuffel, Panzer, and Schabinger '20; and full quark and gluon form factors Lee, Manteuffel, Schabinger, Smirnov, Smirnov, and M. Steinhauser '22.
- four-loop cusp Henn, Korchemsky, Mistlberger '19; Manteuffel, Panzer, and Schabinger '20 + … 5-loop cusp is missing, estimated to have very small effect.

Implementation

hard function: Born + virtual

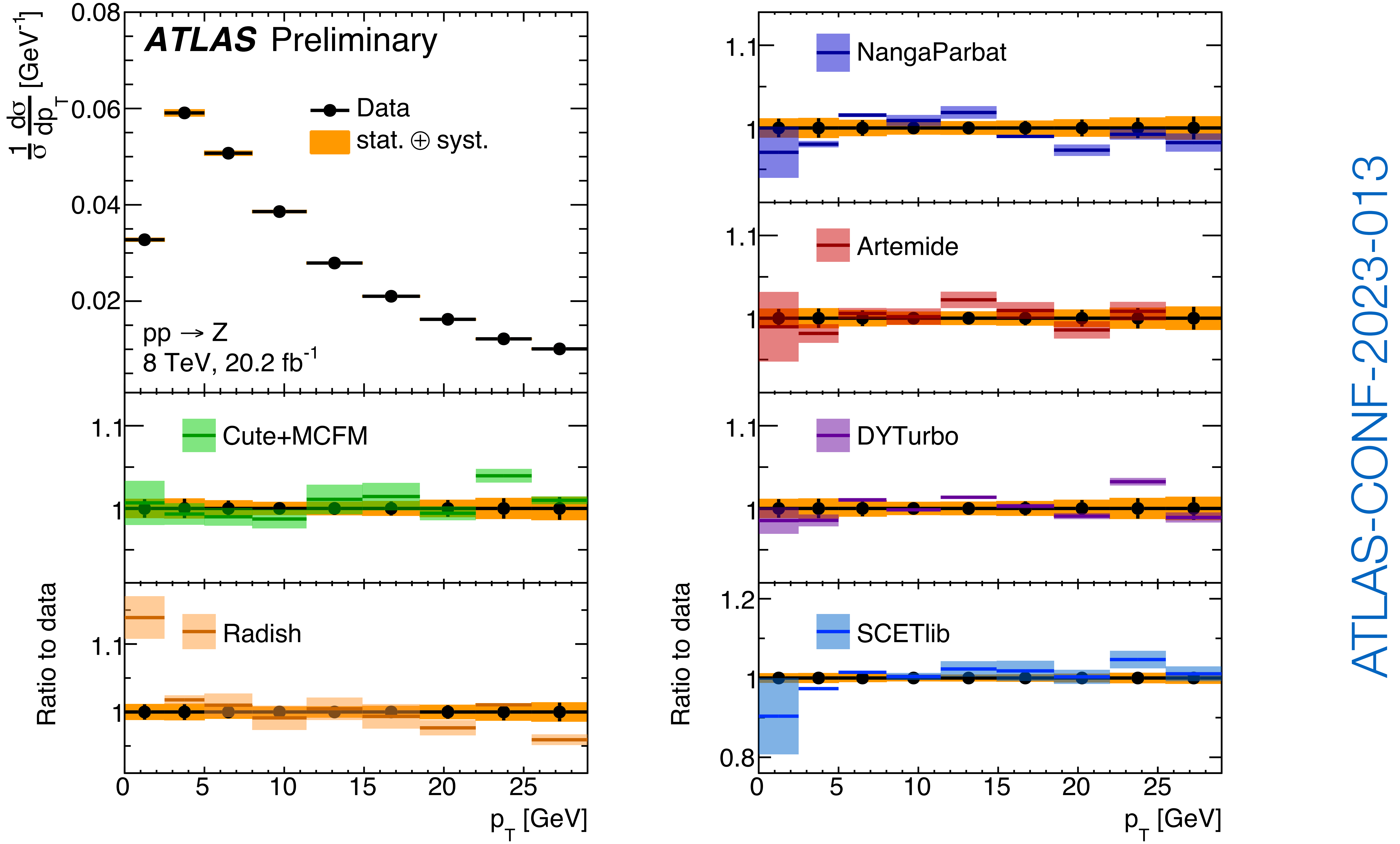
$$\mathrm{d}\sigma_{ij}(p_1,p_2,\{\underline{q}\}) = \int_0^1 \mathrm{d}\xi_1 \int_0^1 \mathrm{d}\xi_2\, \mathrm{d}\sigma^0_{ij}(\xi_1 p_1, \xi_2 p_2, \{\underline{q}\})\, \mathcal{H}_{ij}(\xi_1 p_1, \xi_2 p_2, \{\underline{q}\}, \mu)\, \cdot$$

$$\frac{1}{4\pi}\int \mathrm{d}^2 x_\perp\, e^{-iq_\perp x_\perp} \left(\frac{x_T^2 Q^2}{b_0^2}\right)^{-F_{ij}(x_\perp,\mu)} B_i(\xi_1, x_\perp, \mu)\cdot B_j(\xi_2, x_\perp, \mu)$$

collinear anomaly

beam functions

- Structure of resummation is the same as born-level + virtual in fixed-order computation
  - Resummation can piggyback on existing fixed-order codes MATRIX+RadISH Kallweit, Re, Rottoli, Wiesemann '20, CuTe-MCFM TB, Neumann '20, to get resummed fiducial cross sections.
  - Same for jet-veto cross section MadGraph5_aMC@NLO TB, Frederix, Neubert Rothen '14; MCFM-RE Arpino, Banfi, Jäger, Kauer '19; MCFM Campbell, Ellis, Neumann, Seth '23, → Keith's talk; → Matthew's talk



- aN⁴LL resummations from several groups with different formalisms (public N⁴LL: CuTe-MCFM Campbell, Neumann '22, DYTurbo Camarda, Cieri, Ferrera '23; ARTEMIDE Scimemi, Vladimirov '23)
- All results (except ARTEMIDE) include α_s^3 fixed order from MCFM

Comparison and uncertainties

Resummed computations are performed in a variety of (equivalent) formalisms and with different of scheme choices

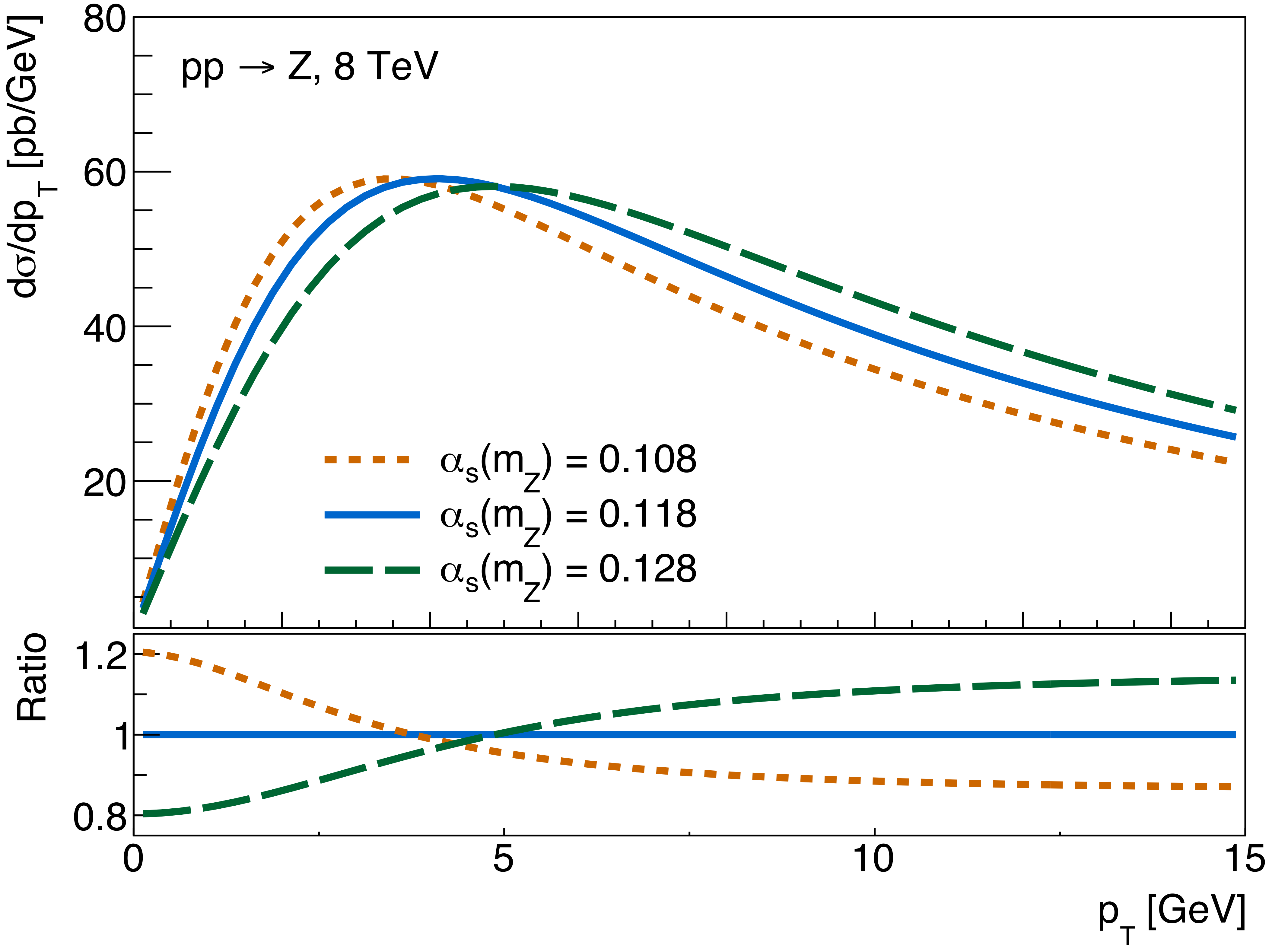
- Scale setting in momentum space (CuTe, Radish) versus impact parameter space (everyone else)
- Different formalisms for rapidity logs (CSS, collinear anomaly, RRG) and associated uncertainty
- Different matching schemes / transition to fixed order

Uncertainty estimates are much less standardized than for fixed-order computations!

- Ongoing comparison/benchmark efforts by LHC EW sub-group

ATLAS α_s extraction

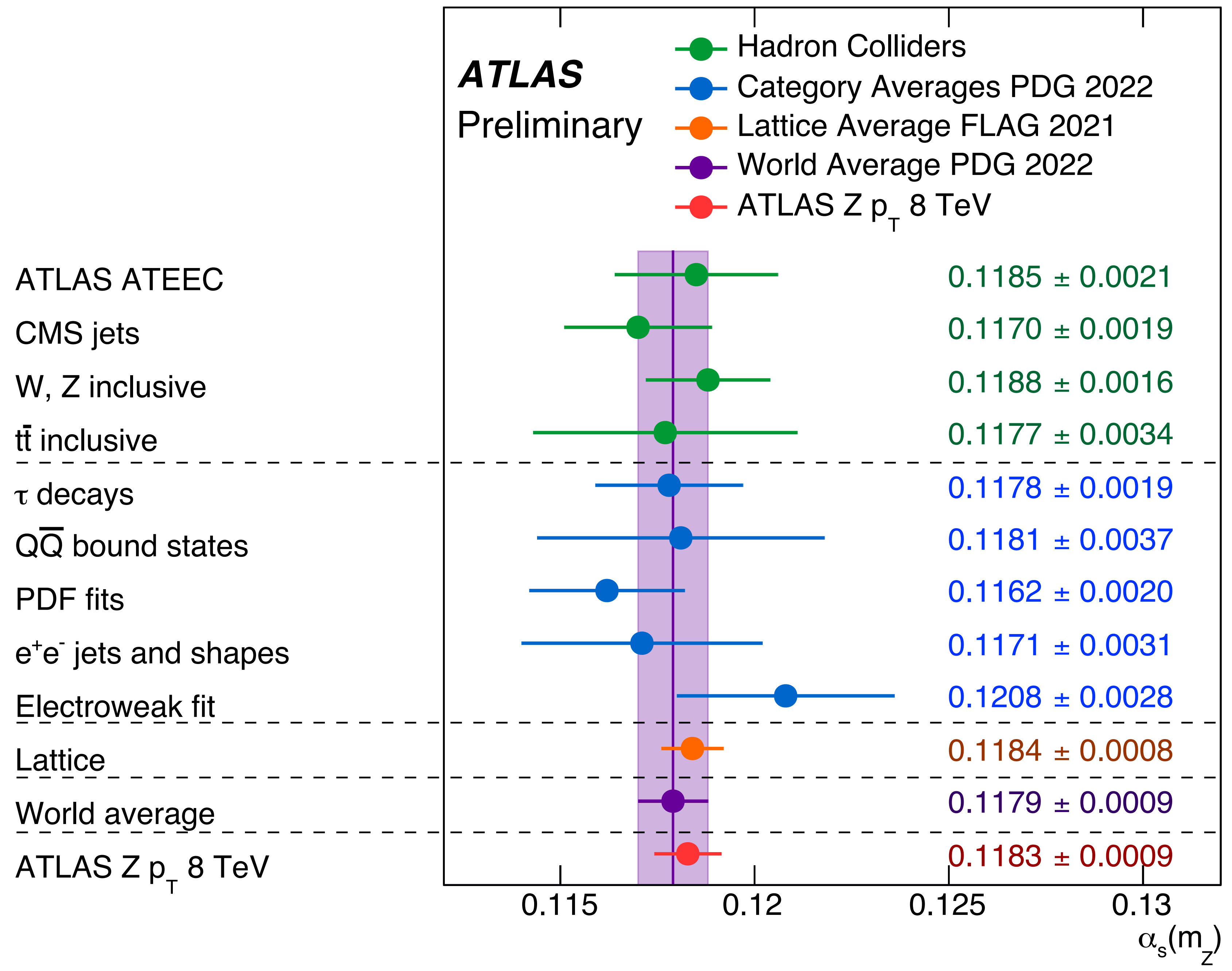
ATLAS-CONF-2023-015



- Reconstruct inclusive spectrum rate from angular coefficients ATLAS-CONF-2023-013
- α_s from fit to DYTurbo
- MSHT20 approximate N^{3}LO PDFs
  - cross checks with NNLO sets
- Non-perturbative effects based on two-parameter ansatz by Collins Rogers '14

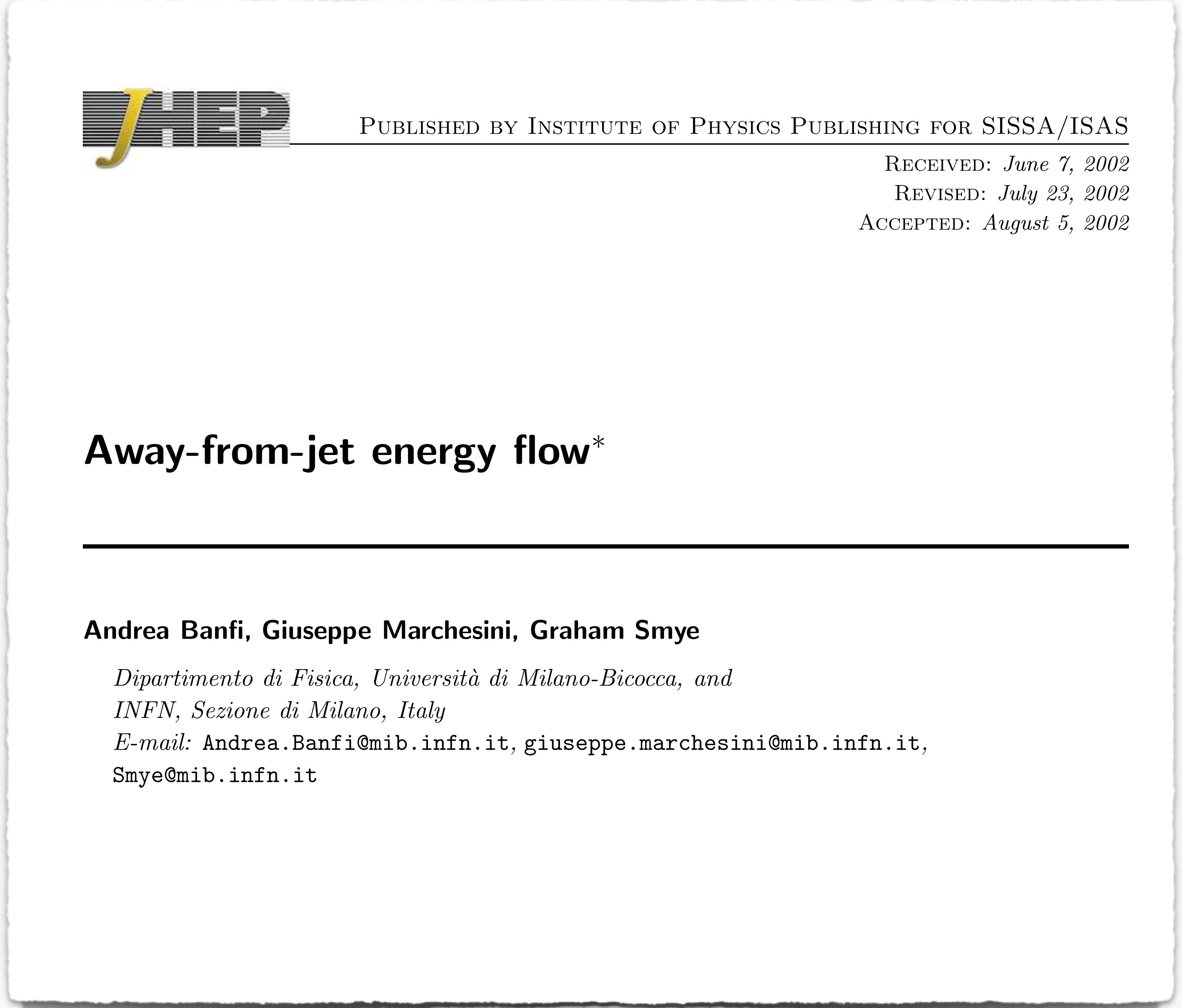
ATLAS α_s extraction

ATLAS-CONF-2023-015



Experimental uncertainty	+0.00044	-0.00044
PDF uncertainty	+0.00051	-0.00051
Scale variations uncertainties	+0.00042	-0.00042
Matching to fixed order	0	-0.00008
Non-perturbative model	+0.00012	-0.00020
Flavour model	+0.00021	-0.00029
QED ISR	+0.00014	-0.00014
N4LL approximation	+0.00004	-0.00004
Total	+0.00084	-0.00088

One of the most precise determinations of α_s !

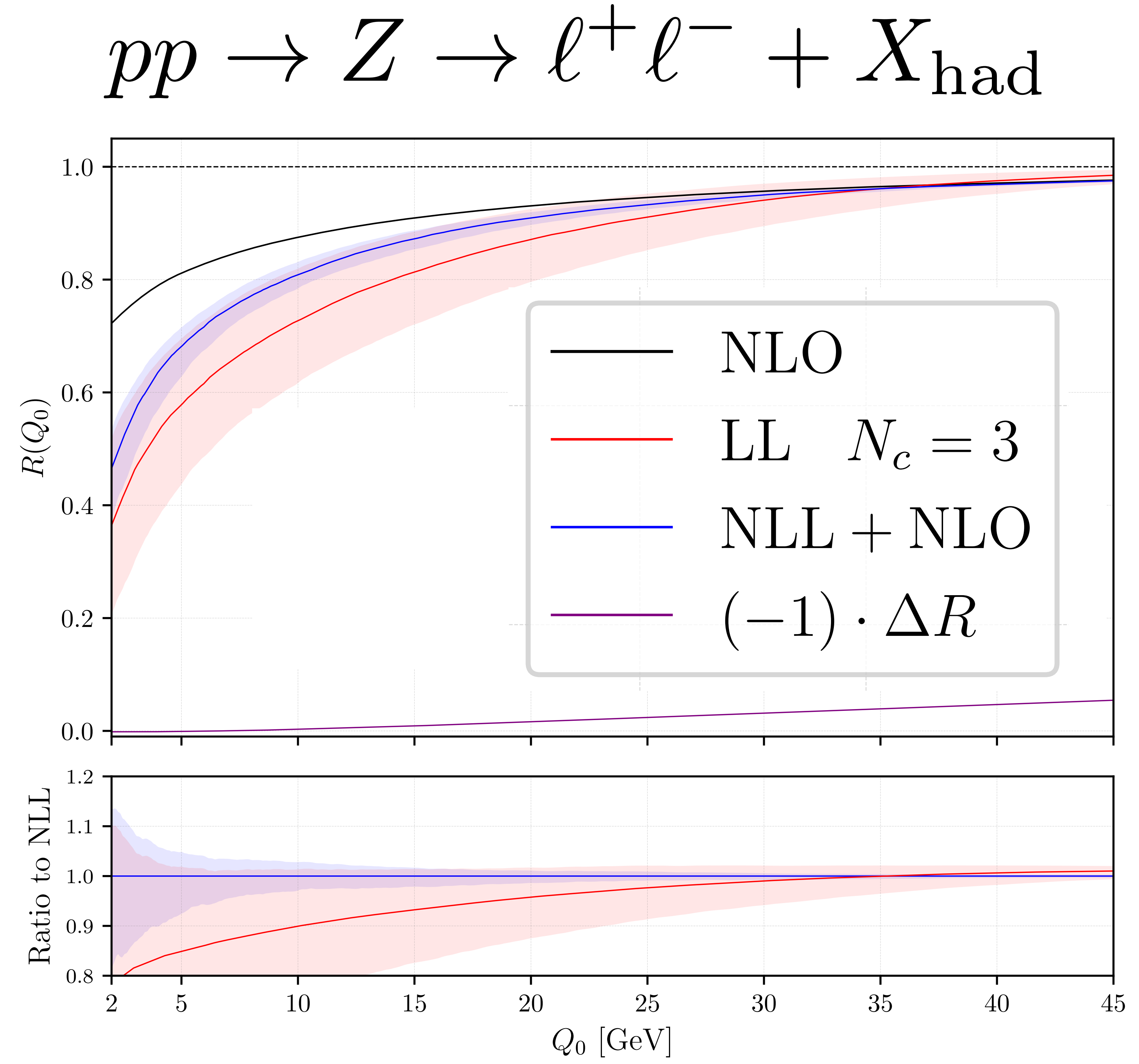


PUBLISHED BY INSTITUTE OF PHYSICS PUBLISHING FOR SISSA/ISAS
RECEIVED: *June 7, 2002*
REVISED: *July 23, 2002*
ACCEPTED: *August 5, 2002*

Away-from-jet energy flow*

Andrea Banfi, Giuseppe Marchesini, Graham Smye

Dipartimento di Fisica, Università di Milano-Bicocca, and
INFN, Sezione di Milano, Italy
E-mail: Andrea.Banfi@mib.infn.it, giuseppe.marchesini@mib.infn.it,
Smye@mib.infn.it



TB, Schalch, Xu, in preparation

Resummation of non-global observables

Traditional resummation methods (such as SCET) restricted to global observables which do not involve angular cuts on hadronic radiation.

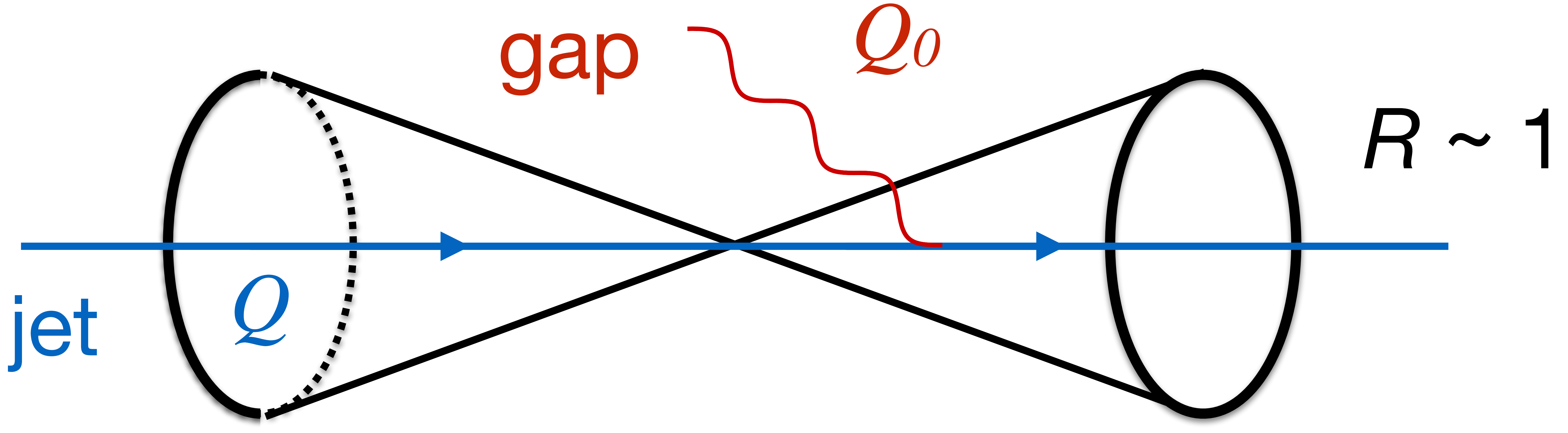
Non-global observables such as

- jet cross sections → Gregory's talk or isolation-cone cross sections relevant for γ production → Xiaofeng's talk

involve very intricate structure of soft radiation

- secondary emissions: non-global logarithms (NGLs) Dasgupta, Salam '01; Banfi, Marchesini, Smye '02
- hadronic collisions: complex phases & breakdown of color coherence: super-leading logarithms SLL Forshaw, Kyrieleis, Seymour '06

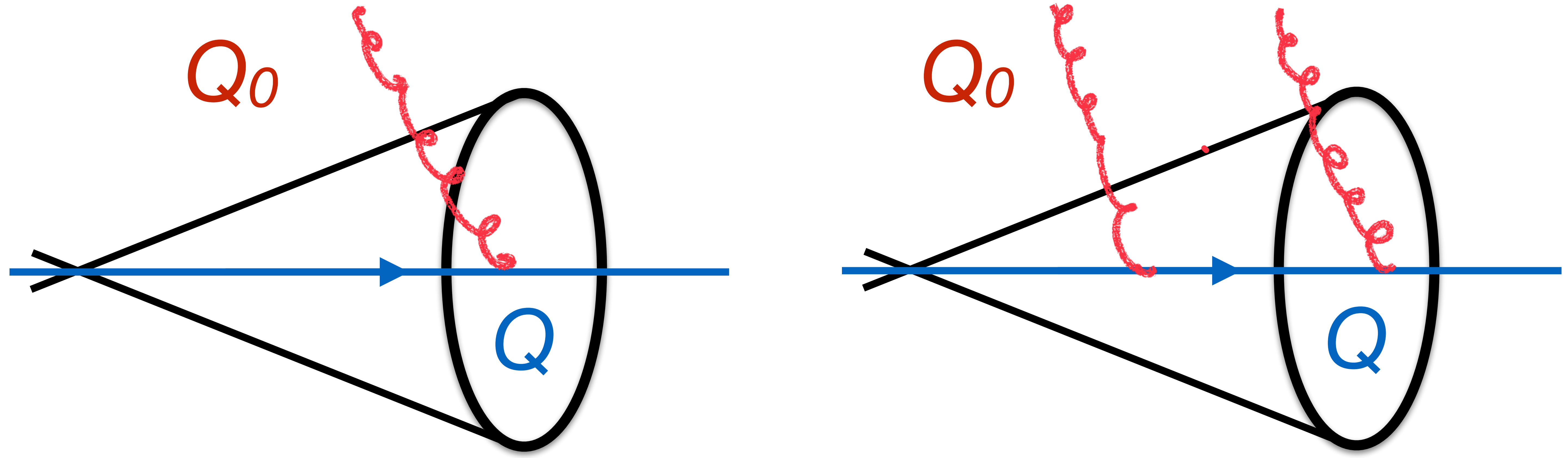
Simplest example of non-global observable: gap between jets aka away from jet energy flow aka interjet energy flow aka rapidity slice



→ large logarithms $\alpha_s^n L^m$ with $L = \ln(Q/Q_0)$

Will discuss case of large cone radius $R \sim 1$.

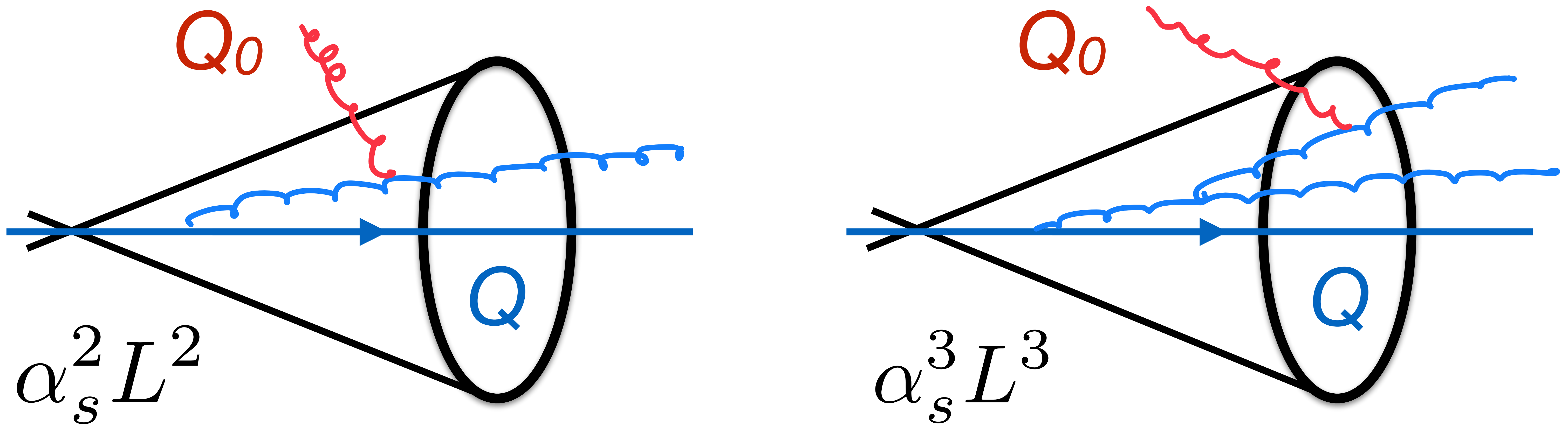
Global Logarithms



$\alpha_s L$ $\alpha_s^2 L^2$

- In (massive) QED, logarithmic terms would exponentiate: full result is exponential of one loop!
- For global observables in QCD, non-abelian higher-order corrections ("non-abelian exponentiation")

Non-global logarithms (NGLs)



- Soft gluons from secondary emissions inside the jets
- Not captured by standard resummation methods. Even leading NGLs $(\alpha_s L)^n$ do not simply exponentiate!
- At large N_c leading NGLs can be obtained with parton shower Dasgupta, Salam '02 or by solving a non-linear integral equation Banfi, Marchesini, Smye '02, the BMS equation

Factorization for gap between jets in e^+e^-

TB, Neubert, Rothen, Shao Phys.Rev.Lett. 116 (2016) 19, 192001, see also Caron-Huot '15

Hard function
m hard partons along
fixed directions $\{n_1, \ldots, n_m\}$
$\boldsymbol{\mathcal{H}}_m \propto |\mathcal{M}_m\rangle\langle\mathcal{M}_m|$

Soft function
squared amplitude
with *m* Wilson lines

$$\sigma(Q, Q_0) = \sum_{m=2}^{\infty} \langle \boldsymbol{\mathcal{H}}_m(\{\underline{n}\}, Q, \mu) \otimes \boldsymbol{\mathcal{S}}_m(\{\underline{n}\}, Q_0, \mu) \rangle$$

color trace

integration over directions

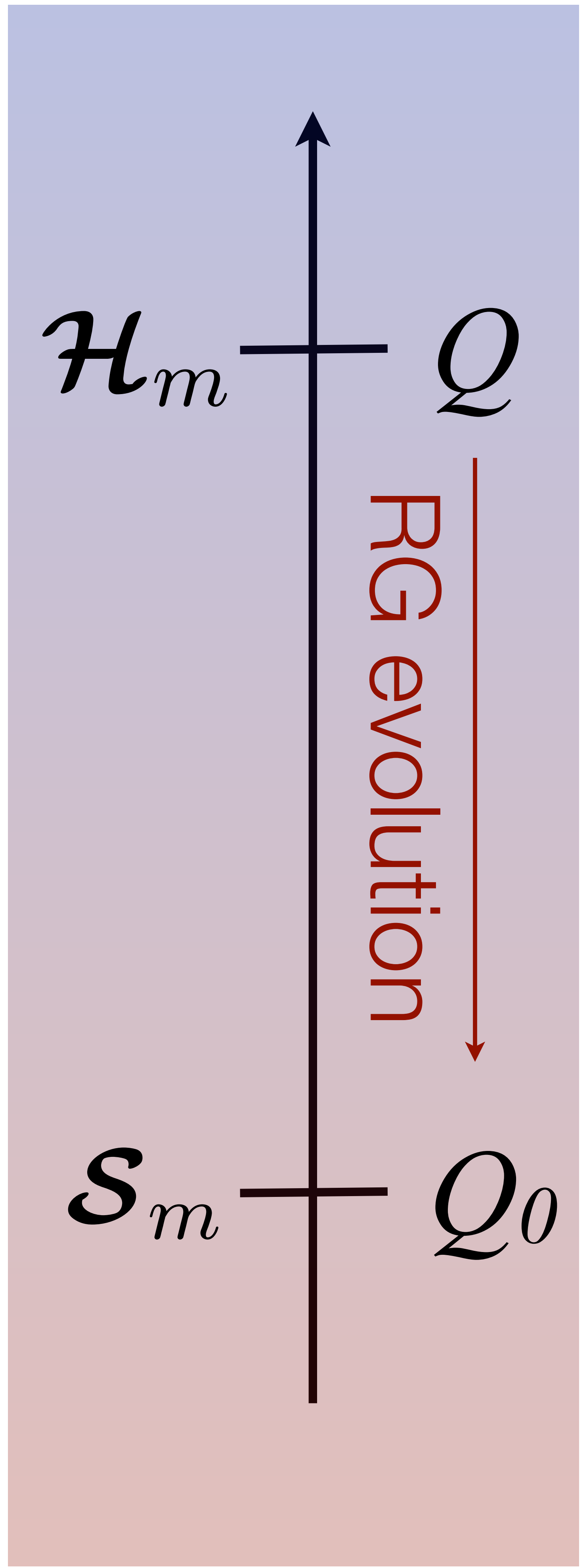
Resummation by RG evolution

Wilson coefficients fulfill RG equations

$$\frac{d}{d\ln\mu}\boldsymbol{\mathcal{H}}_m(Q,\mu) = -\sum_{l=2}^{m}\boldsymbol{\mathcal{H}}_l(Q,\mu)\,\boldsymbol{\Gamma}^H_{lm}(Q,\mu)$$

1. Compute $\mathcal{H}_m$ at a characteristic high scale $\mu_h \sim Q$
2. Evolve $\mathcal{H}_m$ to the scale of low energy physics $\mu_s \sim Q_0$
3. Evaluate $\mathcal{S}_m$ at low scale $\mu_s \sim Q_0$

Avoids large logarithms $\alpha_s^n \ln^n(Q/Q_0)$ of scale ratios which spoil convergence of perturbation theory.



RG = Parton Shower

- Ingredients for LL

$$\boldsymbol{\mathcal{H}}_2(\mu = Q) = \sigma_0$$
$$\boldsymbol{\mathcal{H}}_m(\mu = Q) = 0 \text{ for } m > 2$$
$$\boldsymbol{\mathcal{S}}_m(\mu = Q_0) = 1$$

$$\boldsymbol{\Gamma}^{(1)} = \begin{pmatrix} \boldsymbol{V}_2 & \boldsymbol{R}_2 & 0 & 0 & \dots \\ 0 & \boldsymbol{V}_3 & \boldsymbol{R}_3 & 0 & \dots \\ 0 & 0 & \boldsymbol{V}_4 & \boldsymbol{R}_4 & \dots \\ 0 & 0 & 0 & \boldsymbol{V}_5 & \dots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}$$

- RG

$$\frac{d}{dt}\boldsymbol{\mathcal{H}}_m(t) = \boldsymbol{\mathcal{H}}_m(t)\boldsymbol{V}_m + \boldsymbol{\mathcal{H}}_{m-1}(t)\boldsymbol{R}_{m-1}\,.$$

shower evolution time

$$t \equiv t(\mu_h, \mu_s) = \int_{\alpha_s(\mu_s)}^{\alpha_s(\mu_h)} \frac{d\alpha}{\beta(\alpha)}\frac{\alpha}{4\pi}$$

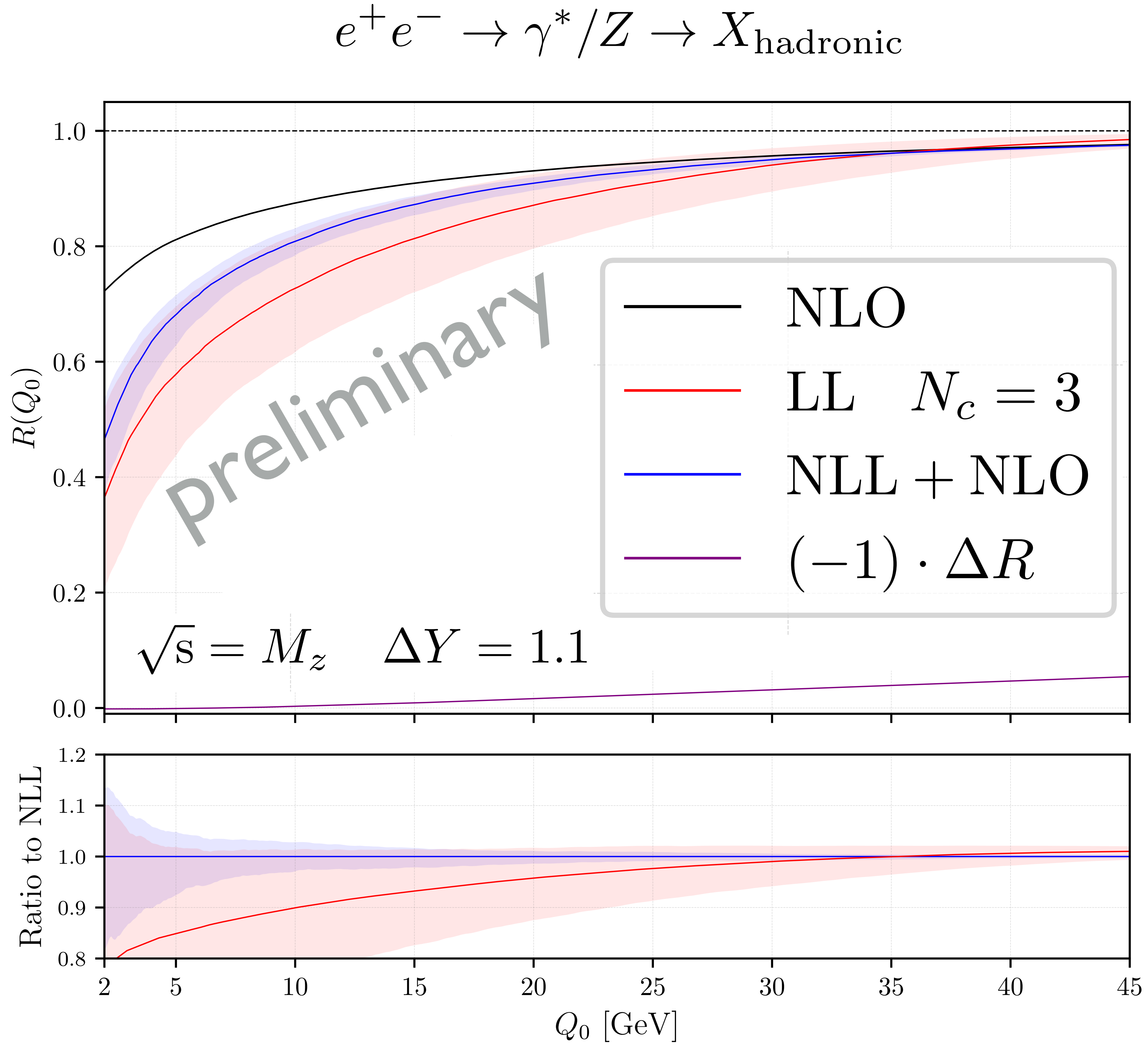
- equivalent to parton shower equation

$$\boldsymbol{\mathcal{H}}_m(t) = \boldsymbol{\mathcal{H}}_m(t_1)e^{(t-t_1)V_n} + \int_{t_1}^{t} dt'\boldsymbol{\mathcal{H}}_{m-1}(t')\boldsymbol{R}_{m-1}e^{(t-t')\boldsymbol{V}_n}$$

Progress on NGLs

- PanScales, a general-purpose shower, which correctly resums leading large-N_c NGLs (and global logs!) Dasgupta, Dreyer, Hamilton, Monni, Salam and Soyez '20, + … ,'21 Alaric Herren, Höche, Krauss, Reichelt, Schoenherr '22 → talks by Silvia and Daniel
- Finite-N_c results for leading NGLs in e^+e^- Hatta, Ueda '13 + Hagiwara '15 based on Weigert '03; De Angelis, Forshaw and Plätzer '20 → talk by Simon
- First NLL numerical results in the large-N_c limit
  - Extension of BMS framework to NLL (2104.06416) and numerical implementation in MC code `Gnole` (2111.02413) Banfi, Dreyer, Monni
  - Two-loop anomalous dimension in factorization framework TB, Rauh, Xu, 2112.02108; implementation into shower code TB, Schalch, Xu, in

Next-to-leading non-global logarithms



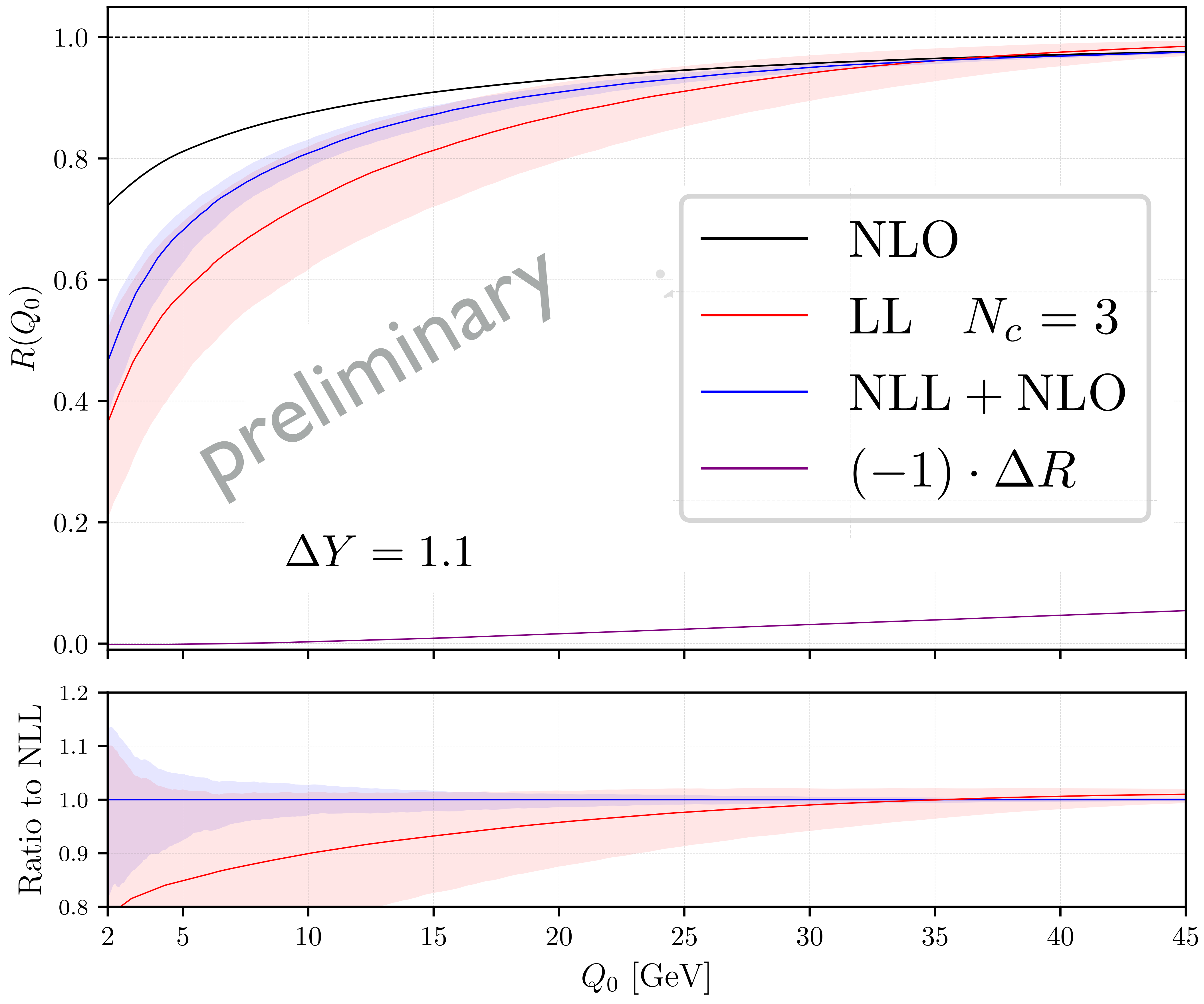
Ingredients:

- $N_c = 3$ leading logs obtained from Hatta, Ueda ’13.
- Two-loop anomalous dimension $\boldsymbol{\Gamma}^{(2)}$ TB, Rauh, Xu, ‘21
- Implementation of $\boldsymbol{\Gamma}^{(2)}$ in parton shower framework TB, Schalch, Xu, in preparation

Corrections scale as $\mathcal{O}(\alpha_s^2)$ or $\mathcal{O}(\alpha_s/N_c^2)$ terms.
First NGL resummation at this accuracy level!

NLL NGLs at hadron collider

$pp \to Z \to \ell^+\ell^- + X_{\rm had}$

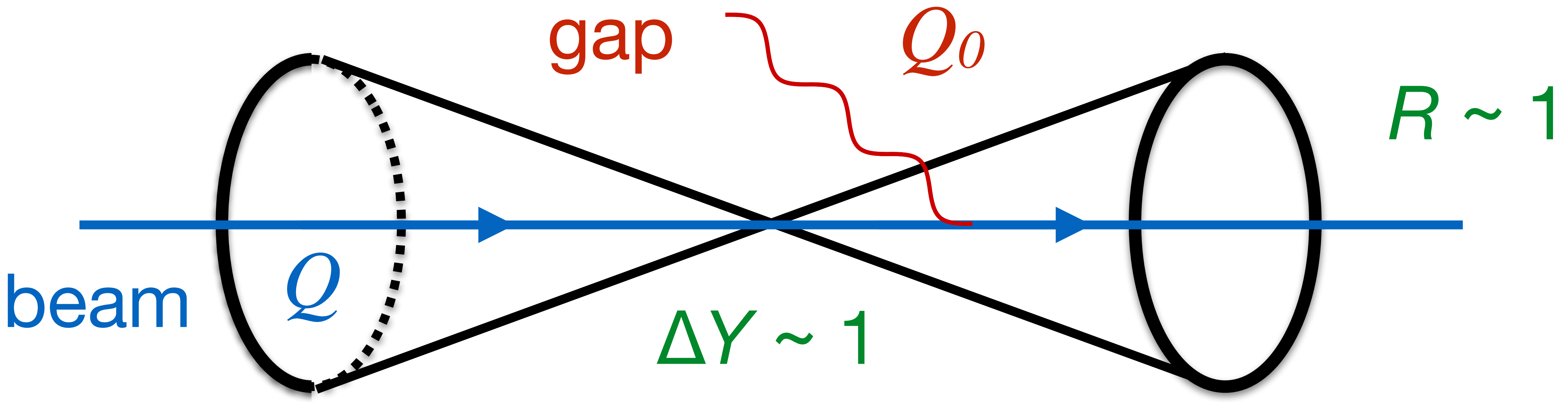


- Many ingredients the same as for e^+e^-
- $N_c = 3$ leading logs again from Hatta, Ueda '13.
  - superleading logs small for $qq \to Z$
- Hard functions related to partonic cross sections for $qq \to Z + g$, $qg \to Z + q$
- It would be great if this would be measured at LHC!
  - currently measured gap between dijet observables are theoretically more complicated, involve also collinear and forward logs

Super-Leading Logs (SLLs)

Forshaw, Kyrieleis, Seymour ’06 ’08

Analyze gap between jets at hadron collider, cone around beam direction



Large logarithms $\alpha_s^n L^m$ with $L = \ln(Q/Q_0)$

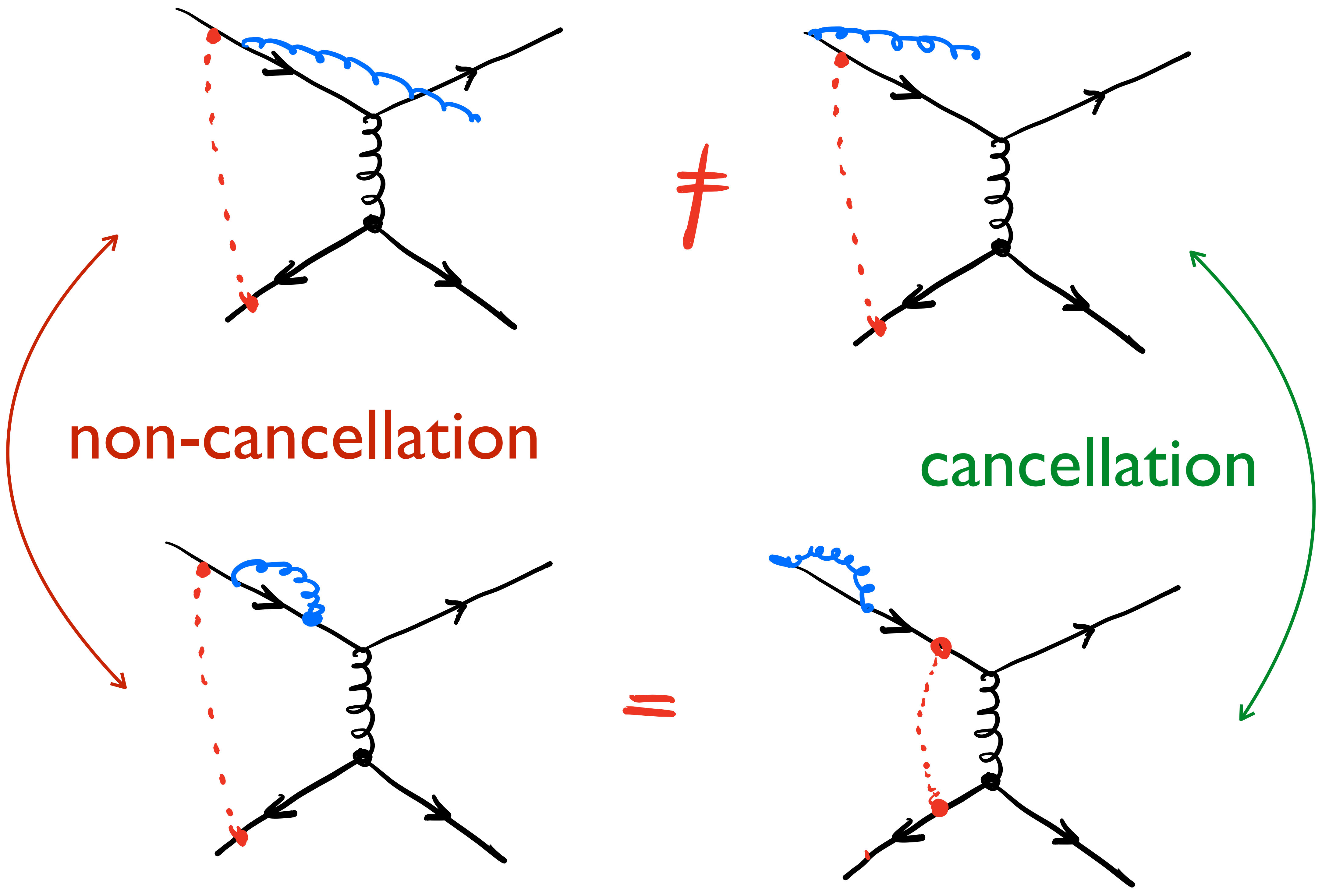
- e^+e^- : $m \leq n$, leading logs $m = n$
- pp : $\alpha_s L,\ \alpha_s^2 L^2,\ \alpha_s^3 L^3,\ \alpha_s^4 L^5 \ldots,\ \alpha_s^{3+n} L^{3+2n}$

missing in large-N_c parton showers!
(Deductor? Soper and Nagy … ’19)

Non-cancellation of collinear logs

Forshaw, Kyrieleis, Seymour ’06 ’08; Catani, de Florian, Rodrigo ’11, Schwartz, Yan, Zhu ’17…

Double logarithms due to soft+collinear configurations.



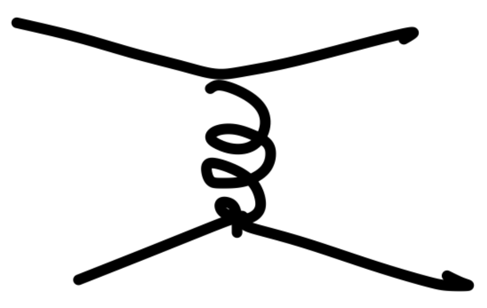
Blue: collinear emission. Red: Glauber/Coulomb phase

Note: Glauber phases cancel in e^+e^- and in large-N_c limit

Earlier results on SLLs

Since effect first arises at $O(\alpha_s^4)$, only few results

- Discovery of effect, computation of first SLL in gaps between jets for $qq \to qq$ Forshaw, Kyrieleis, Seymour '06
- Colour space calculation of leading SLL Forshaw, Kyrieleis, Seymour '08
  - Note that SLLs vanish in the large-N_c limit.
- Diagrammatic calculation, first *two* orders, different channels *qq, qg, gg* Keates and Seymour '09



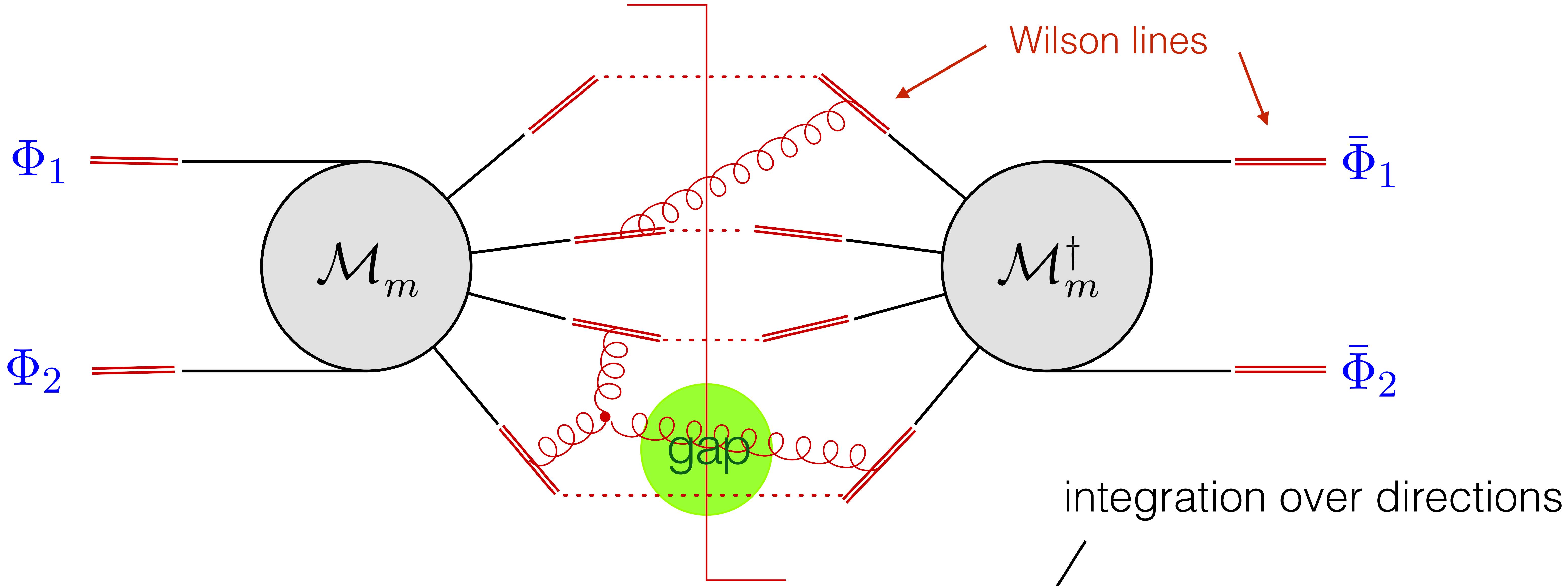
$$S_O^{(4)} = \left(\frac{\alpha_s}{4\pi}\right)^4 L_Q^5 \, \Delta Y \, \pi^2 \, \frac{8}{15} \left(3N_c^2 - 4\right) \sigma_0 \,,$$

$$S_O^{(5)} = \left(\frac{\alpha_s}{4\pi}\right)^5 L_Q^7 \, \Delta Y \, \pi^2 \, \frac{4}{315} N_c \left(-27N_c^2 + 44\right) \sigma_0$$

Glauber $(i\pi)^2$

Factorization for hadronic collisions

TB, Neubert, Shao Phys.Rev.Lett. 127 (2021) 21, 212002 + Stillger, in preparation



$$\sigma_{2\to M}(Q_0) = \int dx_1 \int dx_2 \sum_{m=2+M}^{\infty} \langle \boldsymbol{\mathcal{H}}_m(\{\underline{n}\}, x_1, x_2, s, \mu) \otimes \boldsymbol{\mathcal{W}}_m(\{\underline{n}\}, Q_0, x_1, x_2, \mu) \rangle$$

Hard functions

m hard partons along fixed directions $\{n_1, \ldots, n_m\}$

$\boldsymbol{\mathcal{H}}_m \propto |\mathcal{M}_m\rangle\langle\mathcal{M}_m|$

Soft + collinear function

squared amplitude for m Wilson lines +collinear fields

Remarks

Glauber
s+c interactions
Stewart, Rothstein '16

- Effective theory

$$\mathcal{L}_{\text{eff}} = \mathcal{L}_{\text{SCET}} = \mathcal{L}_{c_1} + \mathcal{L}_{c_2} + \mathcal{L}_s + \mathcal{L}_G$$

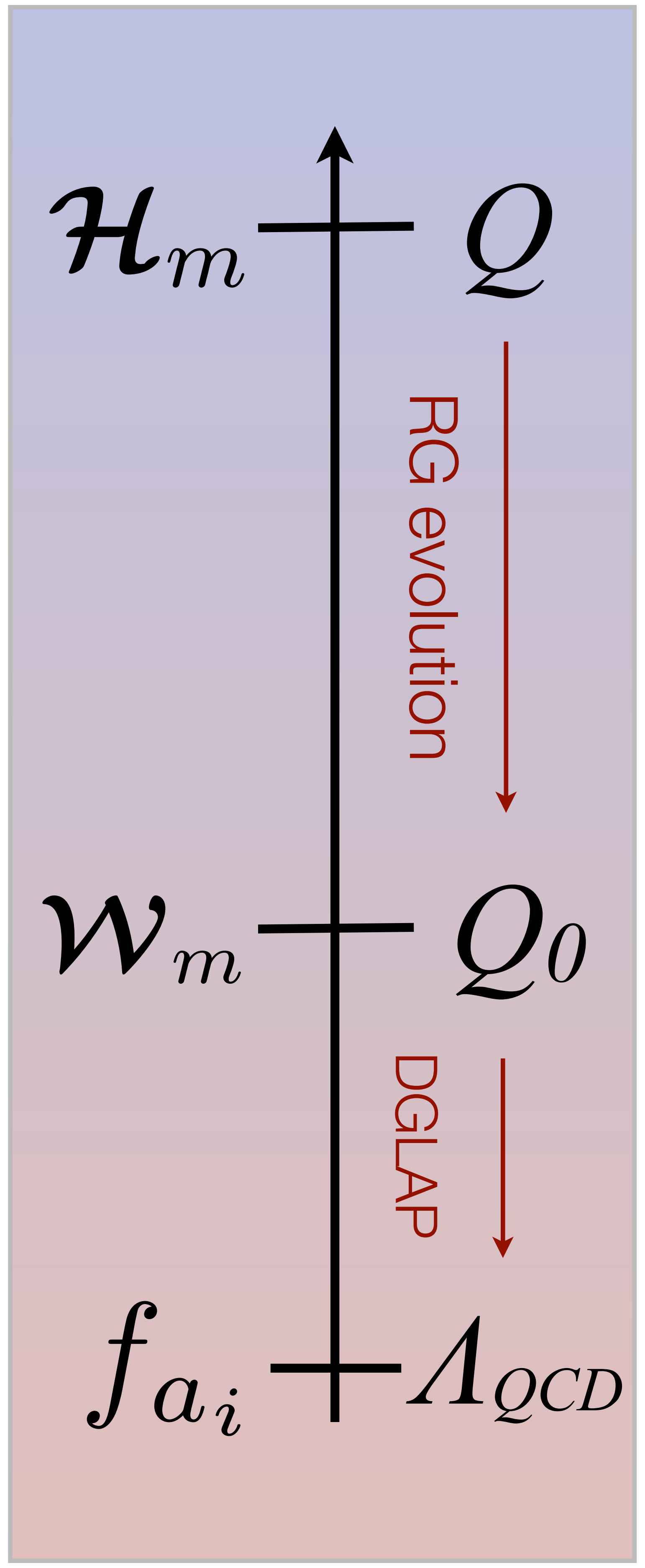
  - Additional regulator
  - Low energy matrix elements $\boldsymbol{\mathcal{W}}_m$ will suffer from rapidity logarithms

- RG evolution

Mellin convolution

$$\frac{d}{d\ln\mu}\boldsymbol{\mathcal{H}}_m(\{\underline{n}\}, s, \mu) = -\sum_{l=2+M}^{m} \boldsymbol{\mathcal{H}}_l(\{\underline{n}\}, s, \mu) \star \mathbf{\Gamma}^H_{lm}(\{\underline{n}\}, s, \mu)$$

  - mixes multiplicities + colors!



One-loop anomalous dimension

$$\mathbf{\Gamma}^H(\{\underline{n}\}, \xi_1, \xi_2, s, \mu) = \frac{\alpha_s}{4\pi}\mathbf{\Gamma}^{(1)} = \frac{\alpha_s}{4\pi}\begin{pmatrix} \boldsymbol{V}_k & \boldsymbol{R}_k & 0 & 0 & \dots \\ 0 & \boldsymbol{V}_{k+1} & \boldsymbol{R}_{k+1} & 0 & \dots \\ 0 & 0 & \boldsymbol{V}_{k+2} & \boldsymbol{R}_{k+2} & \dots \\ 0 & 0 & 0 & \boldsymbol{V}_{k+3} & \dots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}$$

k: number of partons at Born-level

- Split into soft(+collinear) and purely collinear

$$\mathbf{\Gamma}^{(1)}(\xi_1, \xi_2) = \mathbf{\Gamma}_1^C(\xi_1)\delta(1-\xi_2) + \delta(1-\xi_1)\mathbf{\Gamma}_2^C(\xi_2) + \delta(1-\xi_1)\,\delta(1-\xi_2)\,\mathbf{\Gamma}^S$$

- Split soft part

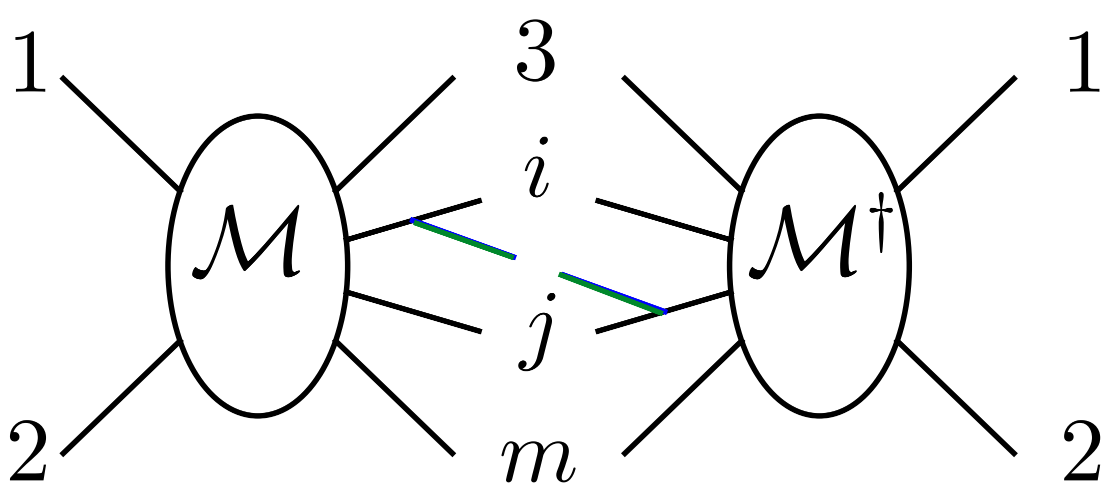
$$\mathbf{\Gamma}^S = \overline{\mathbf{\Gamma}} + \mathbf{\Gamma}^G + \mathbf{\Gamma}^c \ln\frac{\mu^2}{\hat{s}}$$

wide-angle soft — Glauber — cusp: soft+collinear

see also Forshaw, Holguin, and Plätzer '19

Soft wide-angle emissions $\overline{\Gamma}$

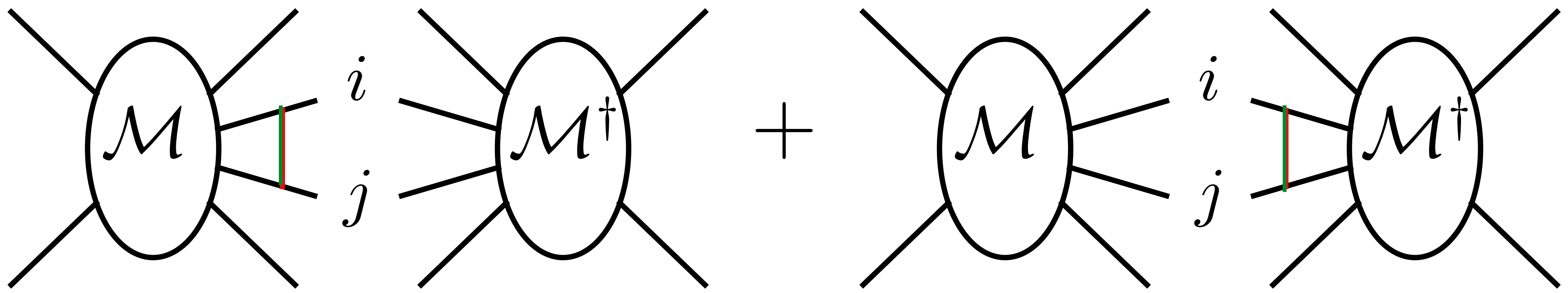
$$\mathcal{H}_m \overline{\boldsymbol{R}}_m = \sum_{(ij)}$$



$$\overline{\boldsymbol{R}}_m = -4 \sum_{(ij)} \boldsymbol{T}_{i,L} \circ \boldsymbol{T}_{j,R}\, \overline{W}_{ij}^{m+1}\, \Theta_{\rm hard}(n_{m+1})$$

extra hard parton!

$$\mathcal{H}_m \overline{\boldsymbol{V}}_m = \sum_{(ij)}$$



$$\overline{\boldsymbol{V}}_m = 2 \sum_{(ij)} \left(\boldsymbol{T}_{i,L} \cdot \boldsymbol{T}_{j,L} + \boldsymbol{T}_{i,R} \cdot \boldsymbol{T}_{j,R} \right) \int \frac{d\Omega(n_k)}{4\pi}\, \overline{W}_{ij}^{k}$$

soft dipole

$$W_{ij}^{q} = \frac{n_i \cdot n_j}{n_i \cdot n_q\, n_j \cdot n_q}$$

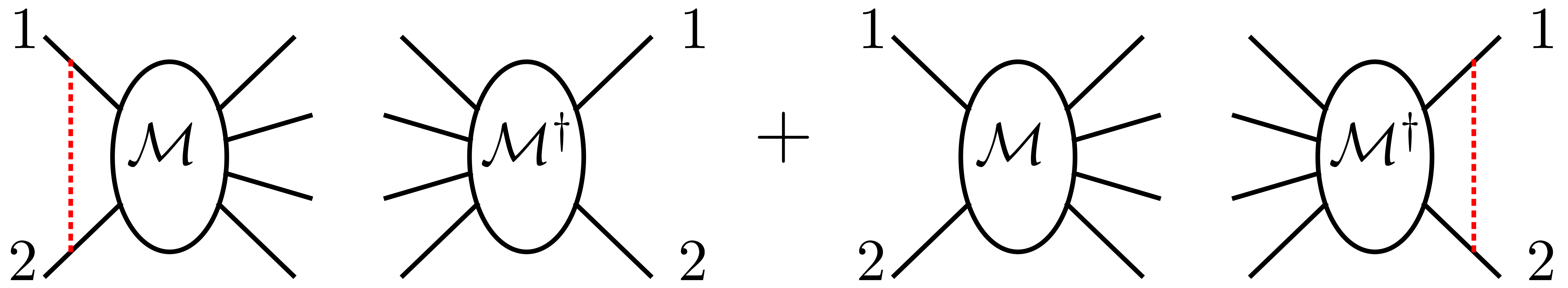
soft dipole with collinear subtraction

$$\overline{W}_{ij}^{q} = W_{ij}^{q} - \frac{1}{n_i \cdot n_q} \delta(n_i - n_q) - \frac{1}{n_j \cdot n_q} \delta(n_j - n_q)$$

see Forshaw, Holguin, and Plätzer ‘19

Glauber term Γ^G

$\mathcal{H}_m \, \boldsymbol{V}^G =$



$$\boldsymbol{V}^G = -8 i\pi \left(\boldsymbol{T}_{1,L} \cdot \boldsymbol{T}_{2,L} - \boldsymbol{T}_{1,R} \cdot \boldsymbol{T}_{2,R} \right)$$

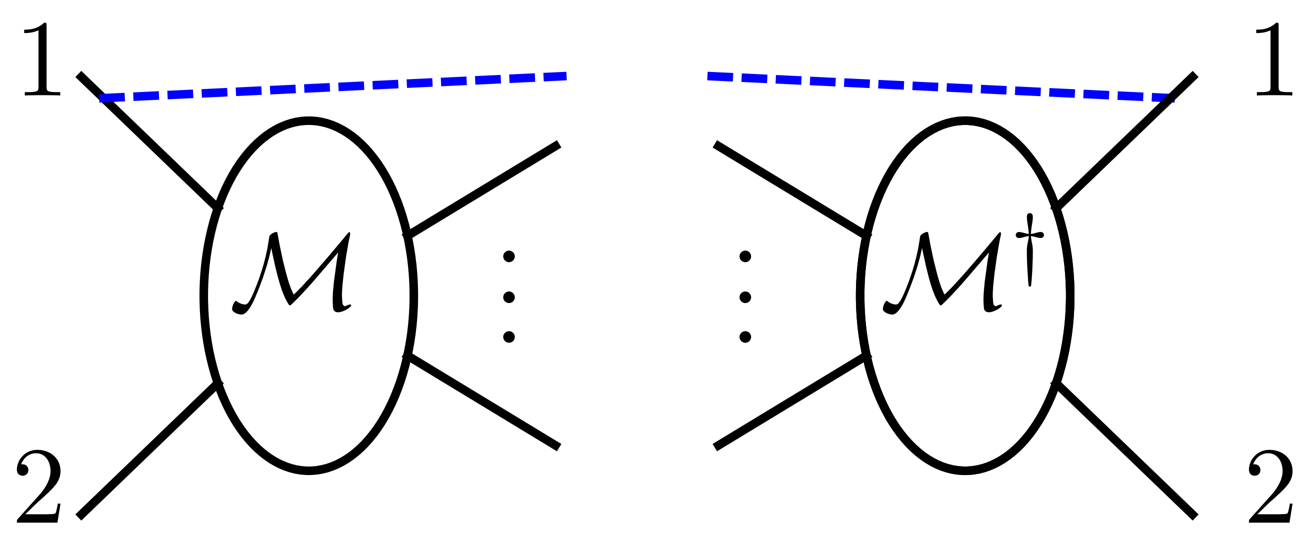
Used color conservation $\sum_i \boldsymbol{T}_i = 0$ to simplify Glauber terms in 1 + 2 → 3 + … + *m*

$\Pi_{ij} = 1$ if both inc./out.

$$\sum_{(ij)} \left(\boldsymbol{T}_{i,L} \cdot \boldsymbol{T}_{j,L} - \boldsymbol{T}_{i,R} \cdot \boldsymbol{T}_{j,R} \right) \Pi_{ij} = 4 \left(\boldsymbol{T}_{1,L} \cdot \boldsymbol{T}_{2,L} - \boldsymbol{T}_{1,R} \cdot \boldsymbol{T}_{2,R} \right)$$

(Soft+)Collinear Cusp Term $\boldsymbol{\Gamma}^c$

$$\boldsymbol{\mathcal{H}}_m \, \boldsymbol{R}_1^c \; =$$



$$\boldsymbol{R}_i^c = -4\boldsymbol{T}_{i,L} \circ \boldsymbol{T}_{i,R} \, \delta(n_{m+1} - n_i)$$

$$\boldsymbol{V}_i^c = 4C_i \, \mathbf{1}$$

- Only present for initial-state partons i=1,2. Final state terms cancel!
- Multiplied by $\ln \frac{\mu^2}{\hat{s}}$ → double logarithms!

Computation of SLLs

Cannot use large N_c: compute order by order

$$\left\langle \boldsymbol{\mathcal{H}}_4\, \boldsymbol{U}(\{\underline{n}\}, \mu_s, \mu_h) \hat{\otimes}\, \mathbf{1} \right\rangle = \left\langle \boldsymbol{\mathcal{H}}_4\, \mathbf{P} \exp\left[\int_{\mu_s}^{\mu_h} \frac{d\mu}{\mu} \boldsymbol{\Gamma}(\{\underline{n}\}, \mu) \right] \hat{\otimes}\, \mathbf{1} \right\rangle$$

$$= \langle \boldsymbol{\mathcal{H}}_4 \rangle + \int_{\mu_s}^{\mu_h} \frac{d\mu}{\mu} \langle \boldsymbol{\mathcal{H}}_4\, \boldsymbol{\Gamma}(Q,\mu) \hat{\otimes} \mathbf{1} \rangle + \int_{\mu_s}^{\mu_h} \frac{d\mu}{\mu} \int_{\mu}^{\mu_h} \frac{d\mu'}{\mu'} \langle \boldsymbol{\mathcal{H}}_4 \boldsymbol{\Gamma}(Q,\mu)\, \boldsymbol{\Gamma}(Q,\mu') \hat{\otimes}\, \mathbf{1} \rangle + \dots$$

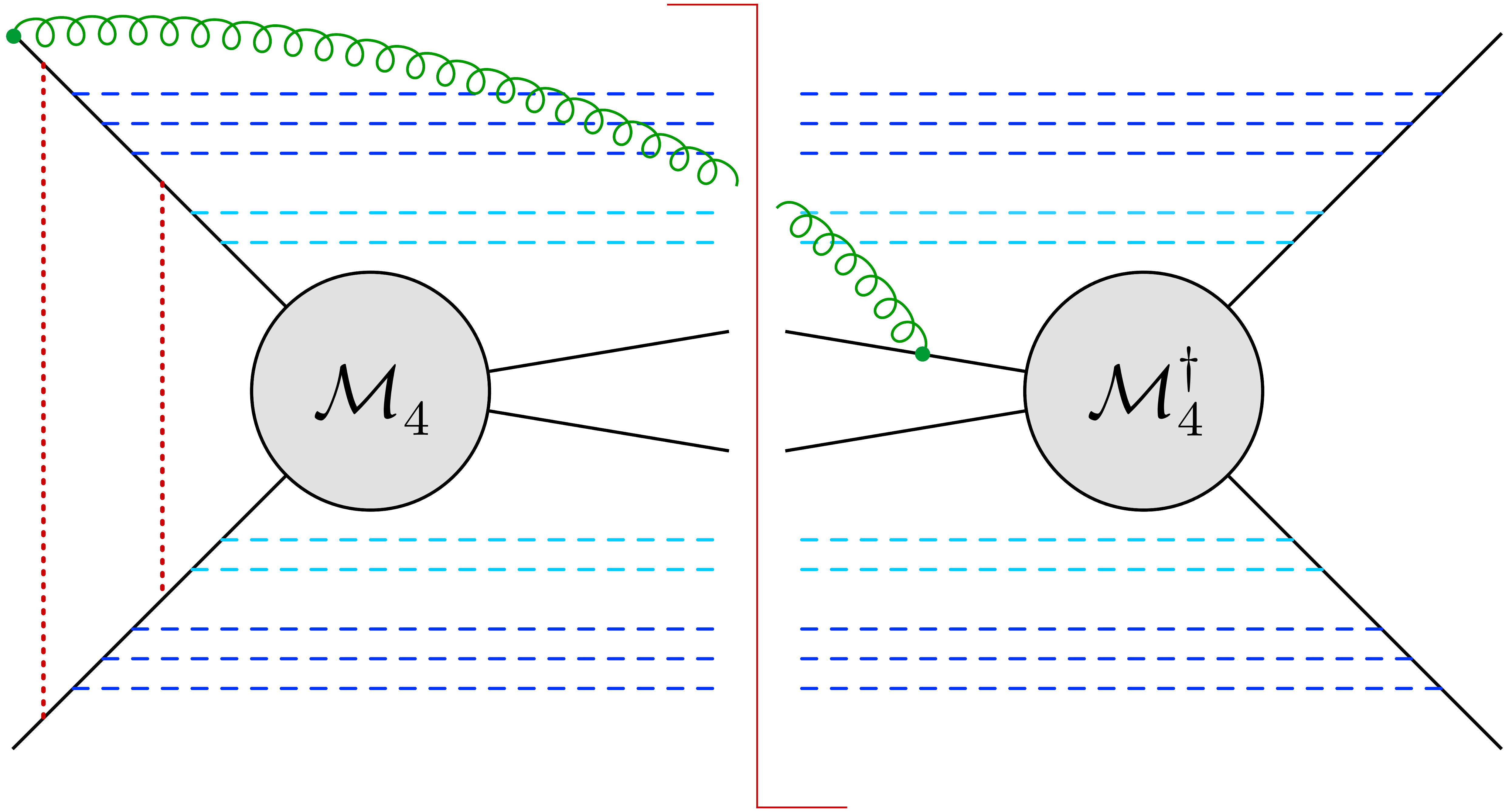
$\hat{\sigma}_{\mathrm{LO}}$ $\quad\quad$ $\alpha_s L$ $\quad\quad$ $\alpha_s^2 L^2$

Need products of anomalous dimensions. Each μ integral produces single log ($\overline{\boldsymbol{\Gamma}}$, $\boldsymbol{\Gamma}^G$) or double logs ($\boldsymbol{\Gamma}^c$), i.e. SLLs!

Will set μ_h=Q and μ_s=Q_0 and ignore running of α_s.

$$C_{r,n} = \langle \mathcal{H}_4 \, (\mathbf{\Gamma}^c)^r \, \mathbf{\Gamma}^G (\mathbf{\Gamma}^c)^{n-r} \, \mathbf{\Gamma}^G \, \overline{\mathbf{\Gamma}} \otimes \mathbf{1} \rangle \qquad 0 \leq r \leq n$$

$$C_{4,10} =$$



Properties

$$[\mathbf{\Gamma}^c, \overline{\mathbf{\Gamma}}] = 0$$

$$\langle \mathcal{H}_m \, \mathbf{\Gamma}^c \otimes \mathbf{1} \rangle = 0$$

$$\langle \mathcal{H}_m \, \boldsymbol{V}^G \otimes \mathbf{1} \rangle = 0$$

+ many more diagrams: Glauber(s) on the right side, different attachents for wide-angle soft, virtuals …

Evaluation of C_{rn}

- Basic strategy: commute $\mathbf{\Gamma}^c$'s and $\mathbf{\Gamma}^G$ to the right where they vanish.
- After a lot of color algebra, one finds

$$C_{rn} = -16\,(4\pi)^2\,(4N_c)^n \sum_{i=1}^{7} v_i^r \,\langle \boldsymbol{\mathcal{H}}_{2\to M}\,\boldsymbol{Q}_i \rangle$$

power-like *n* and *r* dependence

  - Eigenvalues

$$v_1 = 0\,, \quad v_2 = \frac{1}{2}\,, \quad v_3 = 1\,, \quad v_4 = \frac{3N_c - 2}{2N_c}\,, \quad v_5 = \frac{3N_c + 2}{2N_c}\,, \quad v_6 = \frac{2\,(N_c - 1)}{N_c}\,, \quad v_7 = \frac{2\,(N_c + 1)}{N_c}$$

  - Eigenoperators are $\boldsymbol{Q}_i$ are combinations color 10 basic structures.

Eigenoperators, color structures O_i and S_i

$$\boldsymbol{Q}_1 = J_{12}\left[\frac{4N_c}{N_c^2-1}\,C_1C_2\,\boldsymbol{S}_6\right],$$

$$\boldsymbol{Q}_2 = \sum_{j=3}^{M+2} J_j\left[-\frac{N_c}{N_c^2-1}\,\boldsymbol{O}_4^{(j)}\right] + J_{12}\left[\frac{2N_c}{N_c^2-1}\,(C_1+C_2)\,\boldsymbol{S}_5 - \frac{4N_c}{N_c^2-1}\,C_1C_2\,\boldsymbol{S}_6\right],$$

$$\boldsymbol{Q}_3 = \sum_{j=3}^{M+2} J_j\left[-\frac{N_c^2}{2(N_c^2-4)}\,\boldsymbol{O}_2^{(j)}\right] + J_{12}\left[\frac{N_c^2}{N_c^2-4}\,\boldsymbol{S}_3 - \frac{N_c^2}{3}\,\boldsymbol{S}_5\right]$$

$$\boldsymbol{Q}_4 = \sum_{j=3}^{M+2} J_j\left[\frac{1}{2}\,\boldsymbol{O}_1^{(j)} + \frac{N_c}{4(N_c-2)}\,\boldsymbol{O}_2^{(j)} - \frac{1}{2}\,\boldsymbol{O}_3^{(j)} + \frac{1}{2(N_c-1)}\,\boldsymbol{O}_4^{(j)}\right]$$
$$+ J_{12}\left[\frac{1}{2}\,\boldsymbol{S}_1 + \frac{N_c}{4(N_c-2)}\,\boldsymbol{S}_2 - \frac{N_c}{2(N_c-2)}\,\boldsymbol{S}_3 - \frac{1}{2}\,\boldsymbol{S}_4\right.$$
$$\left. + \left((C_1+C_2)\,\frac{N_c-2}{N_c-1} + \frac{N_c(N_c-4)}{6}\right)\boldsymbol{S}_5 + \frac{2C_1C_2}{N_c-1}\,\boldsymbol{S}_6\right],$$

$$\boldsymbol{Q}_5 = \sum_{j=3}^{M+2} J_j\left[\frac{1}{2}\,\boldsymbol{O}_1^{(j)} + \frac{N_c}{4(N_c+2)}\,\boldsymbol{O}_2^{(j)} + \frac{1}{2}\,\boldsymbol{O}_3^{(j)} + \frac{1}{2(N_c+1)}\,\boldsymbol{O}_4^{(j)}\right]$$
$$+ J_{12}\left[\frac{1}{2}\,\boldsymbol{S}_1 + \frac{N_c}{4(N_c+2)}\,\boldsymbol{S}_2 - \frac{N_c}{2(N_c+2)}\,\boldsymbol{S}_3 + \frac{1}{2}\,\boldsymbol{S}_4\right.$$
$$\left. + \left(-(C_1+C_2)\,\frac{N_c+2}{N_c+1} + \frac{N_c(N_c+4)}{6}\right)\boldsymbol{S}_5 + \frac{2C_1C_2}{N_c+1}\,\boldsymbol{S}_6\right],$$

$$\boldsymbol{Q}_6 = -J_{12}\left[\frac{1}{2}\,\boldsymbol{S}_1 + \frac{N_c}{4(N_c-2)}\,\boldsymbol{S}_2 - \frac{1}{2}\,\boldsymbol{S}_4 + \frac{2C_1C_2}{N_c-1}\,\boldsymbol{S}_6\right],$$

$$\boldsymbol{Q}_7 = -J_{12}\left[\frac{1}{2}\,\boldsymbol{S}_1 + \frac{N_c}{4(N_c+2)}\,\boldsymbol{S}_2 + \frac{1}{2}\,\boldsymbol{S}_4 + \frac{2C_1C_2}{N_c+1}\,\boldsymbol{S}_6\right].$$

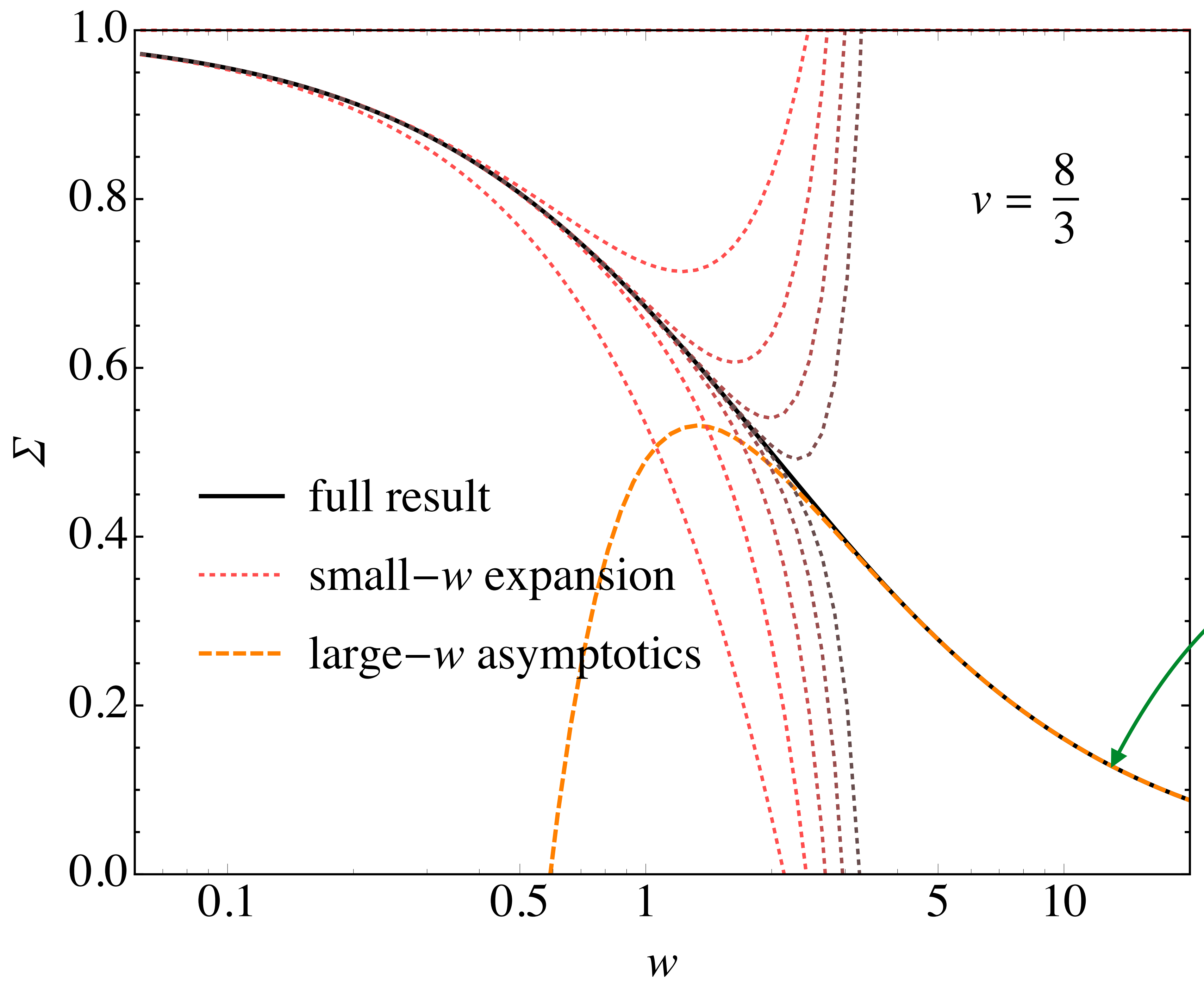
$$\boldsymbol{S}_1 = f_{abe}\,f_{cde}\,\{\boldsymbol{T}_1^b,\boldsymbol{T}_1^c\}\,\{\boldsymbol{T}_2^a,\boldsymbol{T}_2^d\}\,,$$
$$\boldsymbol{S}_2 = d_{ade}\,d_{bce}\,\{\boldsymbol{T}_1^b,\boldsymbol{T}_1^c\}\,\{\boldsymbol{T}_2^a,\boldsymbol{T}_2^d\}\,,$$
$$\boldsymbol{S}_3 = d_{ade}\,d_{bce}\left[\boldsymbol{T}_2^a\left(\boldsymbol{T}_1^b\boldsymbol{T}_1^c\boldsymbol{T}_1^d\right)_+ + (1\leftrightarrow 2)\right],$$
$$\boldsymbol{S}_4 = \{\boldsymbol{T}_1^a,\boldsymbol{T}_1^b\}\,\{\boldsymbol{T}_2^a,\boldsymbol{T}_2^b\}\,,$$
$$\boldsymbol{S}_5 = \boldsymbol{T}_1\cdot\boldsymbol{T}_2\,,$$
$$\boldsymbol{S}_6 = \boldsymbol{1}\,.$$

$$\boldsymbol{O}_1^{(j)} = f_{abe}\,f_{cde}\,\boldsymbol{T}_2^a\,\{\boldsymbol{T}_1^b,\boldsymbol{T}_1^c\}\,\boldsymbol{T}_j^d - (1\leftrightarrow 2)\,,$$
$$\boldsymbol{O}_2^{(j)} = d_{ade}\,d_{bce}\,\boldsymbol{T}_2^a\,\{\boldsymbol{T}_1^b,\boldsymbol{T}_1^c\}\,\boldsymbol{T}_j^d - (1\leftrightarrow 2)\,,$$
$$\boldsymbol{O}_3^{(j)} = \boldsymbol{T}_2^a\,\{\boldsymbol{T}_1^a,\boldsymbol{T}_1^b\}\,\boldsymbol{T}_j^b - (1\leftrightarrow 2)\,,$$
$$\boldsymbol{O}_4^{(j)} = 2C_1\,\boldsymbol{T}_2\cdot\boldsymbol{T}_j - 2C_2\,\boldsymbol{T}_1\cdot\boldsymbol{T}_j\,.$$

$$J_j = \int\frac{d\Omega(n_k)}{4\pi}\left(W_{1j}^k - W_{2j}^k\right)\Theta_{\text{gap}}(n_k)$$

TB, Neubert, Stillger, Shao, in preparation

All-order summation



$$w = \frac{N_c \alpha_s(\bar{\mu})}{\pi} \ln^2\left(\frac{\mu_h}{\mu_s}\right)$$

$$\Sigma(v, w) \sim \frac{1}{w}$$

- Combine C_{rn} with μ-Integrals, we can sum SLLs to all orders using

$$\Sigma(v,w) = \sum_{n=0}^{\infty} \sum_{r=0}^{n} \frac{(-4)^n \, 3! \, n!}{(2n+3)!} \frac{(2r)!}{4^r \, (r!)^2} v^r w^n = {}^{1+1}F_{2+0}\left(\begin{matrix} 1 : 1, \frac{1}{2} ; \\ 2, \frac{5}{2} : \quad ; \end{matrix} - w, -vw \right)$$

Resummed result

TB, Neubert, Shao Phys.Rev.Lett. 127 (2021) 21, 212002

Combine C_{rn} with μ integrals and carry out the sums.

Simplest case is $qq \to qq$ scattering with photon exchange

$J_i = \pm \Delta Y$

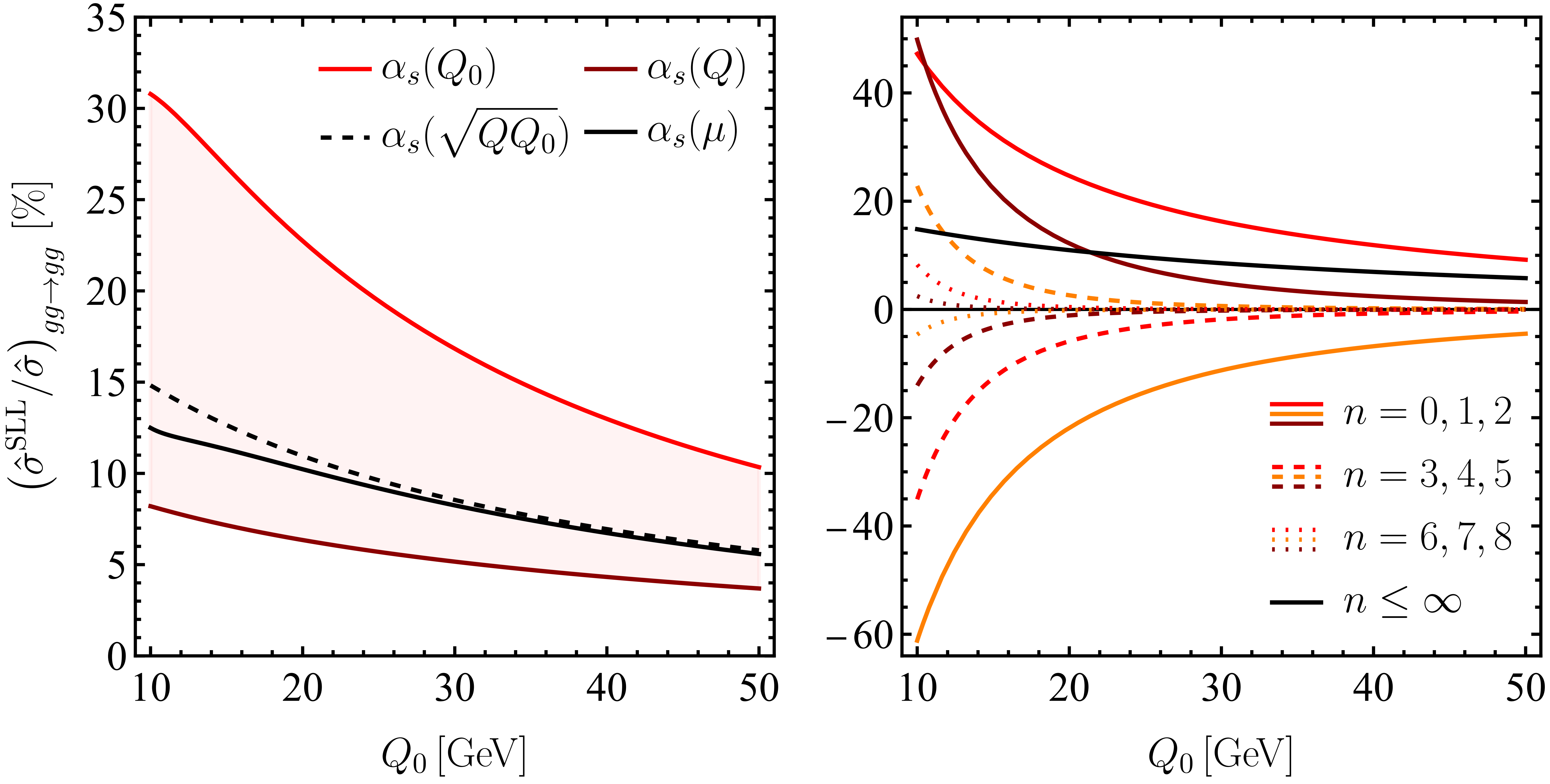
$$\Delta\hat{\sigma}^{(S)} = -\hat{\sigma}_B \frac{4C_F}{3\pi} \alpha_s^3 L^3 \Delta Y \, {}_2F_2\left(1, 1; 2, \tfrac{5}{2}; -w\right)$$

with $w = \frac{N_c \alpha_s}{\pi} L^2$.

$\sim \frac{\ln w}{w}$ for large w

Note: Standard Sudakov has form e^{-cw}.

Numerical results



- LL has *O(1)* uncertainty, e.g. running of α_s is beyond LL but significant!
- Strong cancellations among orders, especially for 2⟶0 and 2⟶1
  - typical for badly convergent expansions

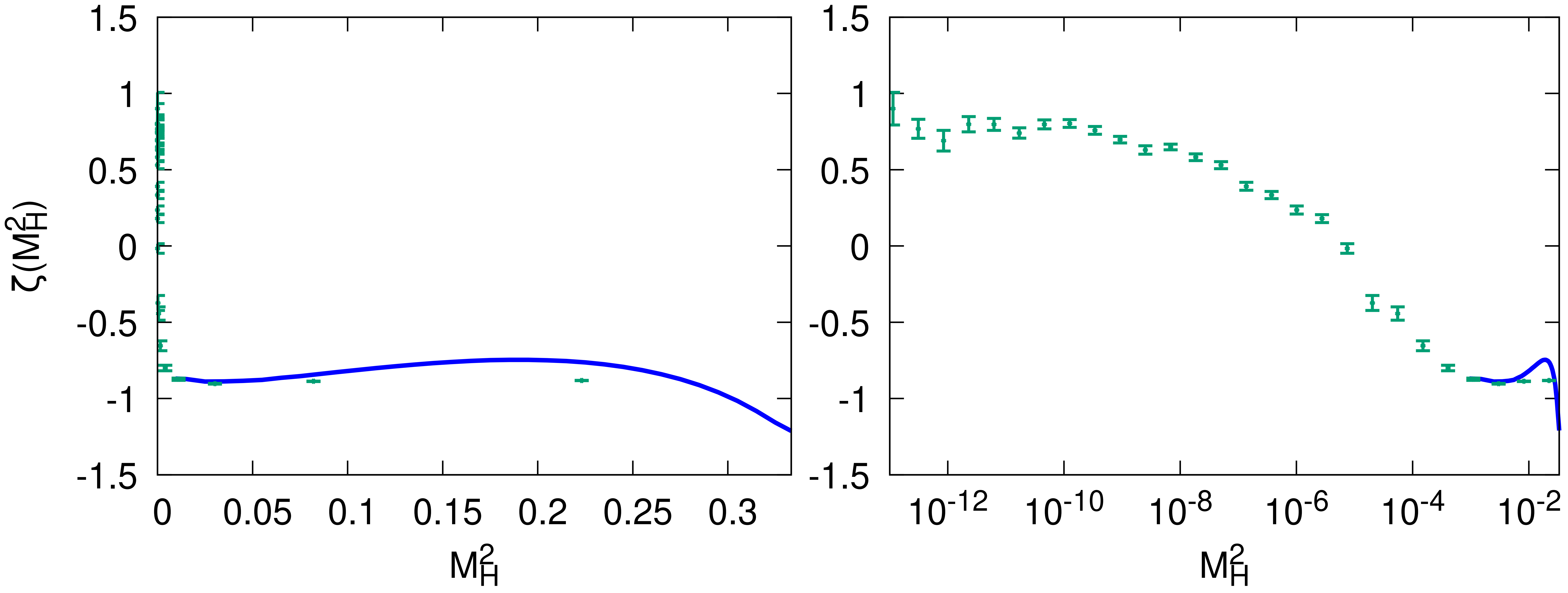
Summary and Outlook

- Interesting new insights into event shapes in e^+e^-, but problems in α_s determination not fully settled, in my opinion
- Precise α_s from q_T spectrum in Z-production
- Energy correlators are a promising new tool for collider physics
    - many new calculations and ideas + first measurements at LHC
- First resummations of subleading NGLs and leading SLLs are becoming available. Next steps and open questions
    - phenomenological applications
    - analysis of low-energy matrix elements, Glauber contributions, NGL $\times$ SLL, ….

Extra slides

Variation	$\alpha_s(M_Z)$	α_0	χ^2	χ^2/N_{deg}
Default setup	**0.1182**	**0.64**	**7.3**	**0.17**
Renormalization scale $Q/4$	0.1202	0.60	9.1	0.21
Renormalization scale Q	0.1184	0.68	8.7	0.20
NP scheme (b)	0.1198	0.77	7.0	0.16
NP scheme (c)	0.1206	0.80	5.4	0.12
NP scheme (d)	0.1194	0.66	5.8	0.13
P-scheme	0.1158	0.62	10.7	0.24
D-scheme	0.1198	0.79	5.7	0.13
Standard scheme	0.1176	0.58	9.2	0.21
No heavy-to-light correction	0.1186	0.67	6.8	0.16
`Herwig6`	0.1180	0.59	15.9	0.36
`Herwig7`	0.1180	0.60	12.0	0.27
Ranges (2)	0.1174	0.62	12.7	0.23
Ranges (3)	0.1188	0.69	2.7	0.08
Replica method (around average)	0.1192	0.61	7.0	0.16
Replica method (around default)	0.1192	0.61	7.0	0.16
y_3 clustered	0.1174	0.66	8.2	0.19
C	0.1256	0.48	1.3	0.07
τ	0.1194	0.64	0.8	0.04
y_3	0.1214	1.81	0.2	0.02
C, τ	0.1238	0.51	2.6	0.07

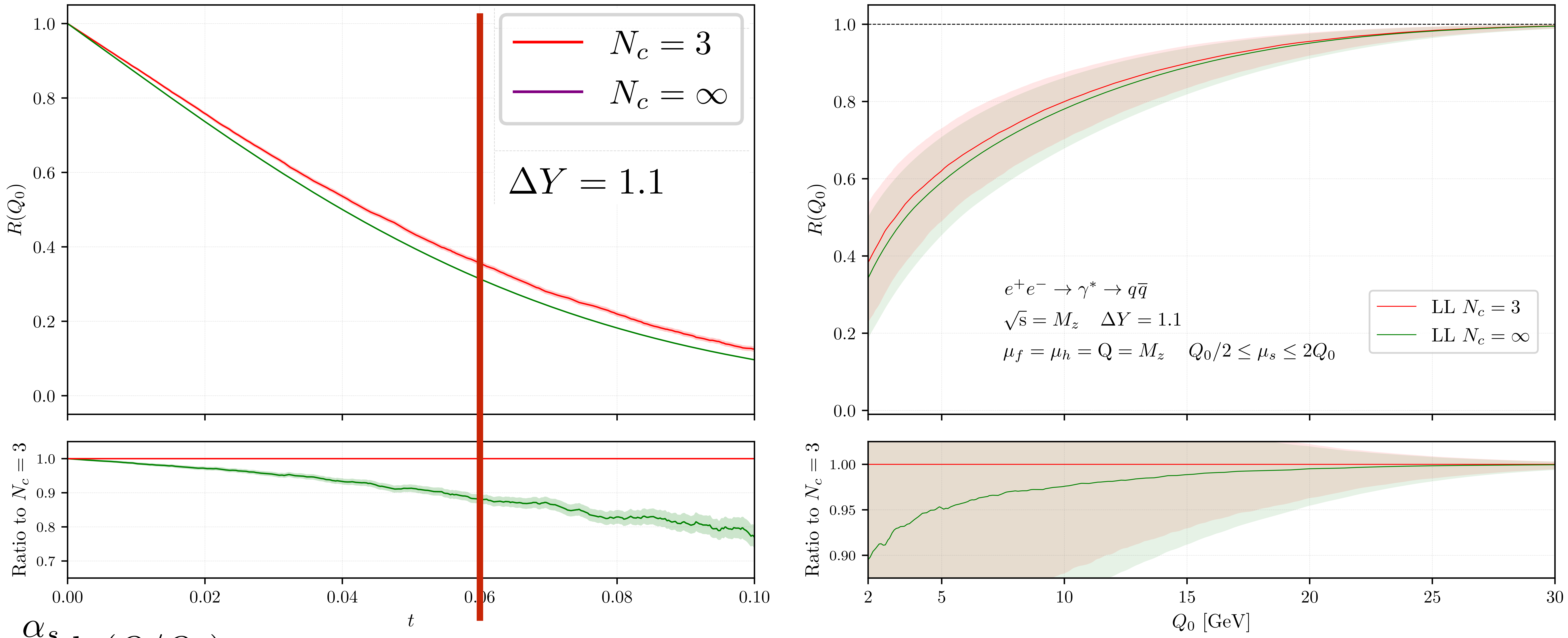
Nason, Zanderighi ‘23



Nason, Zanderighi '23

Large N_c versus $N_c = 3$

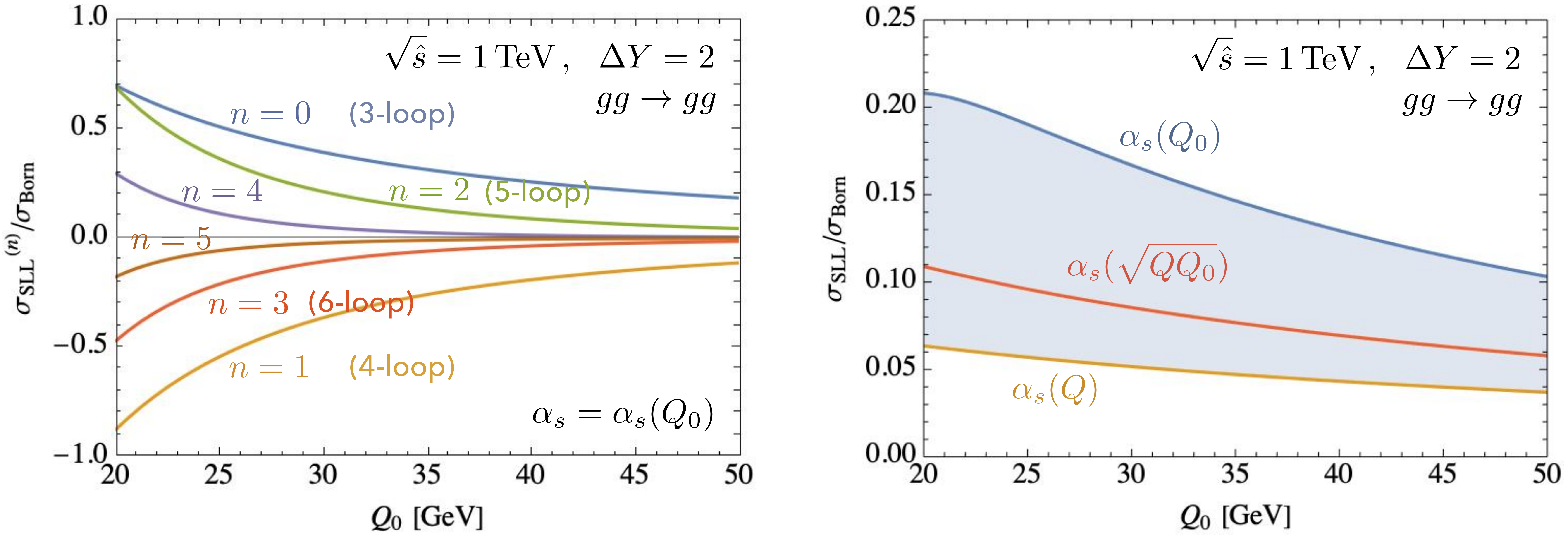
Hatta, Ueda '13



$$t = \frac{\alpha_s}{4\pi} \ln(Q/Q_0)$$

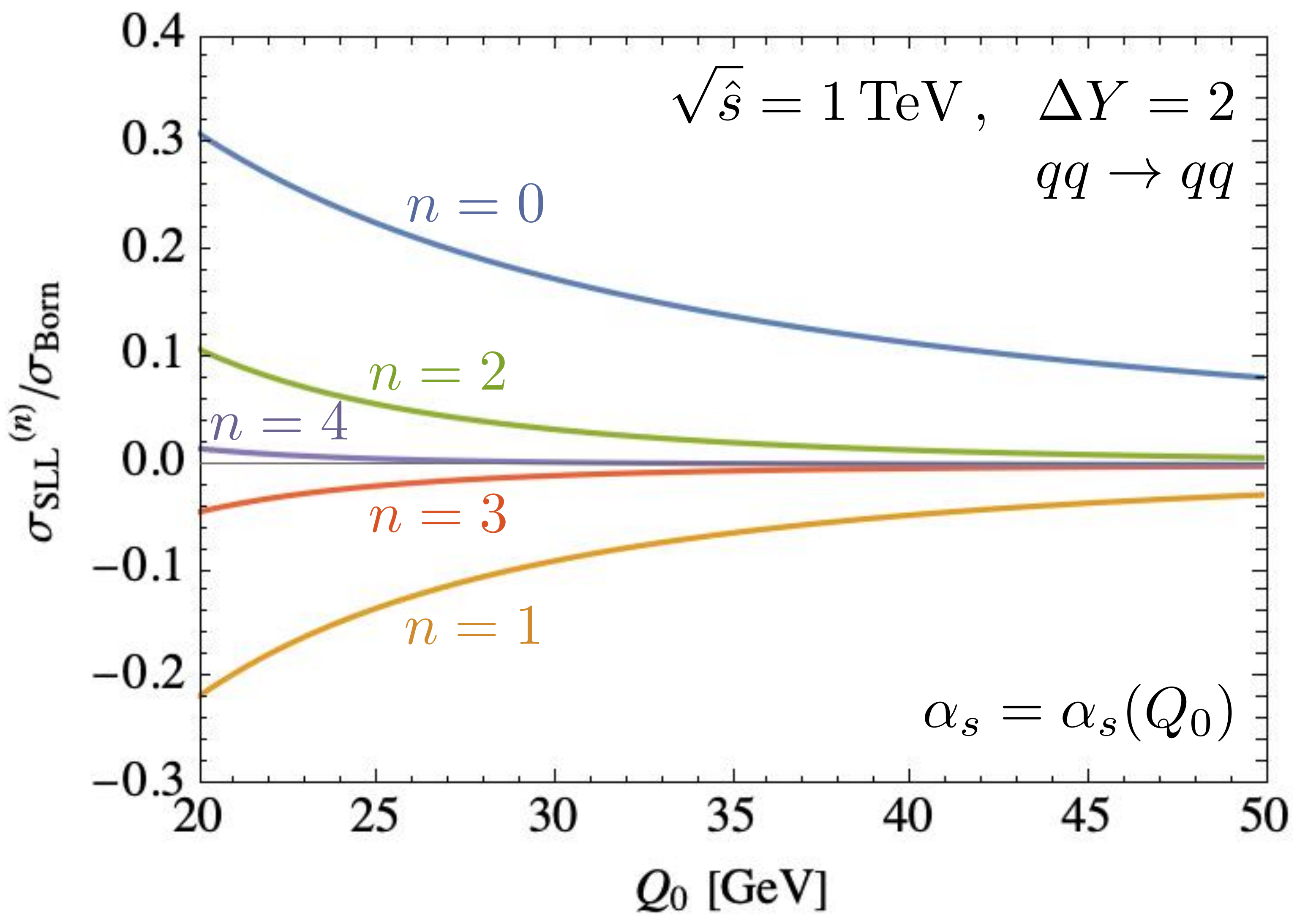
$Q_0 \sim 2$

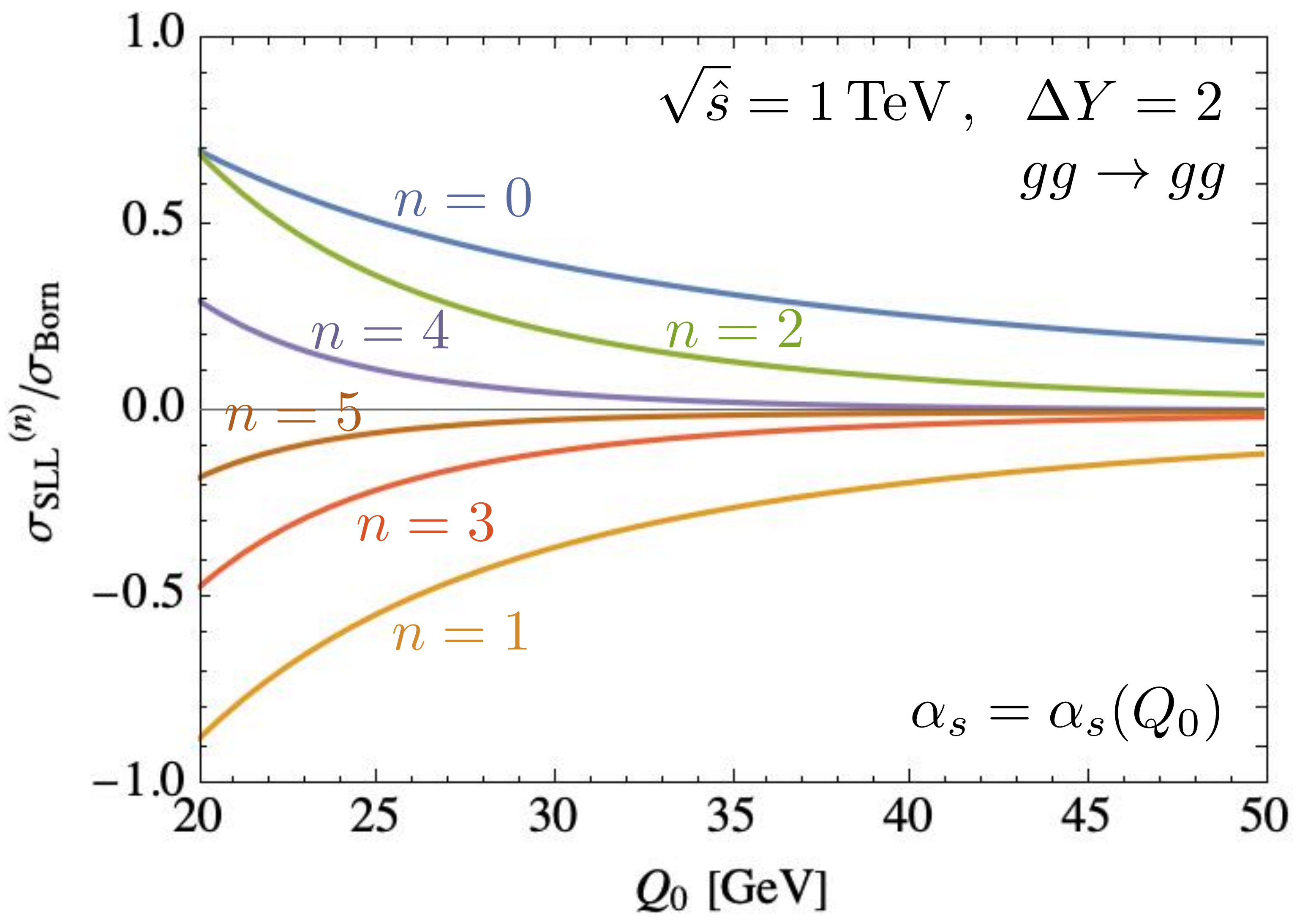
Forward gluon-gluon scattering



- Slow convergence (w~2): necessary to include eight terms (10 loops!) to converge to resummed result
- *Very sensitive* to choice of μ in α_s: should include running!

Terms of order $\alpha_s^{n+3} L^{2n+3}$





- Hard function for gluon exchange in t-channel.
- *n*=0 term is not SLL, but missing in large N_c limit.

Soft emissions are obtained from the matrix elements of the Wilson-line operators

$$\boldsymbol{S}_1(n_1)\,\boldsymbol{S}_2(n_2)\,\ldots\,\boldsymbol{S}_m(n_m)|\mathcal{M}_m(\{\underline{p}\})\rangle$$

Wilson lines: $\boldsymbol{S}_i(n_i) = \exp\left[ig_s \int_0^\infty ds\, n_i \cdot A^a(sn_i)\,\boldsymbol{T}_i^a\right]$

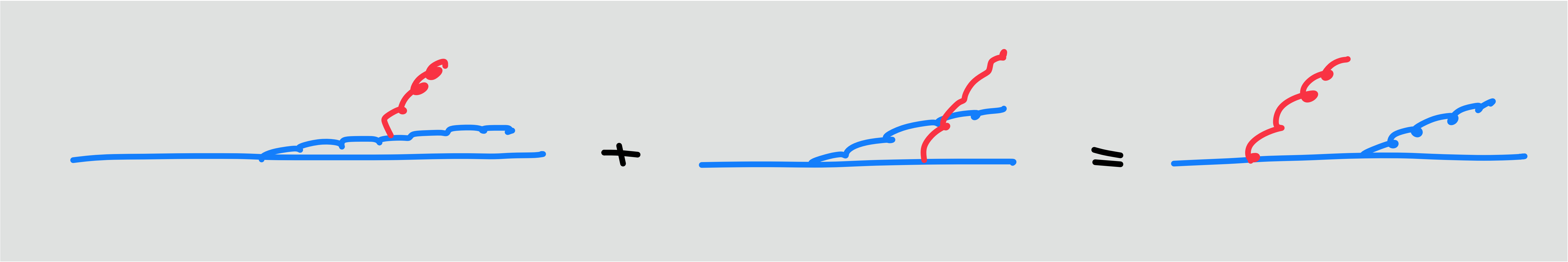
(For outgoing particle! Incoming has integration from -∞ to 0)

To get the amplitudes with additional soft partons, one takes the matrix element of the multi-Wilson-line operators:

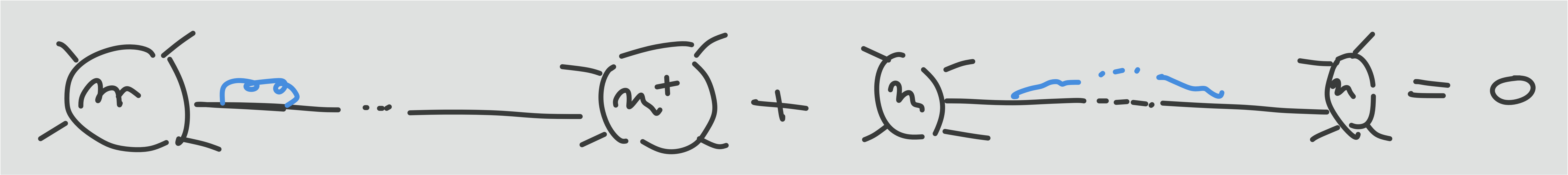
$$\langle X_s|\,\boldsymbol{S}_1(n_1)\,\ldots\,\boldsymbol{S}_m(n_m)\,|0\rangle$$

Basic properties of Γ

- Color coherence: $\mathcal{H}_m \, \boldsymbol{\Gamma}^c \, \overline{\boldsymbol{\Gamma}} = \mathcal{H}_m \, \overline{\boldsymbol{\Gamma}} \, \boldsymbol{\Gamma}^c$



- Collinear safety $\langle \mathcal{H}_m \, \boldsymbol{\Gamma}^c \otimes \mathbf{1} \rangle = 0$



$$\langle \mathcal{H}_m \, (\boldsymbol{R}_i^c + \boldsymbol{V}_i^c) \otimes 1 \rangle \propto \langle \boldsymbol{T}_i^a \, \mathcal{H}_m \, \boldsymbol{T}_i^a - C_i \, \mathcal{H}_m \rangle = 0$$ cyclicity of trace

- $\langle \mathcal{H}_m \, \boldsymbol{V}^G \otimes \mathbf{1} \rangle = 0$ cyclicity of trace

Leading SLLs

1. Want maximum number of $\boldsymbol{\Gamma}^c$'s at given order.
2. Need $\boldsymbol{\Gamma}^G$ to prevent $\boldsymbol{\Gamma}^c$ from commuting to the right and vanishing. Two insertions of $\boldsymbol{\Gamma}^G$ since cross section is real.
3. Need one emission $\overline{\boldsymbol{\Gamma}}$ at the end to prevent $\boldsymbol{\Gamma}^G$ from vanishing

Taken together, this implies that the leading SLLs at (n+3)-rd order arise from matrix elements

$$C_{rn} = \left\langle \boldsymbol{\mathcal{H}}_4 \left(\boldsymbol{\Gamma}^c\right)^r \boldsymbol{V}^G \left(\boldsymbol{\Gamma}^c\right)^{n-r} \boldsymbol{V}^G \,\overline{\boldsymbol{\Gamma}} \otimes \boldsymbol{1} \right\rangle \qquad 0 \le r \le n$$