JOHANNES GUTENBERG UNIVERSITÄT MAINZ

QCD anatomy of photon isolation

Xiaofeng Xu

2208.01554 with Thomas Becher and Samuel Favrod

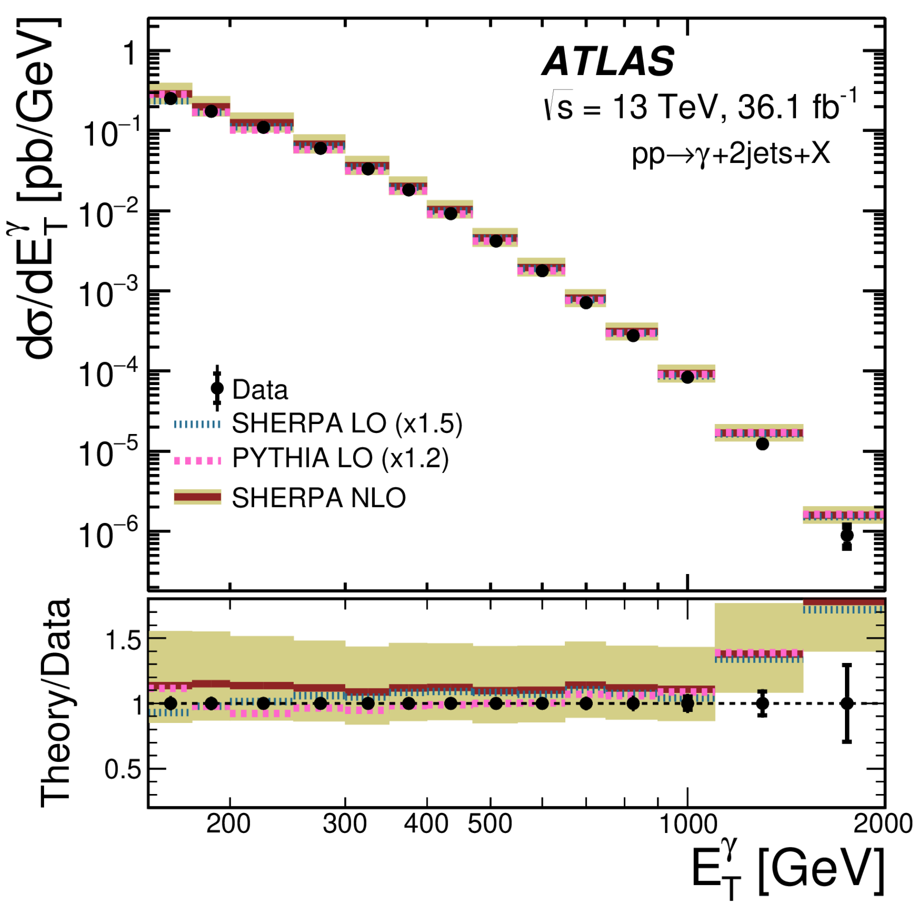
Parton Showers and Resummation 2023

Outline

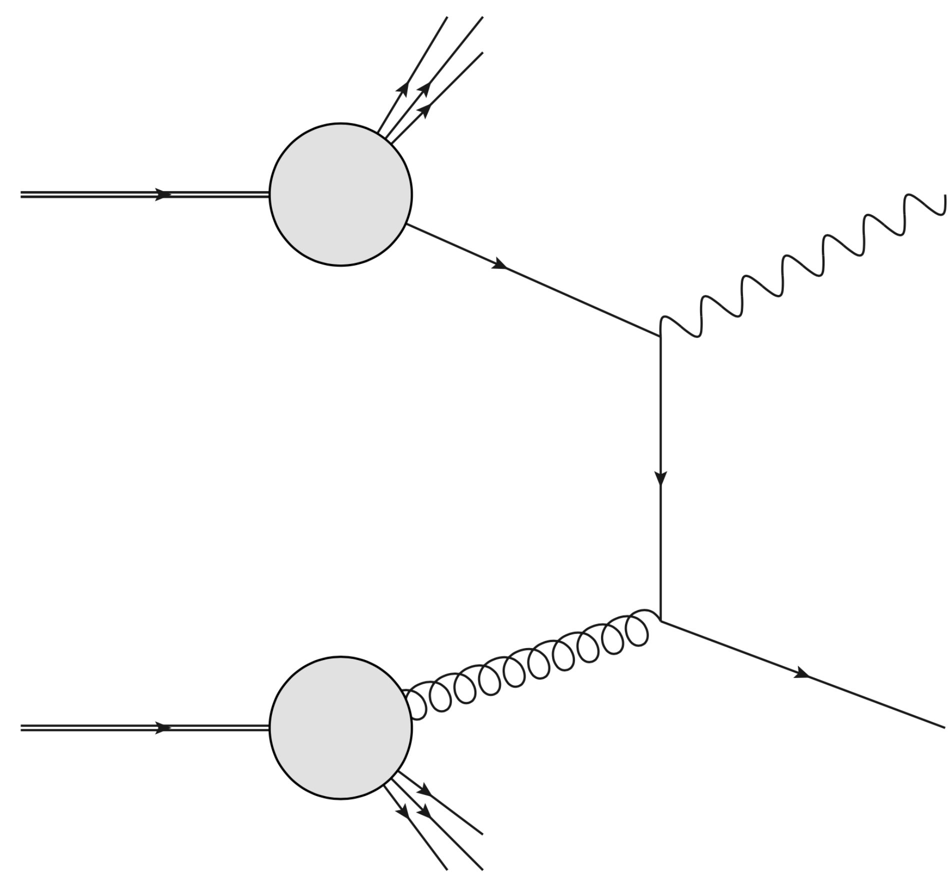
- Introduction and photon isolation
- Factorization theorem
  - Parameter dependence of cross section
  - Resummation of $\ln(R)$
  - Resummation of $\ln(\epsilon_\gamma)$
- Conclusion and outlook

Motivation

1. Test SM and probe BSM physics
2. Study gluon PDF with photon production

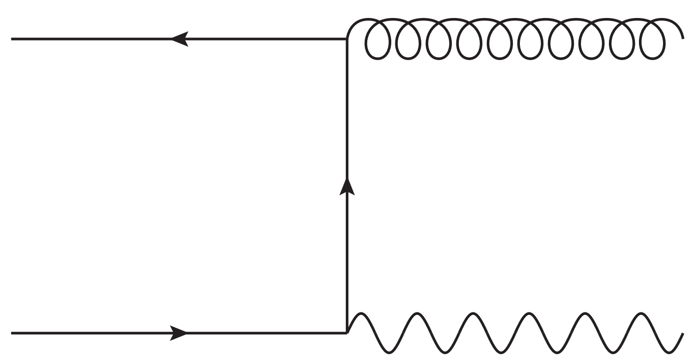


photon + 2 jets

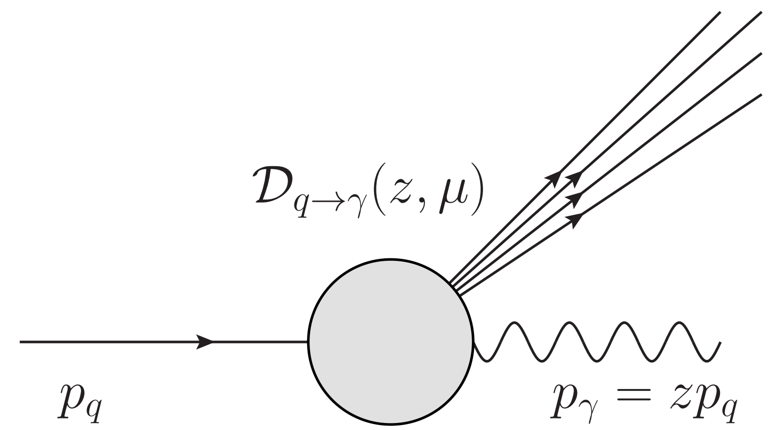


Motivation

- Photon production at hadron collider



direct

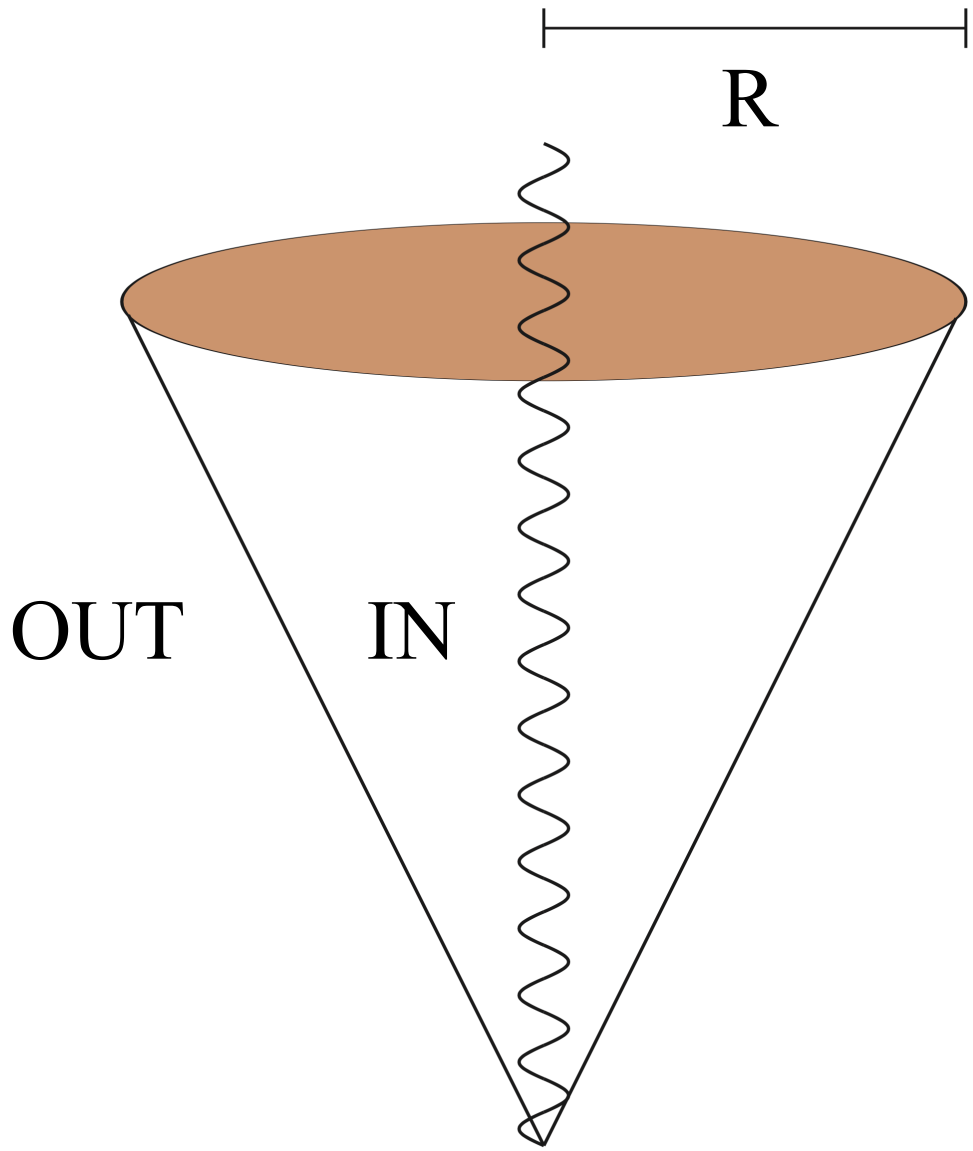


fragmentation

Non-perturbative

- Background from energetic hadron decay of π^0 and η
  - Put photon isolation constraints to suppress the background

Photon isolation cone



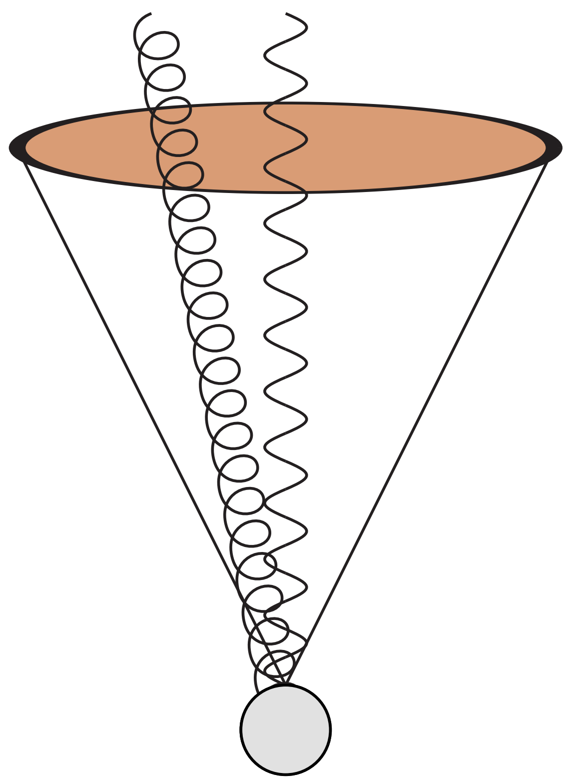
- Isolated in outside region

- Energy constraint for inside region:

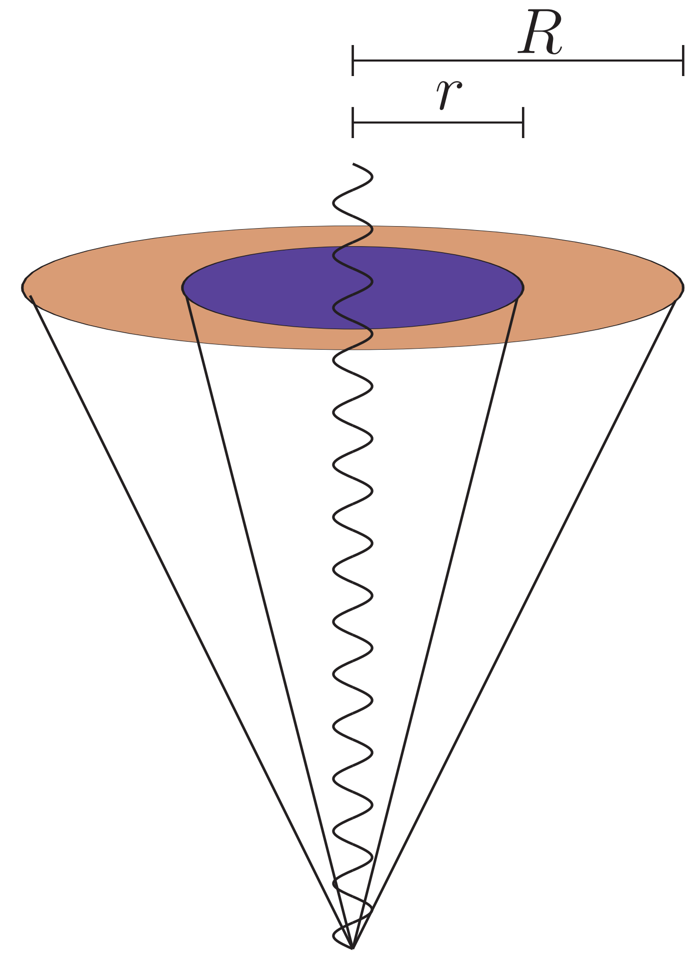
$$E^T_{\text{cone}} < E^T_{\text{iso}}$$

- Different types of photon isolation

Photon isolation cone



Fixed-cone isolation



Frixone cone isolation

$$E^T_{\text{cone}}(R) < E^T_{\text{iso}} = \epsilon E^T_\gamma + E^T_{th}$$

ATLAS sets $\epsilon = 0.0042$ and $E^T_{th} = 4.8 \text{ GeV}$

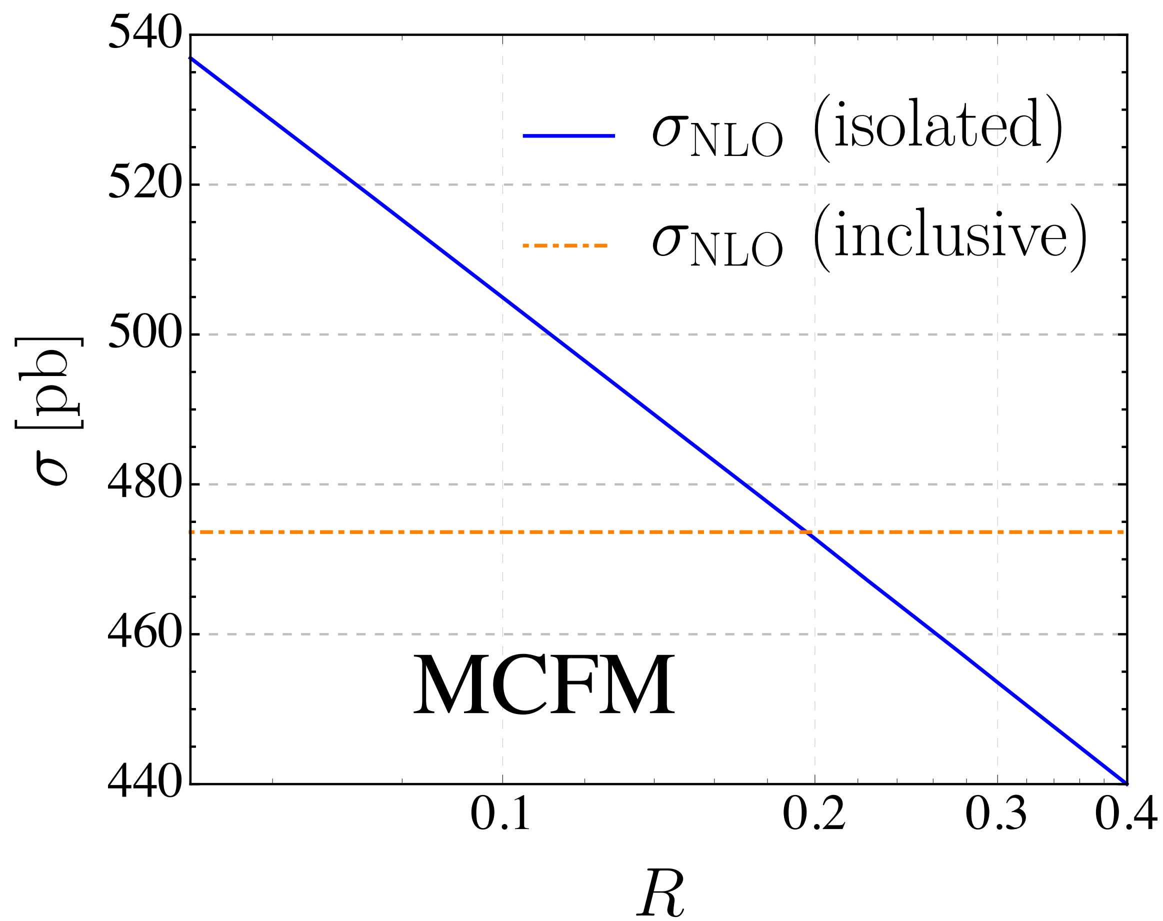
Involve non-perturbative effects

$$E^T_{\text{cone}}(R) < E^T_{\text{iso}} = \epsilon_\gamma E^T_\gamma \left(\frac{1-\cos r}{1-\cos R} \right)^n$$

No non-perturbative fragmentation

- Large logarithm ln(R) in NLO cross section

Gehrmann de Ridder, Glover '98



σ(isolated) with Frixone-cone, $n = 1$, $\varepsilon_\gamma = 1$

σ(inclusive) with GdRG fragmentation functions

- The cross section with isolation is proportional to ln(R)

- The Frixone-cone isolation breaks down for $R < 0.2$

should have: $\sigma(\text{isolated}) < \sigma(\text{inclusive})$

Same problem also for fixed-cone isolation Catani, Fontannaz, Guillet and Pilon in JHEP 05, 028 (2002)

Isolation radius	Total
R	NLO
1.0	3765.1
0.7	4098.0
0.4	4524.5
0.1	5431.1
Without isolation	5217.9

Tevatron cross section

$$\frac{d\sigma}{dE_\gamma^T}\,[\mathrm{pb/GeV}]$$

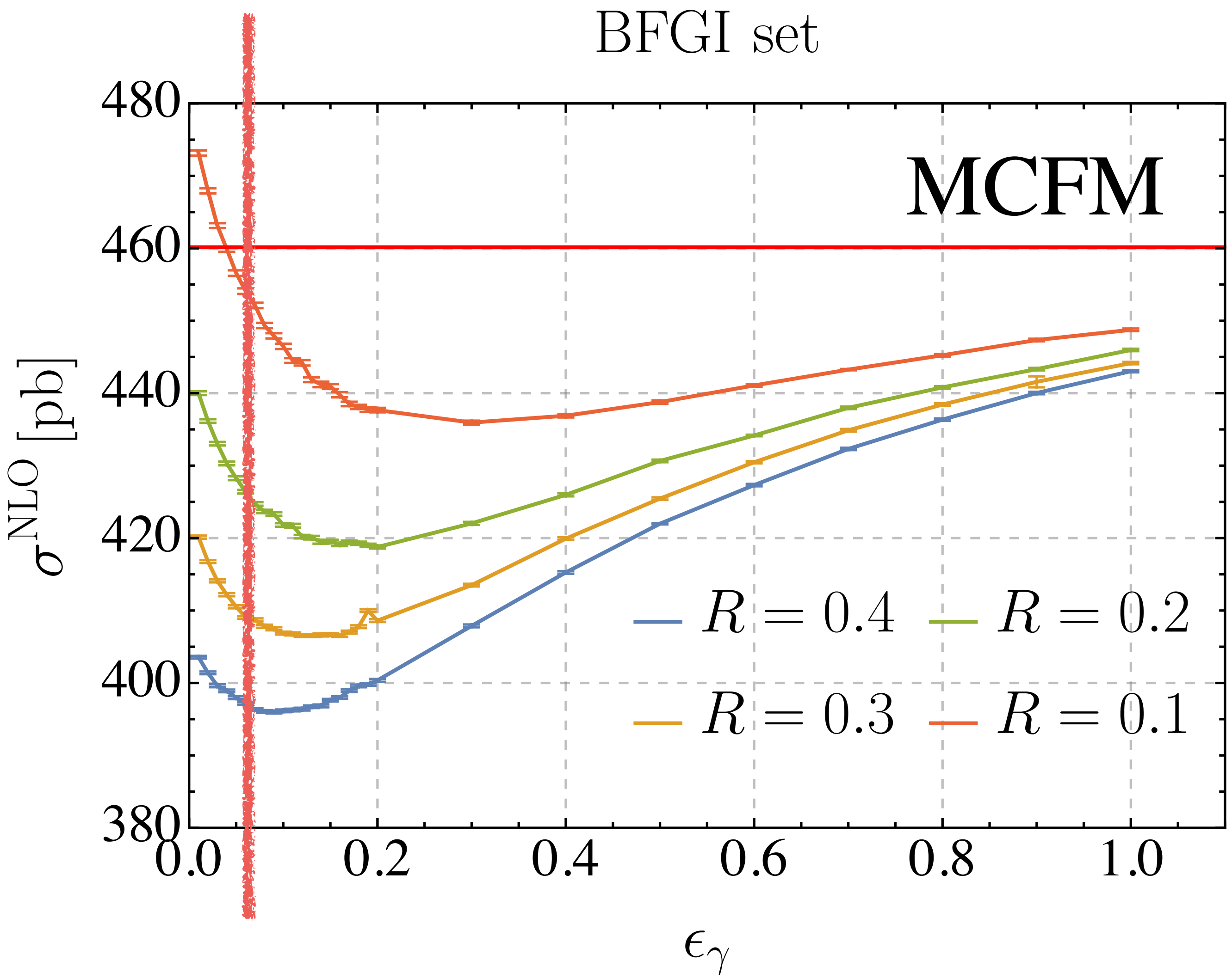
with $E_\gamma^T = 15\,\mathrm{GeV}$

Fixed-cone isolation,

$\varepsilon_\gamma = 0.133$

Also σ(isolated) depends fragmentation functions.

- Pathological behavior in ϵ_γ



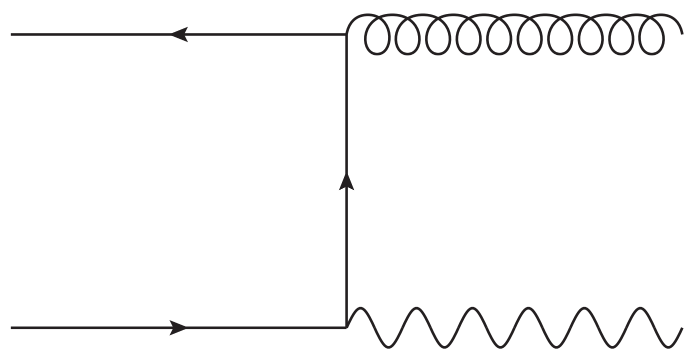
(Bourhis, Fontannaz and Guillet, '98)

σ(isolated) with fixed-cone isolation and BFG fragmentation functions

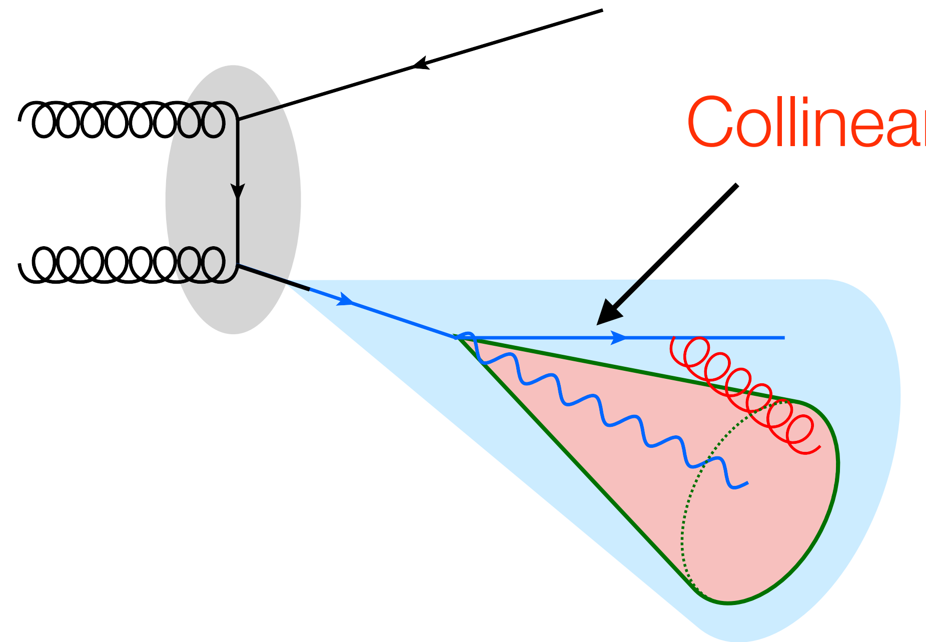
- The σ(isolated) should decrease as ε_γ is lowered
- The fixed-cone isolation breaks down for small R and ε_γ

Factorization theorem

Becher, Favrod, Xu, 2208.01554



direct



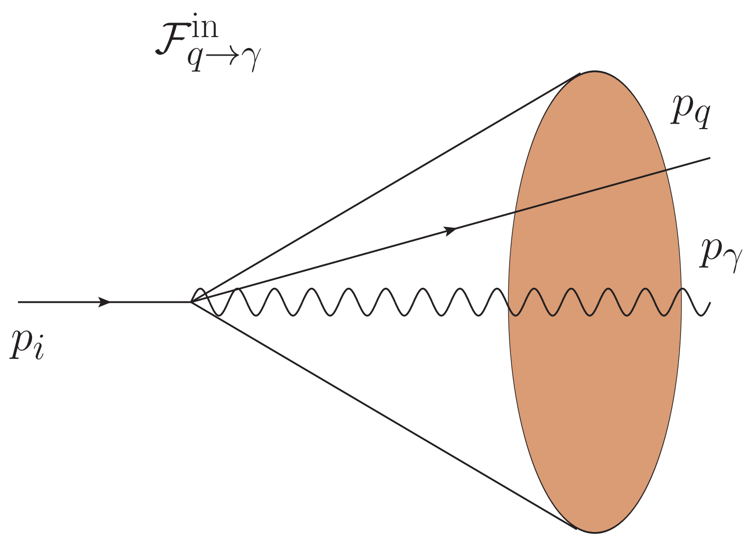
- For small R, the cross sections are factorized as

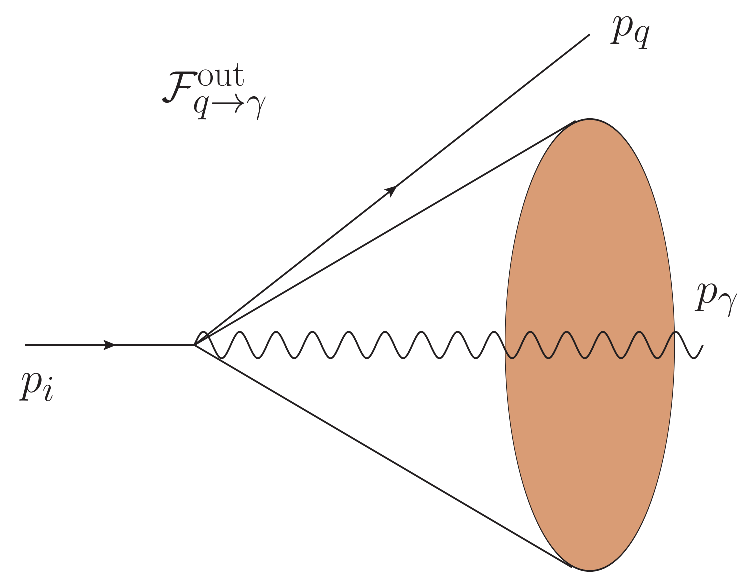
$$\frac{\mathrm{d}\sigma(E_0,R)}{\mathrm{d}E_\gamma} = \frac{\mathrm{d}\sigma^{\mathrm{dir}}_{\gamma+X}}{\mathrm{d}E_\gamma} + \sum_{i=q,\bar{q},g} \int dz \frac{\mathrm{d}\sigma_{i+X}}{\mathrm{d}E_i} \mathcal{F}_{i\to\gamma}(z,E_\gamma,E_0,R) + \mathcal{O}(R)$$

Cone fragmentation function

Cone fragmentation function

$$\mathcal{F}_{q\to\gamma}(z, E_\gamma, E_0, R, \mu) = \mathcal{F}^{\rm in}_{q\to\gamma}(z, E_\gamma, E_0, R, \mu) + \mathcal{F}^{\rm out}_{q\to\gamma}(z, R\, E_\gamma, \mu)$$





- NLO outside part of cone fragmentation function

$$\mathcal{F}^{\rm out}_{q\to\gamma}(z, R\, E_\gamma) = \frac{\alpha_{\rm EM}\, Q_q^2}{2\pi} \left\{ P(z) \left[\frac{1}{\epsilon} - \ln\!\left(\frac{R^2 (2E_T^\gamma)^2}{\mu^2} (1-z)^2 \right) \right] - z \right\}$$

Cone fragmentation function

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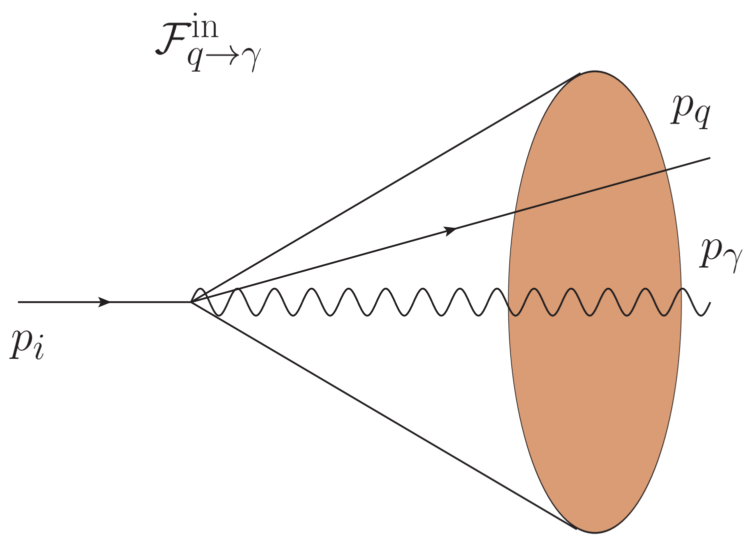

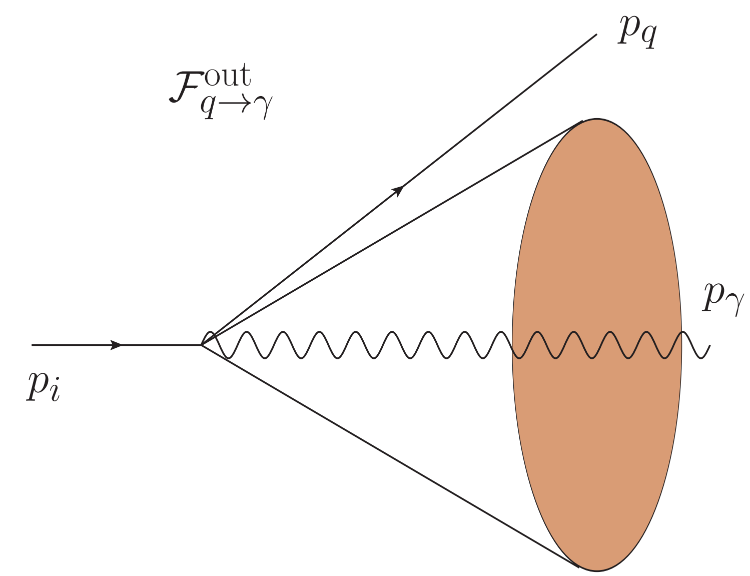

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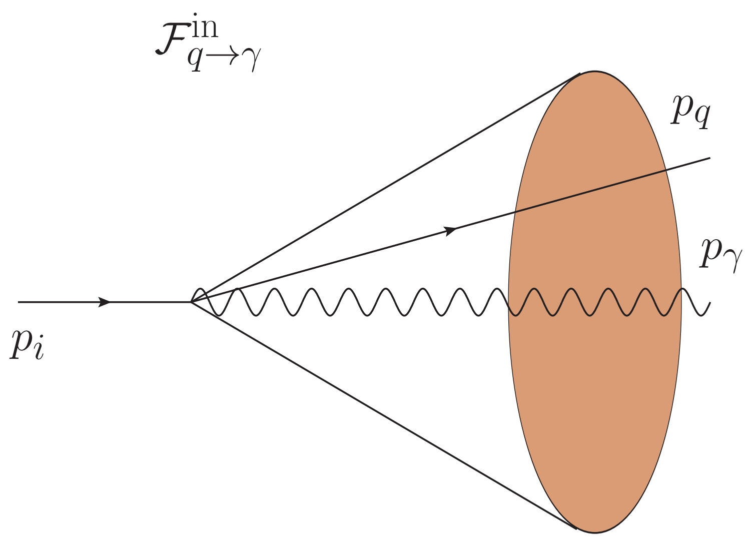
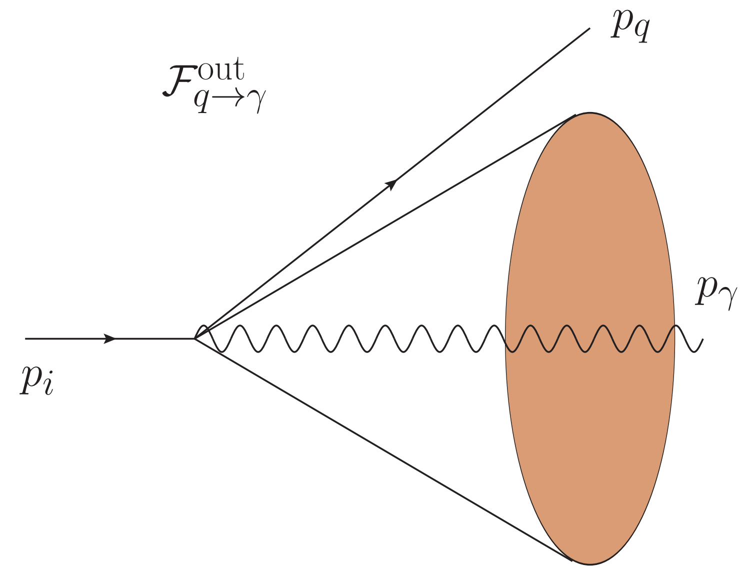
Splitting function $P(z) = \frac{1+(1-z)^2}{z}$ Jet scale $\mu_j = RE_T^\gamma$

Cone fragmentation function

$$\mathcal{F}_{q\to\gamma}(z, E_\gamma, E_0, R, \mu) = \mathcal{F}^{\text{in}}_{q\to\gamma}(z, E_\gamma, E_0, R, \mu) + \mathcal{F}^{\text{out}}_{q\to\gamma}(z, R\,E_\gamma, \mu)$$





- NLO outside part of cone fragmentation function

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Cone fragmentation function

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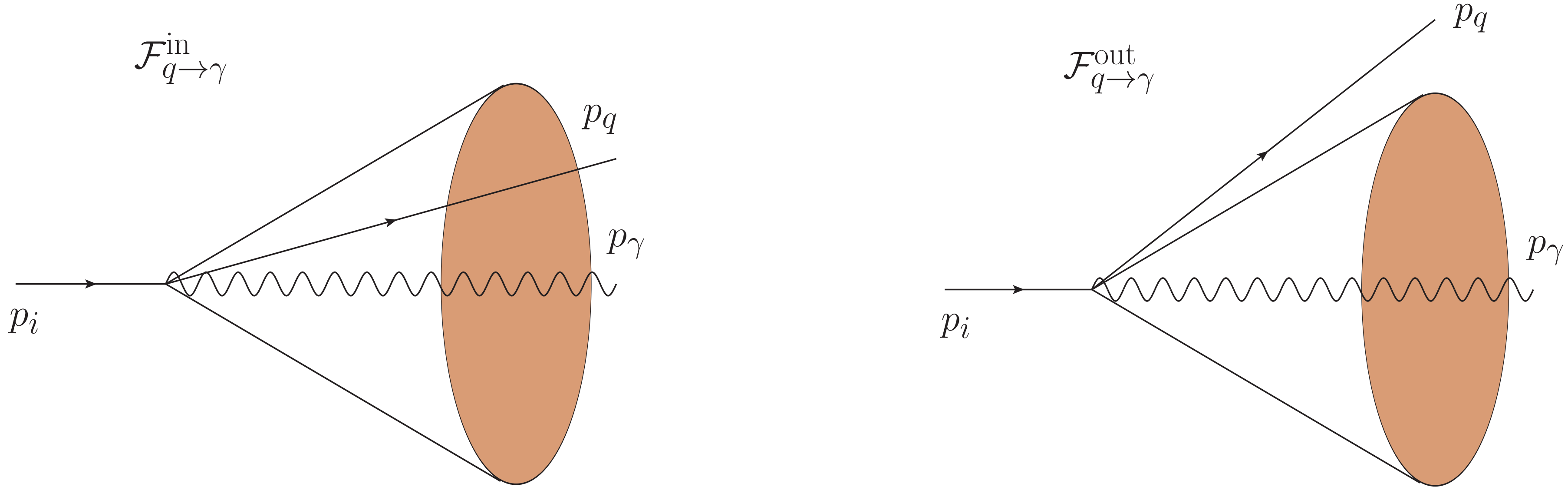

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- It's independent of cone isolation

Cone fragmentation function

- Inside part for Frixone-cone isolation

isolation energy constraint

$$\mathcal{F}^{\rm in}_{q\to\gamma}(z,E_\gamma,E_0,R,\mu)=\frac{\alpha_{\rm EM}Q_q^2}{2\pi}P(z)\frac{1}{n}\ln\left(\frac{z\,\epsilon_\gamma}{1-z}\right)\theta\left(z-\frac{1}{1+\epsilon_\gamma}\right)$$

- Inside part for fixed energy cone isolation

$$\mathcal{F}^{\rm in}_{i\to\gamma}(z,R,E_\gamma,\,E_0,\mu)=\left[\mathcal{D}_{i\to\gamma}(z,\mu)+\sum_{k=q,\bar q}\delta_{ik}\,\mathcal{I}^{\rm in}_{k\to\gamma}(z,R,E_\gamma,\mu)\right]\theta\left(z-\frac{1}{1+\epsilon_\gamma}\right)$$

$$\mathcal{I}^{\rm in}_{q\to\gamma}(z,R,E_\gamma,\mu)=\frac{\alpha_{\rm EM}\,Q_q^2}{2\pi}\left\{P(z)\ln\left(\frac{R^2(2E_T^\gamma)^2}{\mu^2}(1-z)^2\right)+z\right\}$$

- Inside part are power suppressed in the limit $\epsilon_\gamma=0$

parameter dependence

Independent on isolation

$$\frac{\mathrm{d}\sigma(E_0,R)}{\mathrm{d}E_\gamma} = \frac{\mathrm{d}\sigma^{\mathrm{dir}}_{\gamma+X}}{\mathrm{d}E_\gamma} + \sum_{i=q,\bar{q},g}\int dz \frac{\mathrm{d}\sigma_{i+X}}{\mathrm{d}E_i}\mathcal{F}_{i\to\gamma}(z,E_\gamma,E_0,R) + \mathcal{O}(R)$$

- Study the difference of cross sections with different parameters

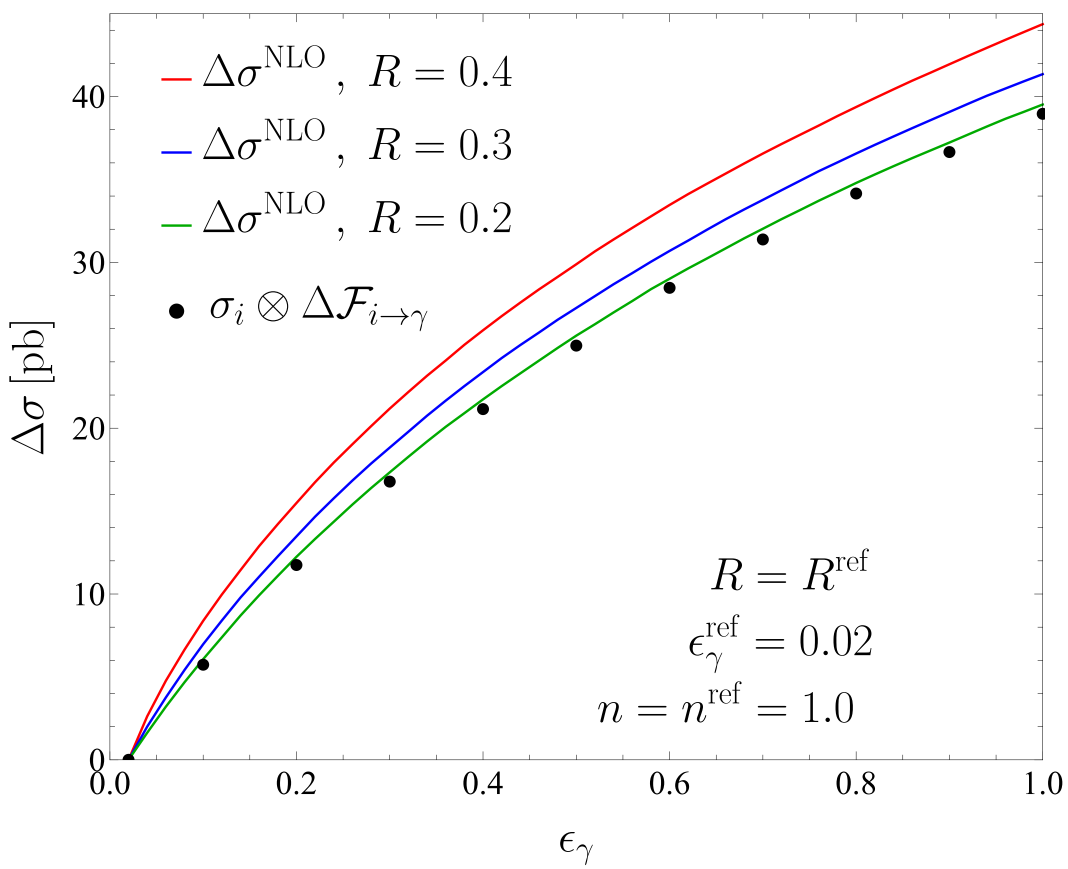
$$\Delta\sigma = \sigma\left(\epsilon_\gamma, n, R\right) - \sigma(\epsilon_\gamma^{\mathrm{ref}}, n^{\mathrm{ref}}, R^{\mathrm{ref}})$$

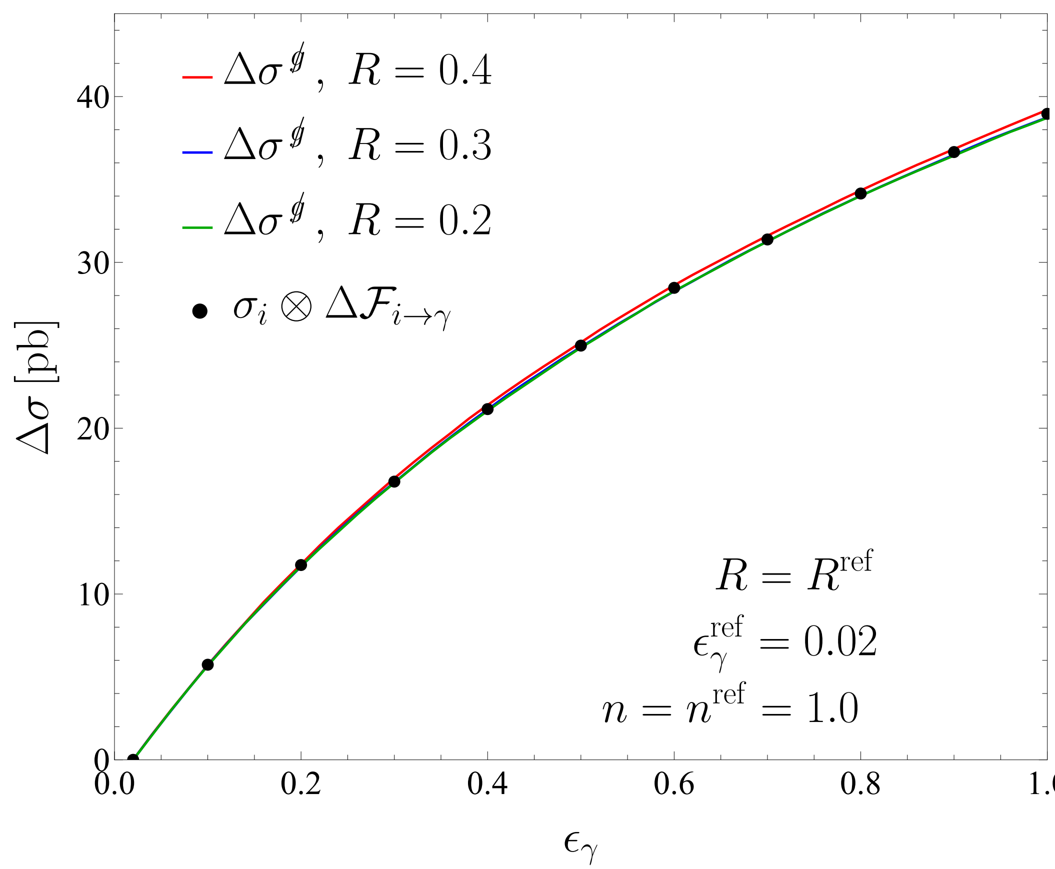
$$\Delta\sigma = \sum_{i=q,\bar{q}}\int_{E_T^{\mathrm{min}}}^{\infty} dE_i \int_{z_{\mathrm{min}}}^{1} dz \frac{d\sigma_{i+X}}{dE_i}\Delta\mathcal{F}_{i\to\gamma}$$

ϵ_γ-dependence (Frixone)

$$\mathcal{F}^{\text{out}}_{q\to\gamma}(z, R\,E_\gamma) = \frac{\alpha_{\text{EM}}\,Q_q^2}{2\pi}\left\{P(z)\left[\frac{1}{\epsilon} - \ln\!\left(\frac{R^2(2E_T^\gamma)^2}{\mu^2}(1-z)^2\right)\right] - z\right\}$$

$$\mathcal{F}^{\text{in}}_{q\to\gamma}(z, E_\gamma, E_0, R, \mu) = \frac{\alpha_{\text{EM}}Q_q^2}{2\pi}P(z)\frac{1}{n}\ln\!\left(\frac{z\,\epsilon_\gamma}{1-z}\right)\theta\!\left(z - \frac{1}{1+\epsilon_\gamma}\right)$$





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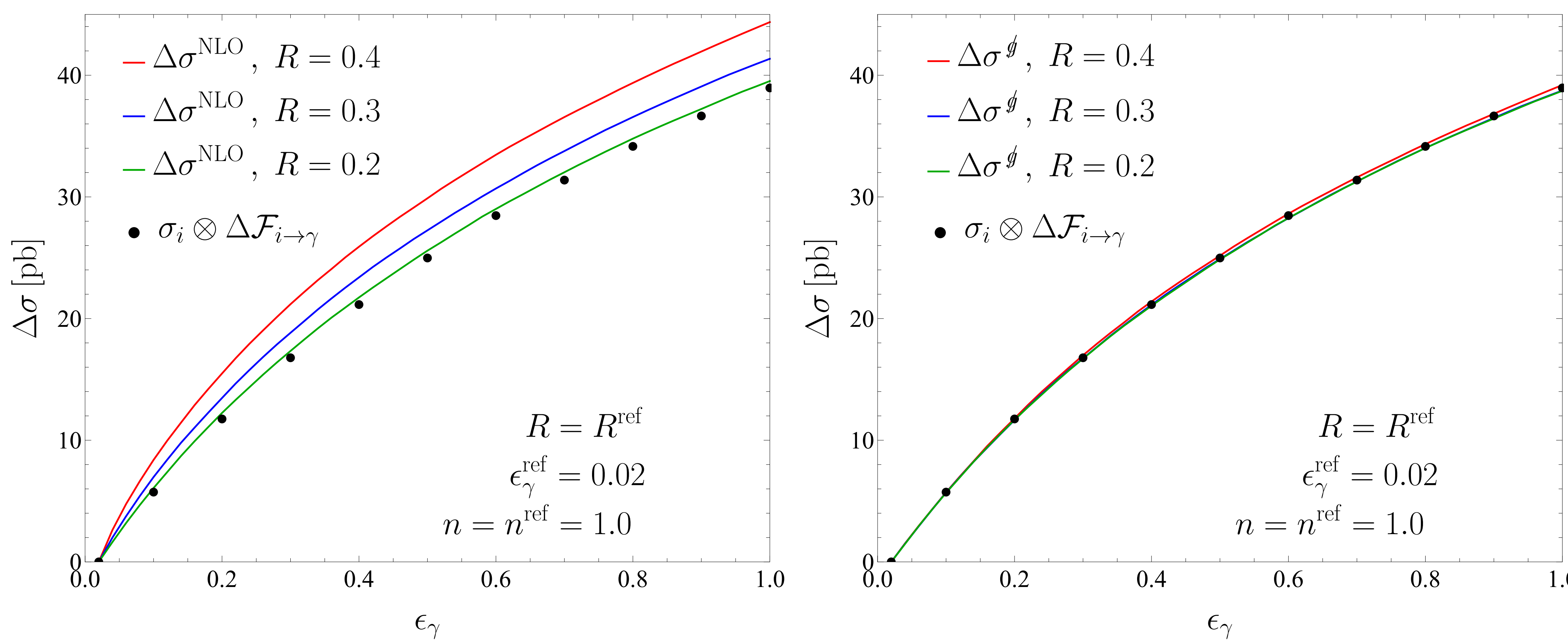
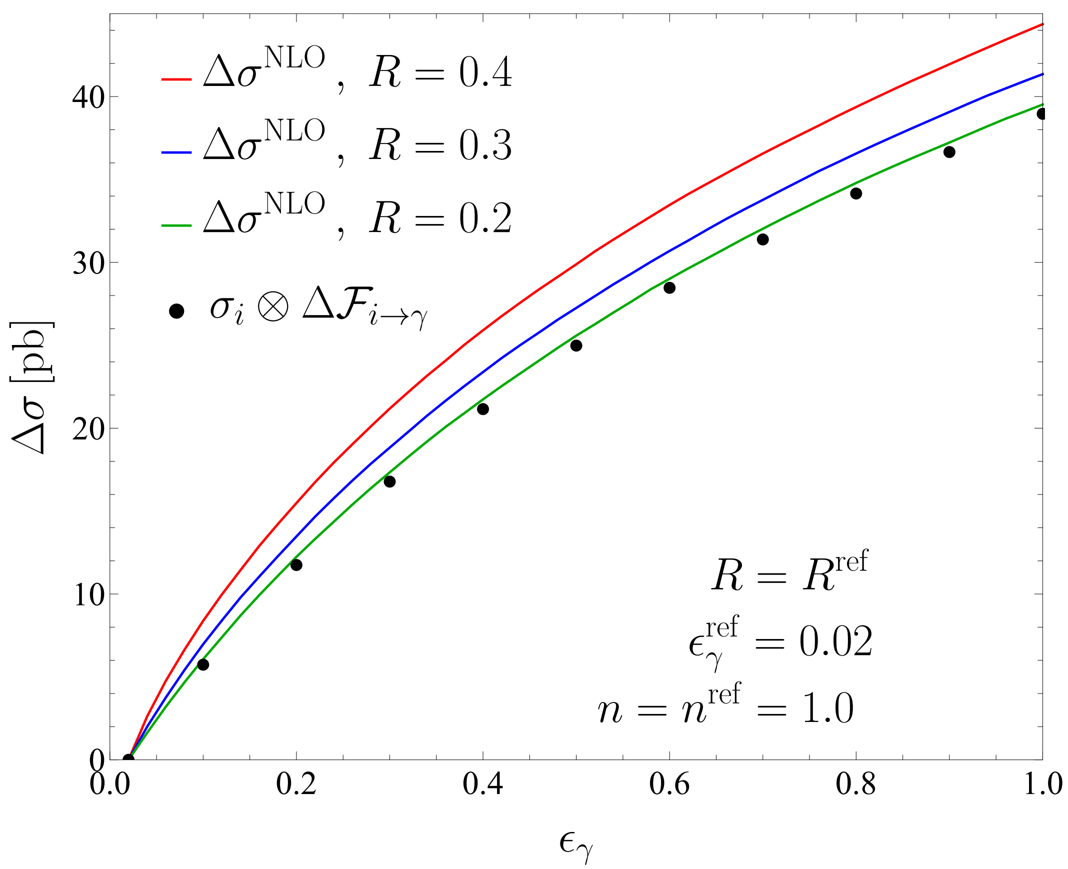


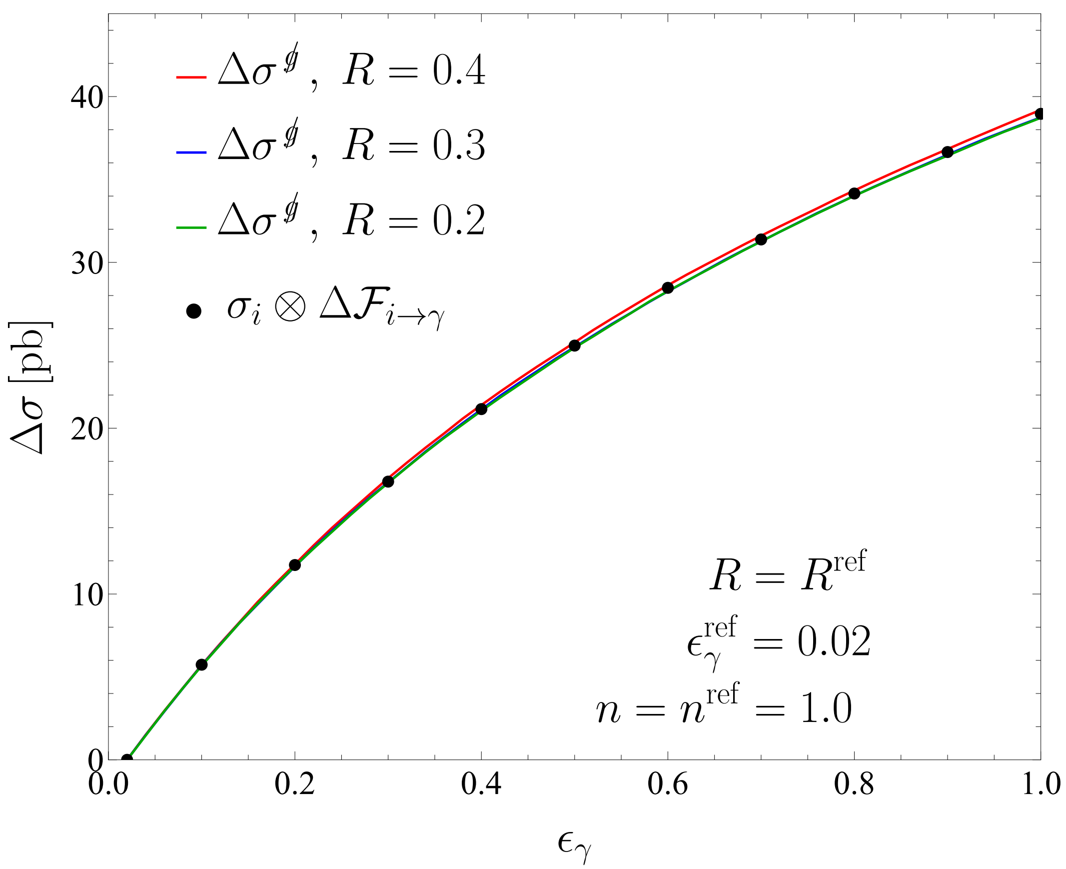
- Good agreement between NLO (solid) and fragmentation approach (dots)

ϵ_γ-dependence (Frixone)

$$\mathcal{F}^{\text{out}}_{q\to\gamma}(z, R\,E_\gamma) = \frac{\alpha_{\text{EM}}\,Q_q^2}{2\pi}\left\{P(z)\left[\frac{1}{\epsilon} - \ln\!\left(\frac{R^2(2E_T^\gamma)^2}{\mu^2}(1-z)^2\right)\right] - z\right\}$$

$$\mathcal{F}^{\text{in}}_{q\to\gamma}(z, E_\gamma, E_0, R, \mu) = \frac{\alpha_{\text{EM}} Q_q^2}{2\pi} P(z)\frac{1}{n}\ln\!\left(\frac{z\,\epsilon_\gamma}{1-z}\right)\theta\!\left(z - \frac{1}{1+\epsilon_\gamma}\right)$$





ϵ_γ-dependence (Frixone)

$$\mathcal{F}^{\rm out}_{q\to\gamma}(z, R\,E_\gamma) = \frac{\alpha_{\rm EM}\,Q_q^2}{2\pi}\left\{P(z)\left[\frac{1}{\epsilon} - \ln\!\left(\frac{R^2(2E_T^\gamma)^2}{\mu^2}(1-z)^2\right)\right] - z\right\}$$

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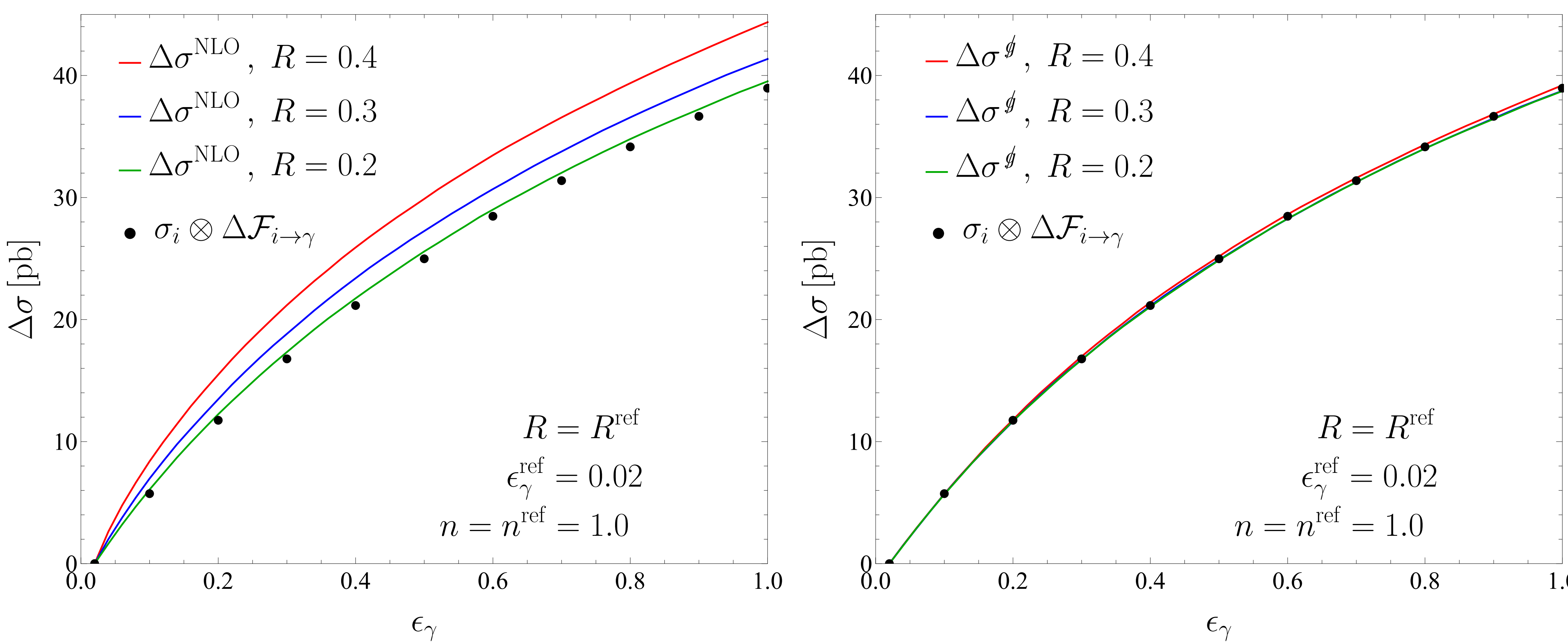
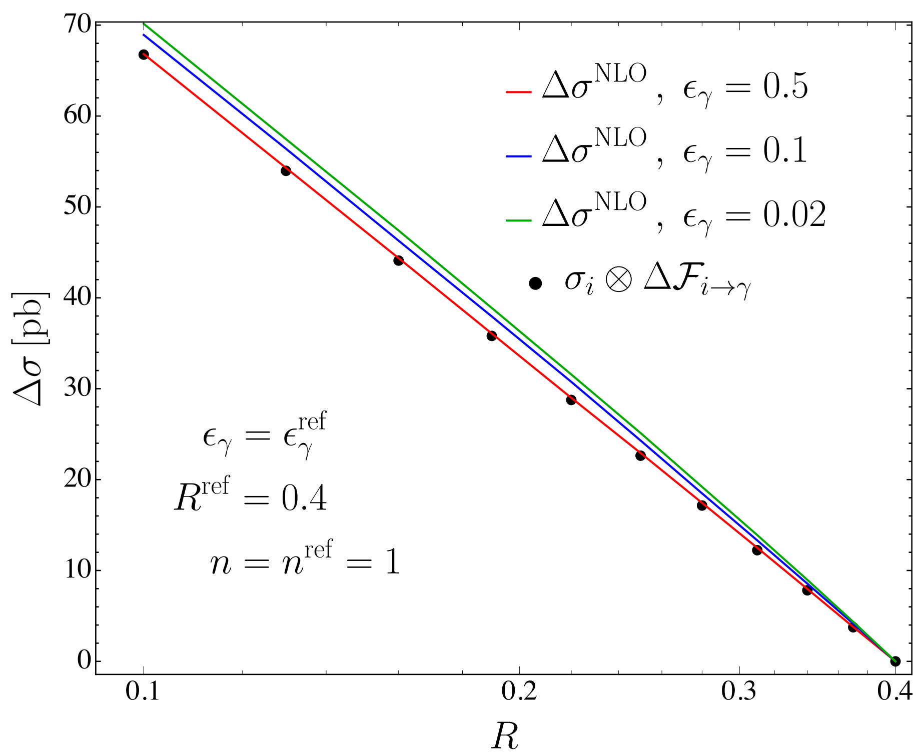


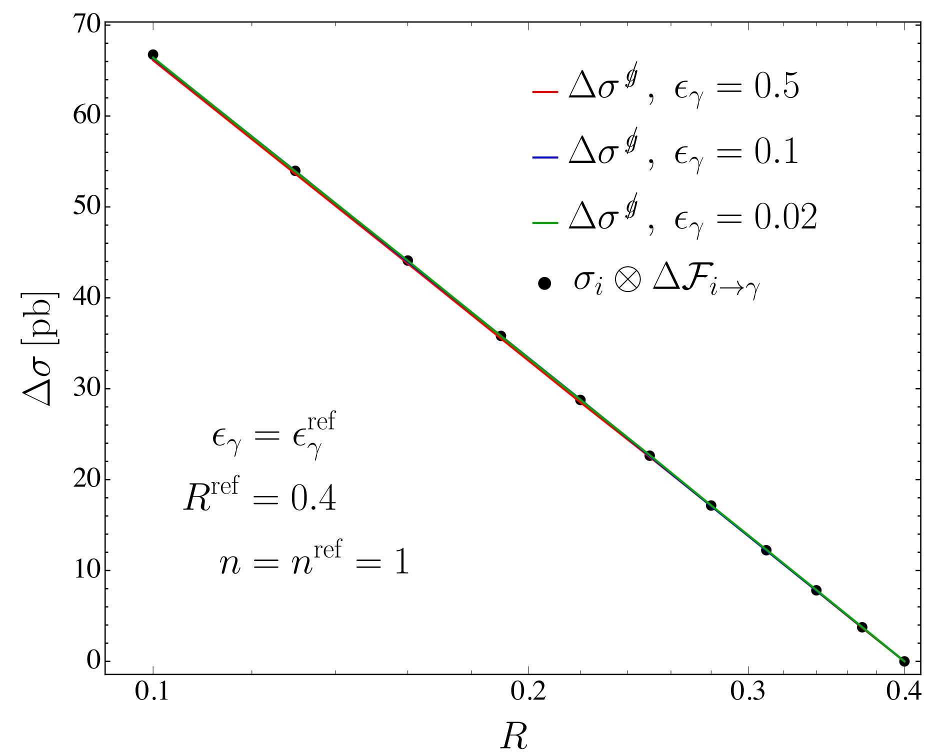
- It has better agreement for smaller R with fragmentation approach (power suppressed in R)

R-dependence (Frixone)

$$\mathcal{F}^{\rm out}_{q\to\gamma}(z, R\,E_\gamma) = \frac{\alpha_{\rm EM}\,Q_q^2}{2\pi}\left\{P(z)\left[\frac{1}{\epsilon} - \ln\!\left(\frac{R^2(2E_T^\gamma)^2}{\mu^2}(1-z)^2\right)\right] - z\right\}$$

$$\mathcal{F}^{\rm in}_{q\to\gamma}(z, E_\gamma, E_0, R, \mu) = \frac{\alpha_{\rm EM} Q_q^2}{2\pi} P(z)\frac{1}{n}\ln\!\left(\frac{z\,\epsilon_\gamma}{1-z}\right)\theta\!\left(z - \frac{1}{1+\epsilon_\gamma}\right)$$





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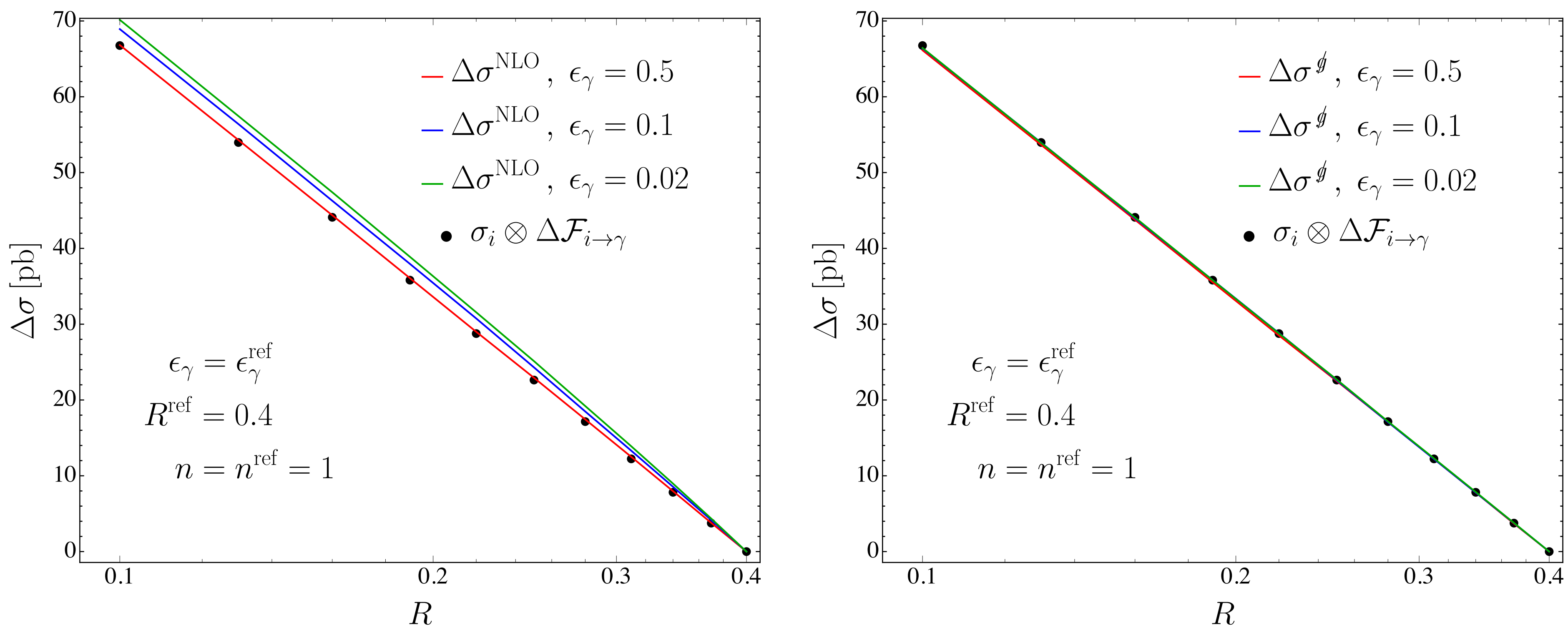
$$\mathcal{F}^{\rm in}_{q\to\gamma}(z, E_\gamma, E_0, R, \mu) = \frac{\alpha_{\rm EM}Q_q^2}{2\pi}P(z)\frac{1}{n}\ln\!\left(\frac{z\,\epsilon_\gamma}{1-z}\right)\theta\!\left(z - \frac{1}{1+\epsilon_\gamma}\right)$$

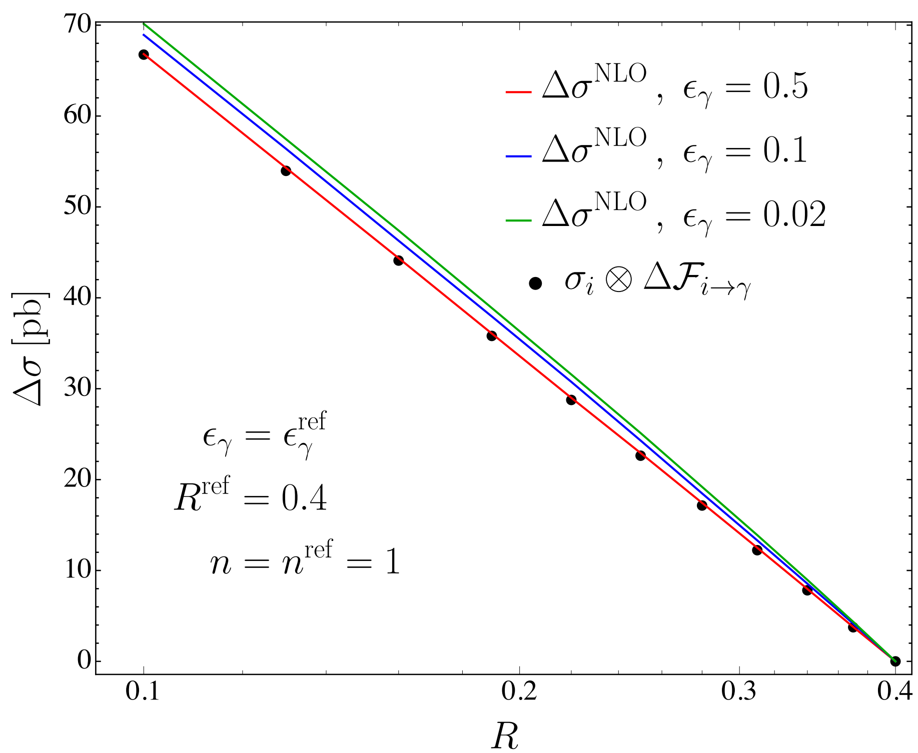


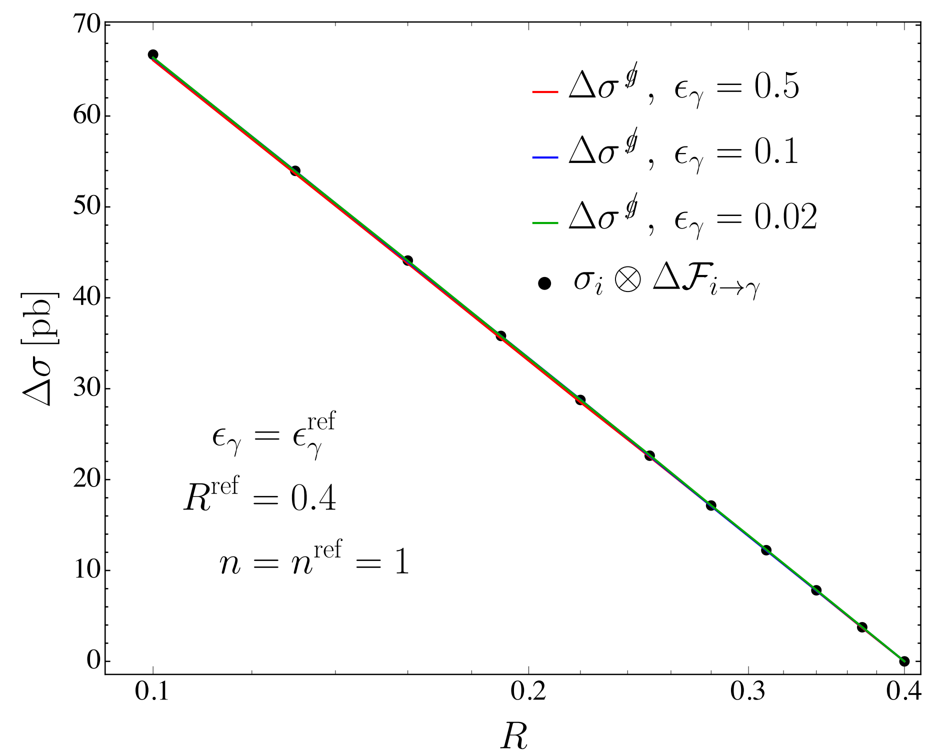
- Proportional to ln(R) $\longrightarrow$ $\Delta\sigma \sim \ln(\frac{R^{\rm ref}}{R})$

R-dependence (Frixone)

$$\mathcal{F}^{\mathrm{out}}_{q\to\gamma}(z, R\,E_\gamma) = \frac{\alpha_{\mathrm{EM}}\,Q_q^2}{2\pi}\left\{P(z)\left[\frac{1}{\epsilon} - \ln\!\left(\frac{R^2(2E_T^\gamma)^2}{\mu^2}(1-z)^2\right)\right] - z\right\}$$

$$\mathcal{F}^{\mathrm{in}}_{q\to\gamma}(z, E_\gamma, E_0, R, \mu) = \frac{\alpha_{\mathrm{EM}}Q_q^2}{2\pi}P(z)\frac{1}{n}\ln\!\left(\frac{z\,\epsilon_\gamma}{1-z}\right)\theta\!\left(z - \frac{1}{1+\epsilon_\gamma}\right)$$





R-dependence (Frixone)

$$\mathcal{F}^{\text{out}}_{q\to\gamma}(z, R\,E_\gamma) = \frac{\alpha_{\text{EM}}\,Q_q^2}{2\pi}\left\{P(z)\left[\frac{1}{\epsilon} - \ln\!\left(\frac{R^2(2E_T^\gamma)^2}{\mu^2}(1-z)^2\right)\right] - z\right\}$$

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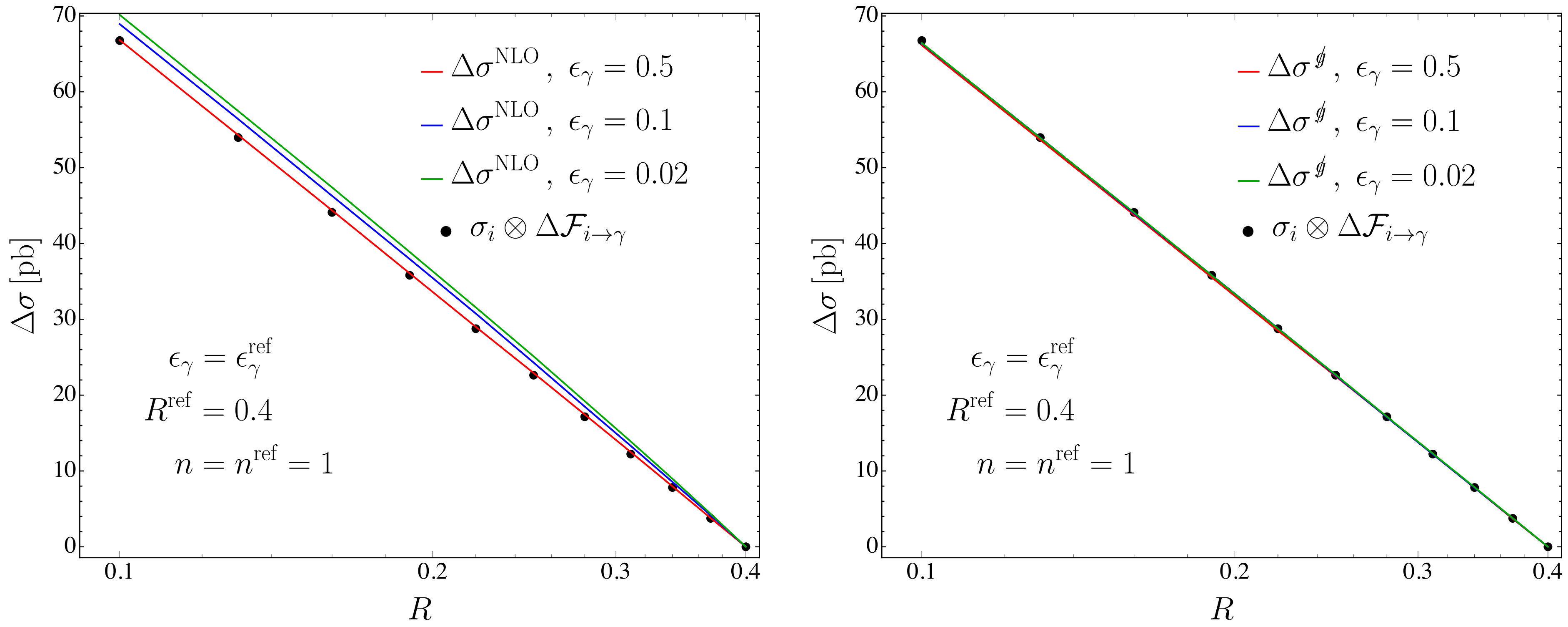
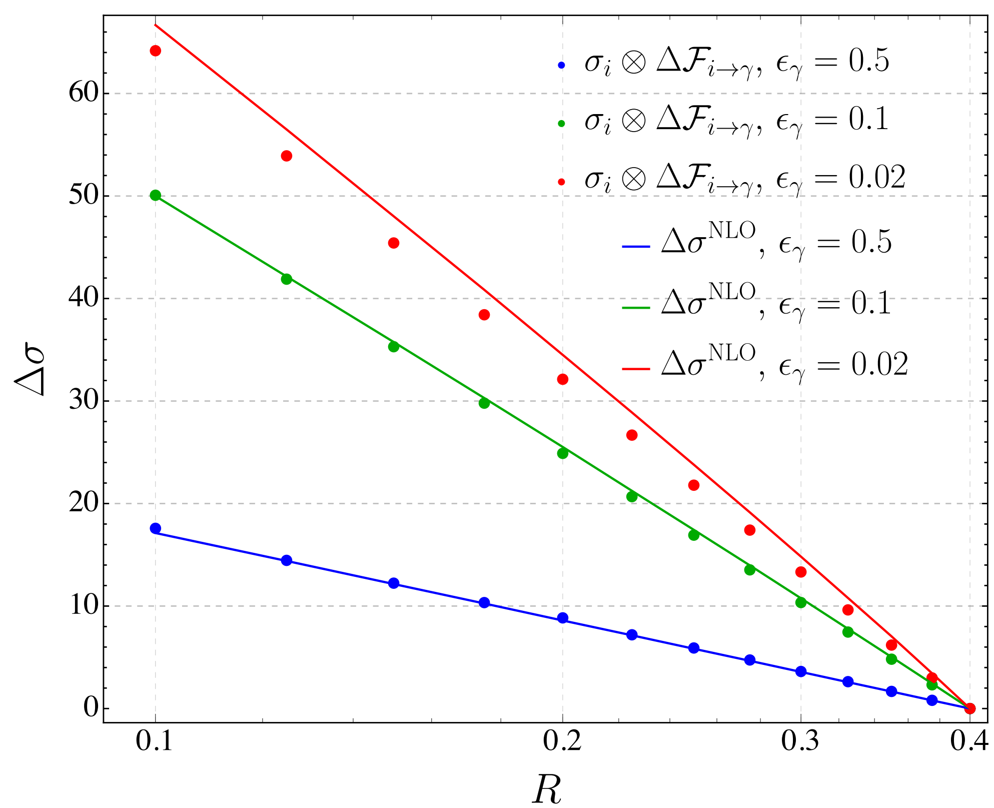
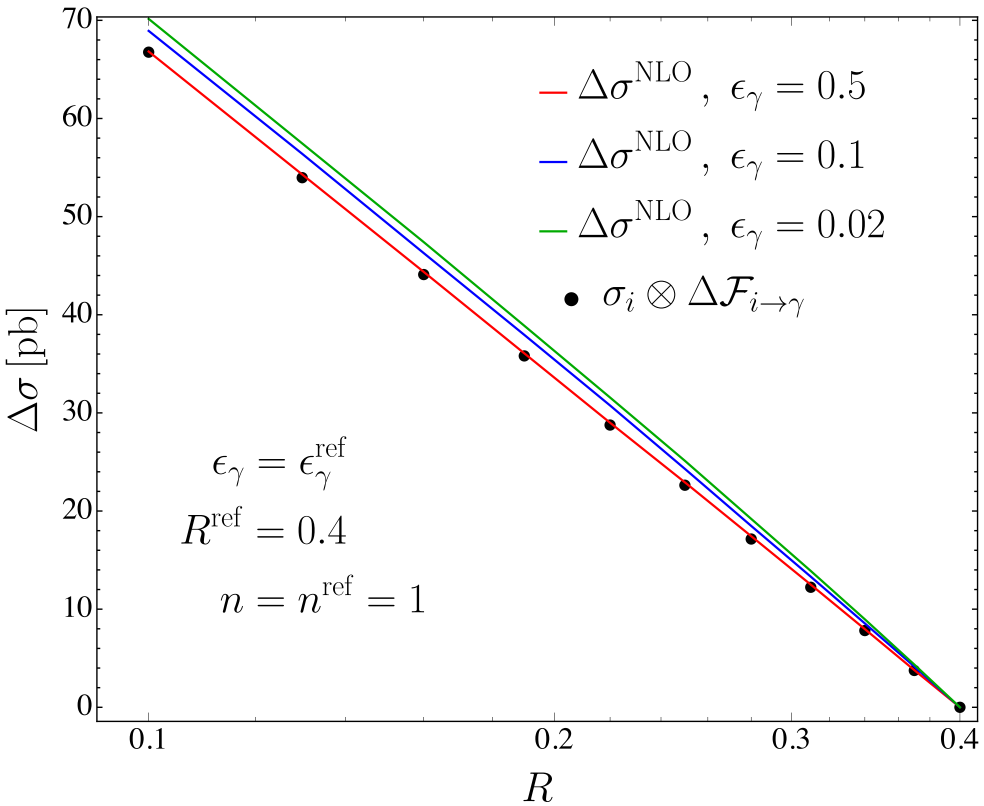


- Better agreement if exclude inside gluons (power suppressed)

R-dependence (Fixed-cone)

$$\Delta\mathcal{F}^{\rm in}_{i\to\gamma} = \frac{Q_i^2\alpha_{\rm EM}}{\pi} P(z)\ln\left(\frac{R^{\rm ref}}{R}\right)\theta\left(\frac{1}{1+\epsilon_\gamma} - z\right)$$





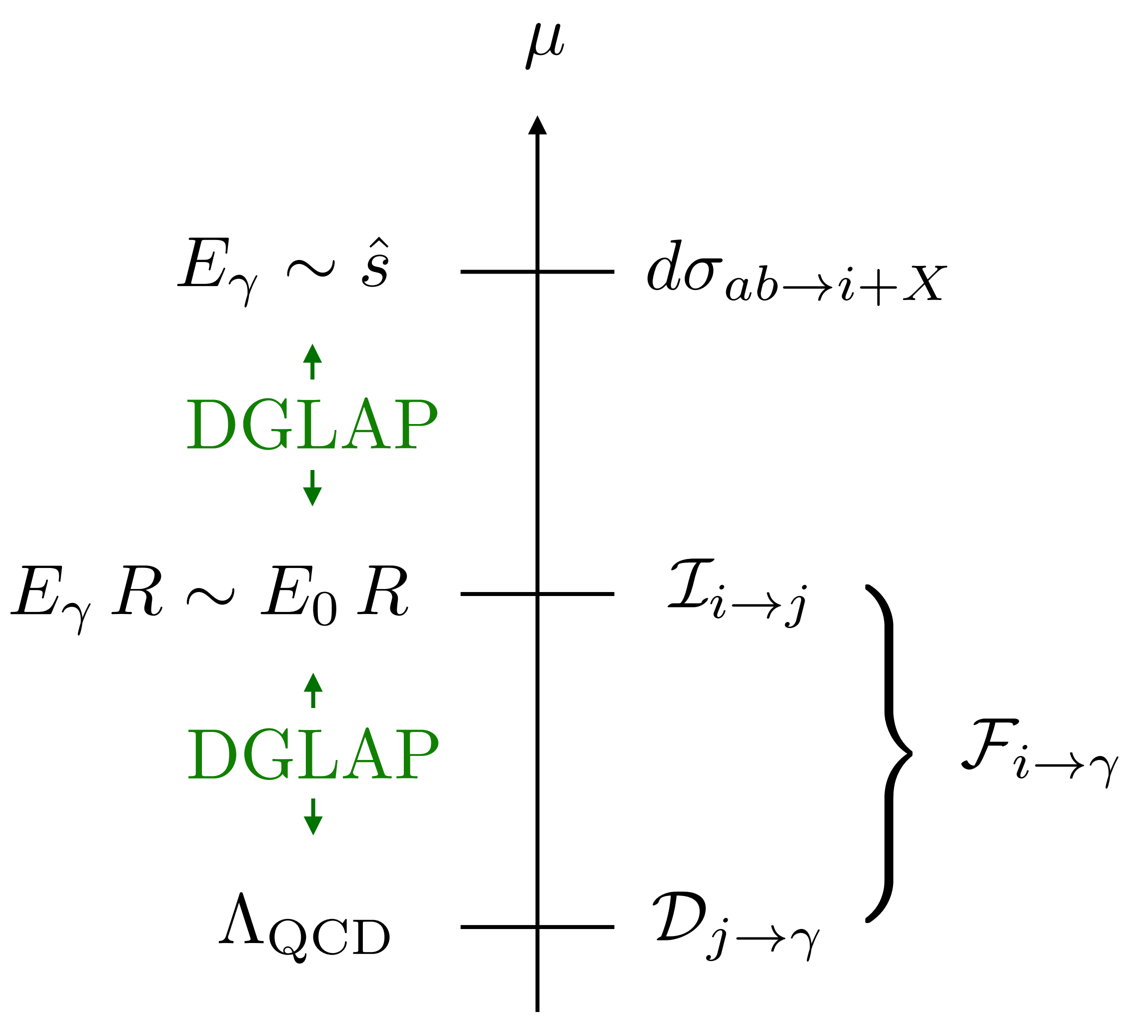
- Inside part for fixed-cone also has ln(R) dependence
- For small ϵ_γ, inside part is suppressed and the cross section recovers ln(R) dependence with Frixone isolation

ln(R) resummation

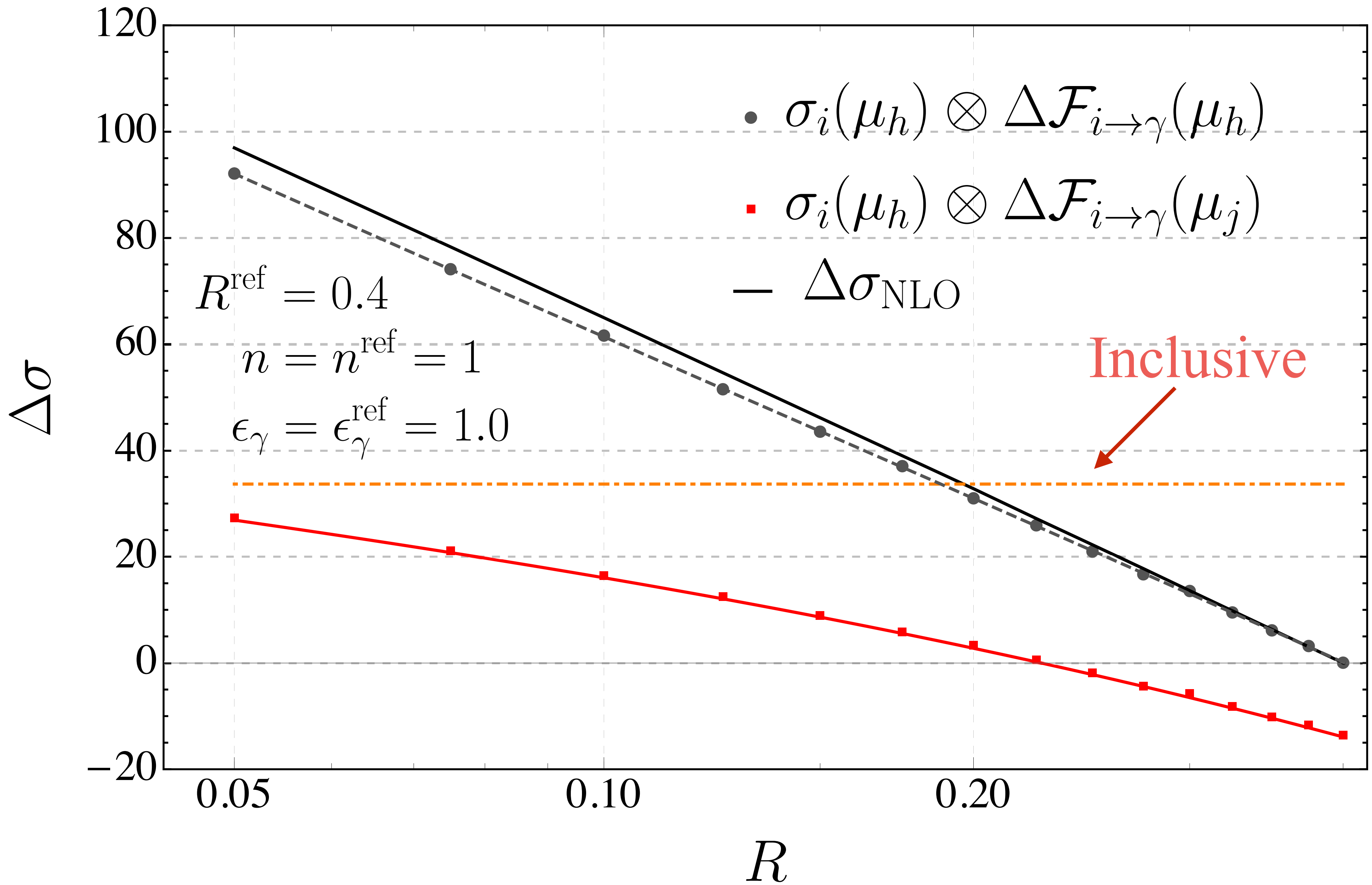
- Fragmentation function fulfill with DGLAP equation

$$\frac{d}{d\ln\mu}\mathcal{F}_{i\to\gamma}(z,\mu) = \sum_{j=\gamma,q,\bar{q},g} \mathcal{P}_{i\to j} \otimes \mathcal{F}_{j\to\gamma}$$

- Evolve from jet scale to hard scale to resum ln(R)

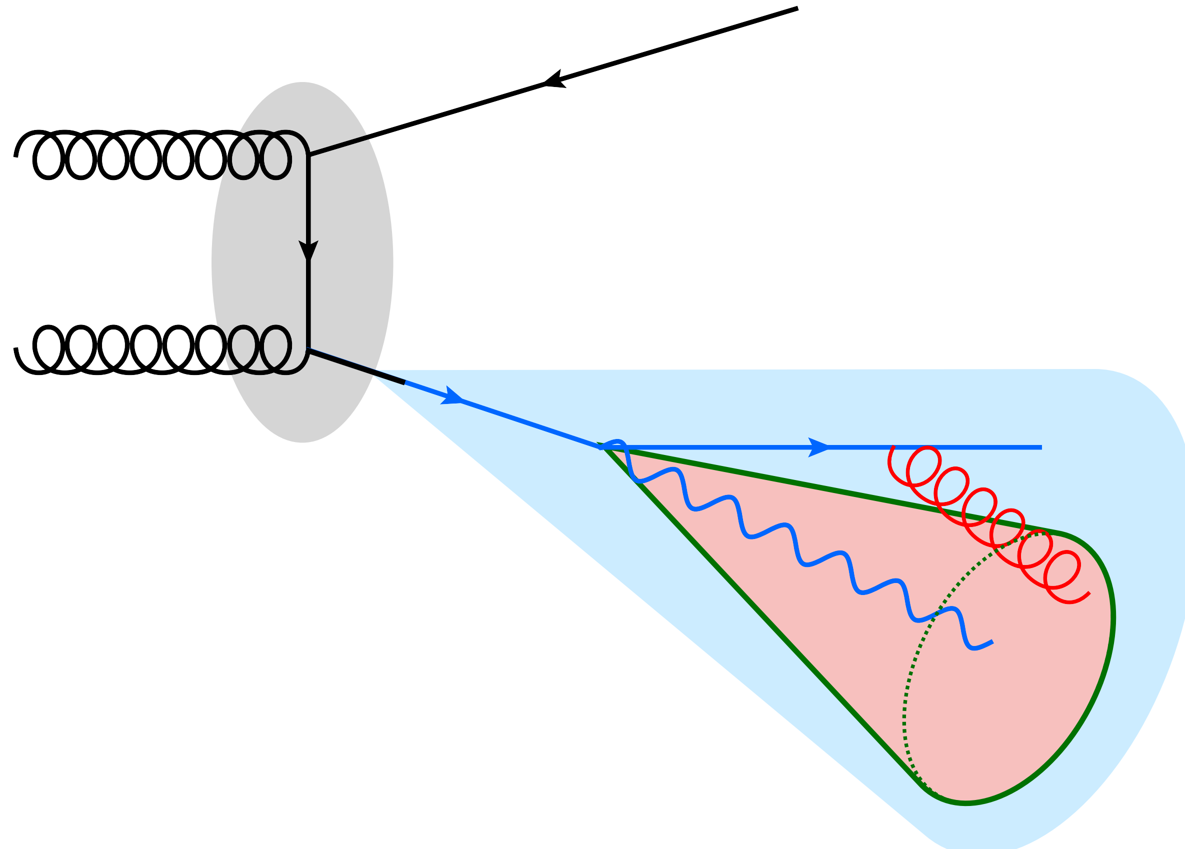


ln(R) resummation



- Show the difference of the cross sections
- Resummation fixes the unphysical behavior

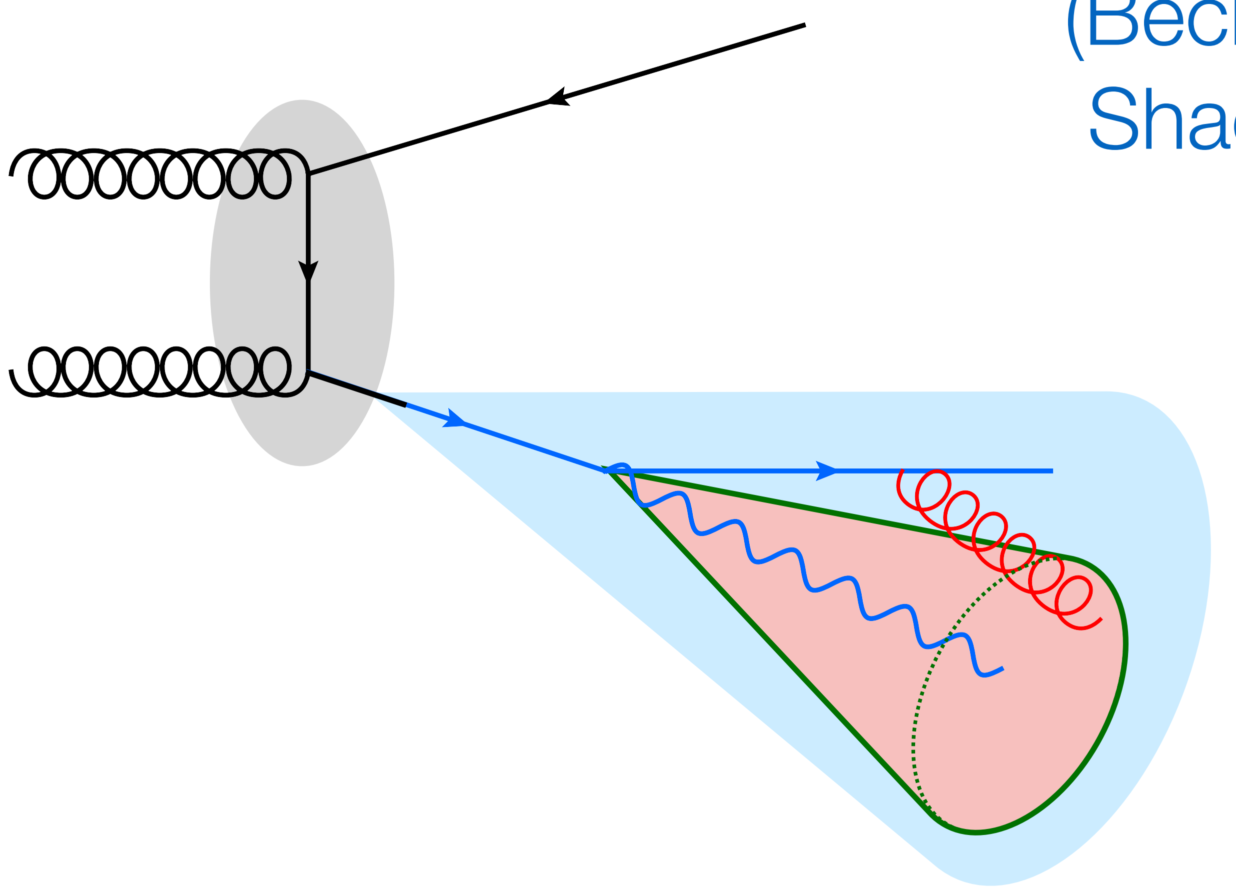
Resummation of $\ln(R)$ and $\ln(\epsilon_\gamma)$



- For ATLAS, $E_0 \sim 5\text{GeV} \longrightarrow \text{small } \epsilon_\gamma$
- Only soft radiation inside the cone
- Large logarithm associated with $\ln(\epsilon_\gamma)$
- Inside part of cone fragmentation function is suppressed
- A typical process with non-global log

Factorization for $\mathcal{F}_{i\to\gamma}$

(Becher, Neubert, Rothen, Shao '15)



- The fragmentation function is factorized as

$$\mathcal{F}_{i\to\gamma}(z, R\,E_\gamma, R\,E_0, \mu) = \sum_{l=1}^{\infty} \langle \boldsymbol{\mathcal{J}}_{i\to\gamma+l}(\{\underline{n}\}, R\,E_\gamma, z, \mu) \otimes \boldsymbol{\mathcal{U}}_l\,(\{\underline{n}\},\, R\,E_0, \mu)\rangle$$

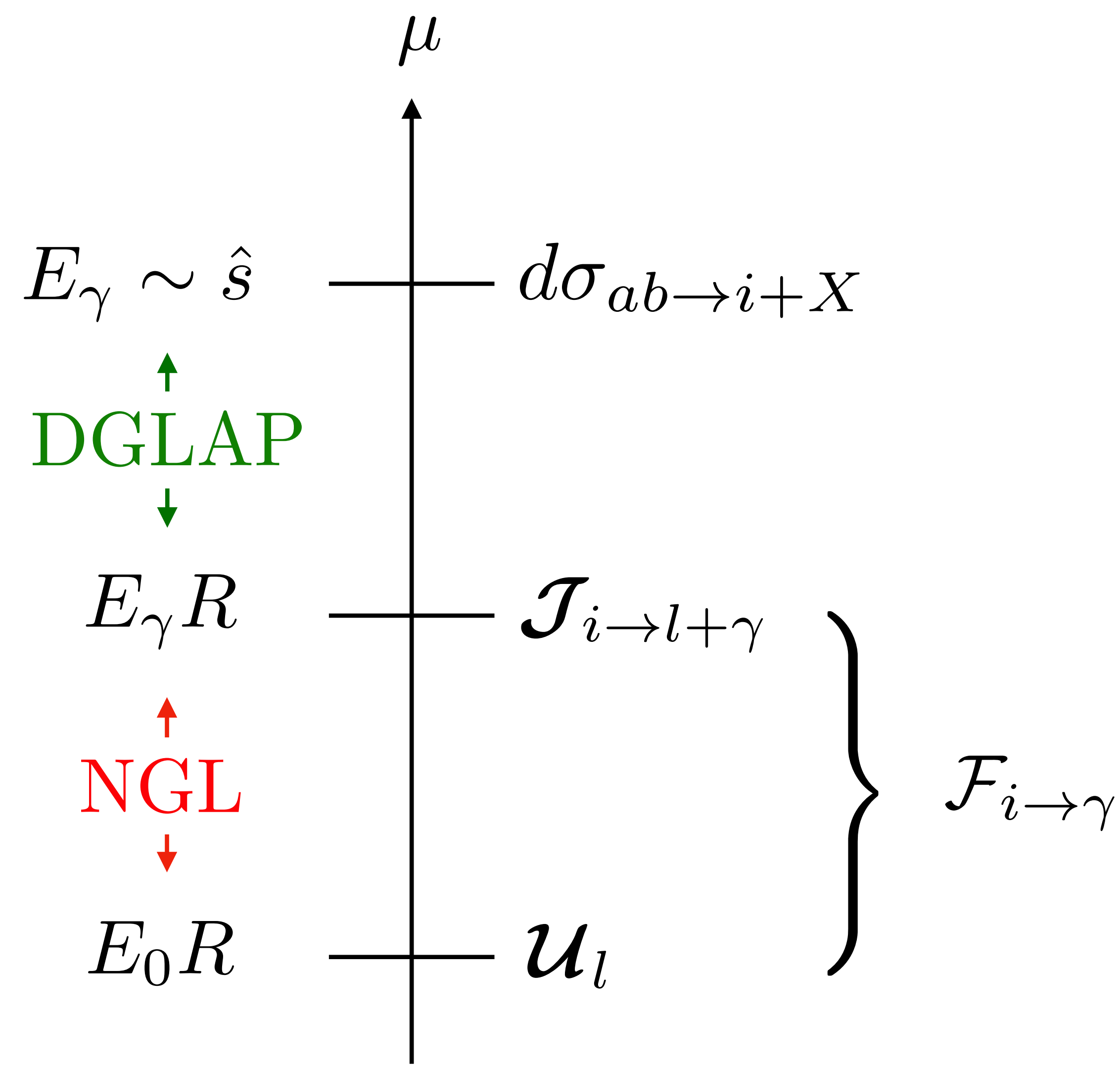
energetic partons outside cone

soft radiation inside cone

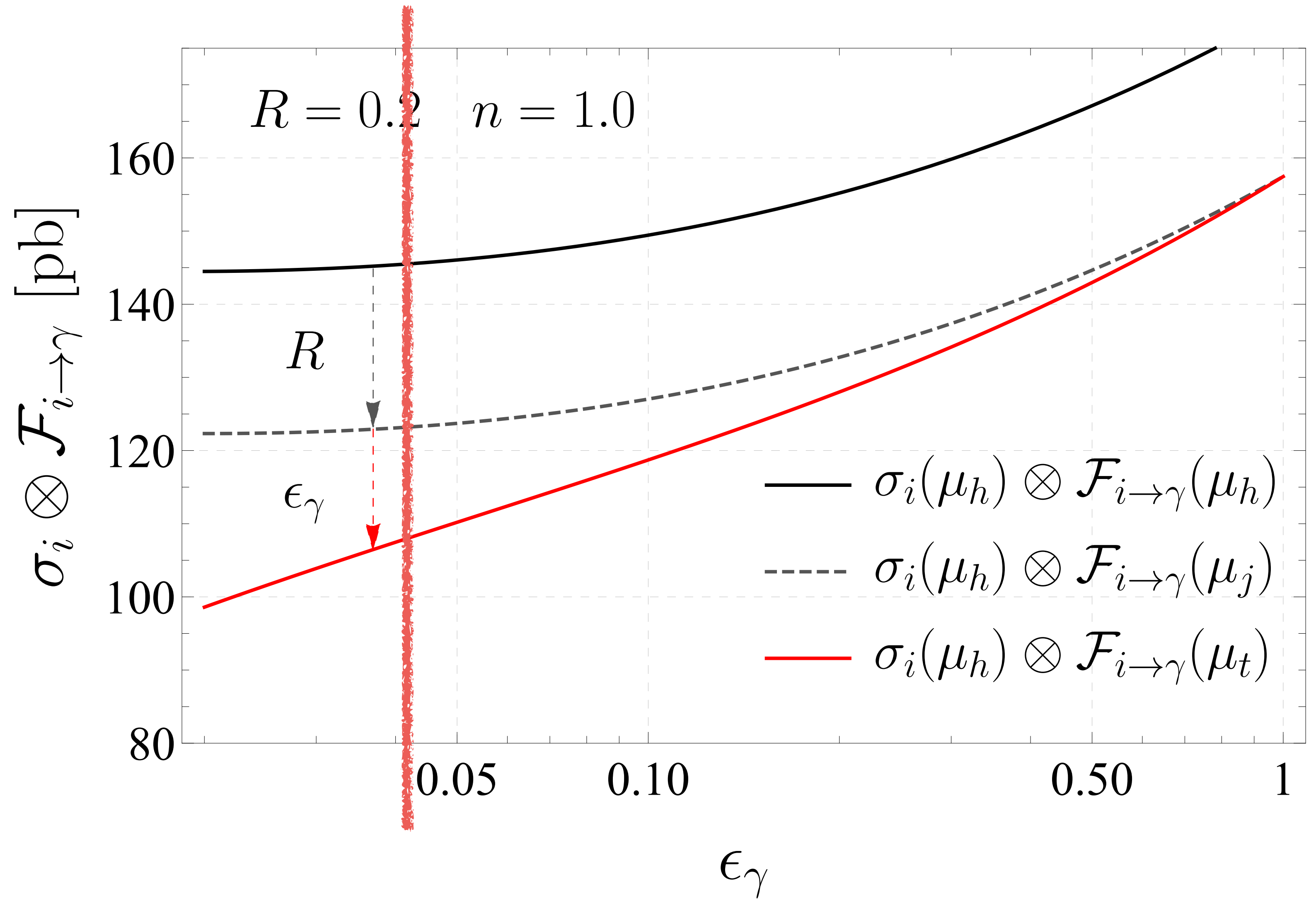
$$\mathcal{F}_{i\to\gamma}(z, R\,E_\gamma, R\,E_0, \mu) = \sum_{l=1}^{\infty} \langle \boldsymbol{\mathcal{J}}_{i\to\gamma+l}(\{\underline{n}\}, R\,E_\gamma, z, \mu) \otimes \boldsymbol{\mathcal{U}}_l\,(\{\underline{n}\},\, R\,E_0, \mu)\rangle$$

- Run the parton shower to resum NGL log

- Solve the DGLAP equations for cone fragmentation function



Resummation of $\ln(R)$ and $\ln(\epsilon_\gamma)$



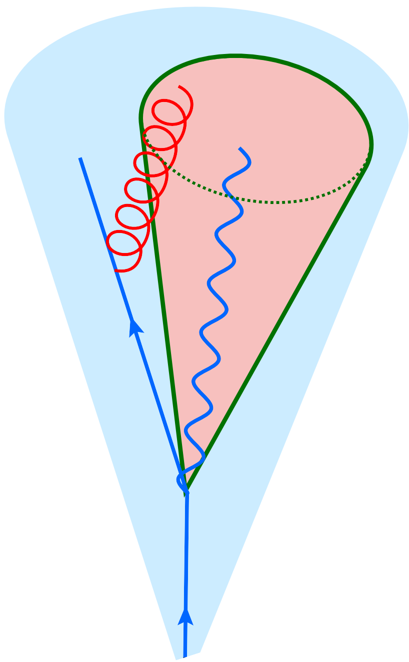
- For small ϵ_γ, the NGL effect is comparable to ln(R) resummation

- For the full cross section, add direct part $\sigma^{\text{dir}} \approx 290\,\text{pb}$

A simple relation

- In the limit $R \to 0$ and $\epsilon_\gamma \to 0$, the inside part is suppressed

$$\mathcal{F}_{i\to\gamma}(z, R\,E_\gamma, R\,E_0, \mu) = \sum_{l=1}^{\infty} \langle \boldsymbol{\mathcal{J}}_{i\to\gamma+l}(\{\underline{n}\}, R\,E_\gamma, z, \mu) \otimes \boldsymbol{\mathcal{U}}_l\,(\{\underline{n}\},\, R\,E_0, \mu)\rangle$$



- We can derive a relation between

$$\Delta\sigma = \sigma_{\text{fixedcone}}(R, \epsilon_\gamma) - \sigma_{\text{Frixonecone}}(R, \epsilon_\gamma^{\text{ref}}, n)$$

$$\Delta\sigma = \sum_{i=q,\bar{q}} \int_{E_T^{\text{min}}}^{\infty} dE_i \int_{z_{\text{min}}}^{1} dz \frac{d\sigma_{i+X}}{dE_i} \frac{Q_q^2 \alpha_{\text{EM}}}{\pi} \frac{C_F \alpha_s}{4\pi} P(z) \left[\frac{\pi^2}{3} \ln\frac{\epsilon_\gamma}{\epsilon_\gamma^{\text{ref}}} + 2n\,\zeta_3\right]$$

- Can be used to convert NNLO smooth-cone into fixed-cone results. For standard setup and $\epsilon_\gamma = \epsilon_\gamma^{\text{ref}}$

$$\Delta\sigma = -1.3\,\text{pb}$$

Outlook and conclusion

- Have performed a detailed analysis of QCD effects associated with photon isolation

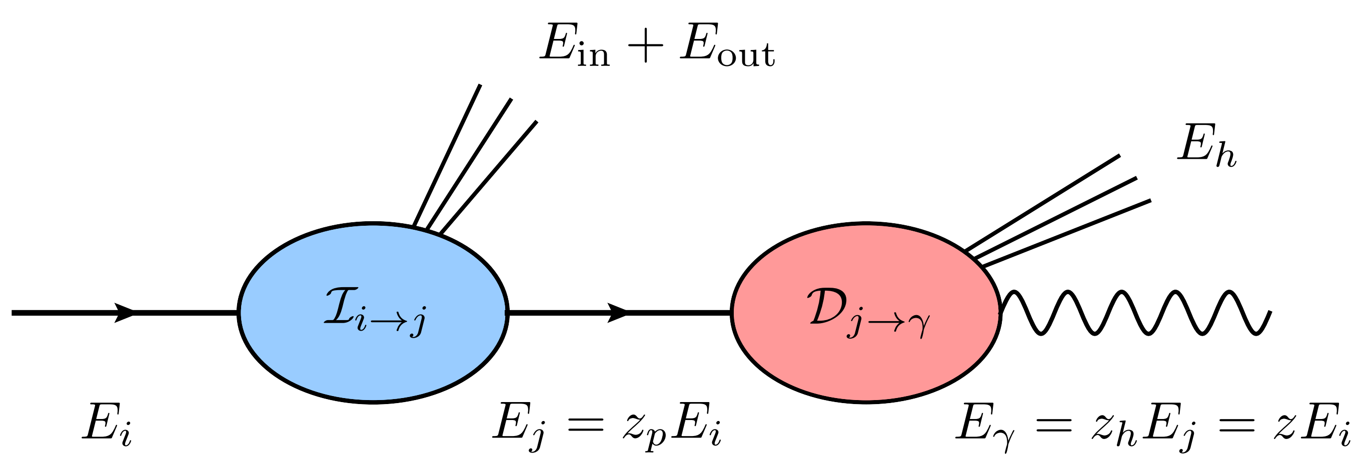
- Understand isolation with analytical formalism

- Resum the effects of $\ln(\varepsilon_\gamma)$ and $\ln(R)$

- Experimental measurements of photon production With different values of R and ε_γ

Appendices

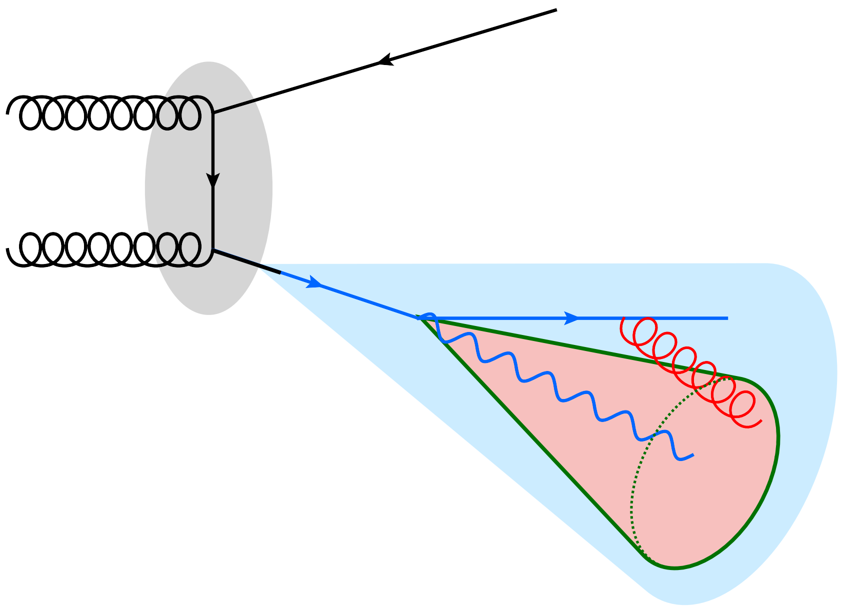
Cone fragmentation function

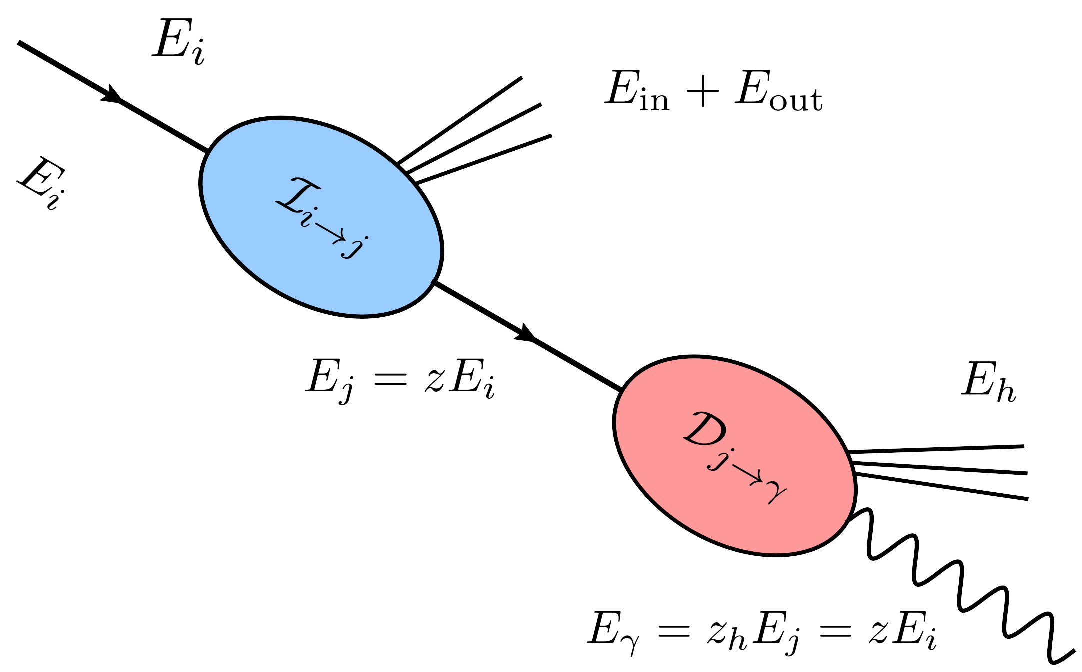


$$\mathcal{F}_{i\to\gamma}(z, E_\gamma, E_0, R, \mu) = \sum_{j=\gamma,q,\bar{q},g} \int_z^1 \frac{dz_h}{z_h} \int dE_{\rm in}\, \theta\Big(E_0 - E_{\rm in} - \frac{1-z_h}{z_h} E_\gamma\Big) \mathcal{I}_{i\to j}(z/z_h, E_\gamma, E_{\rm in}, R, \mu)\, \mathcal{D}_{j\to\gamma}(z_h, \mu)$$

- Should have: $E_0 = \epsilon_\gamma E_\gamma \to z_h > \dfrac{1}{1+\epsilon_\gamma}$
- In the limit of $\epsilon_\gamma = 0$: $\mathcal{D}_{\gamma\to\gamma}(z_h, \mu) = \delta(1 - z_h)$

Cone fragmentation function





The cone fragmentation function reads

Constraint on isolation energy

$$\mathcal{F}_{i\to\gamma}(z, E_\gamma, E_0, R, \mu) = \sum_{j=\gamma,q,\bar{q},g} \int_z^1 \frac{dz_h}{z_h} \int dE_{\rm in}\, \theta\left(E_0 - E_{\rm in} - \frac{1-z_h}{z_h} E_\gamma\right) \mathcal{I}_{i\to j}(z/z_h, E_\gamma, E_{\rm in}, R, \mu)\, \mathcal{D}_{j\to\gamma}(z_h, \mu)$$

perturbative kernel

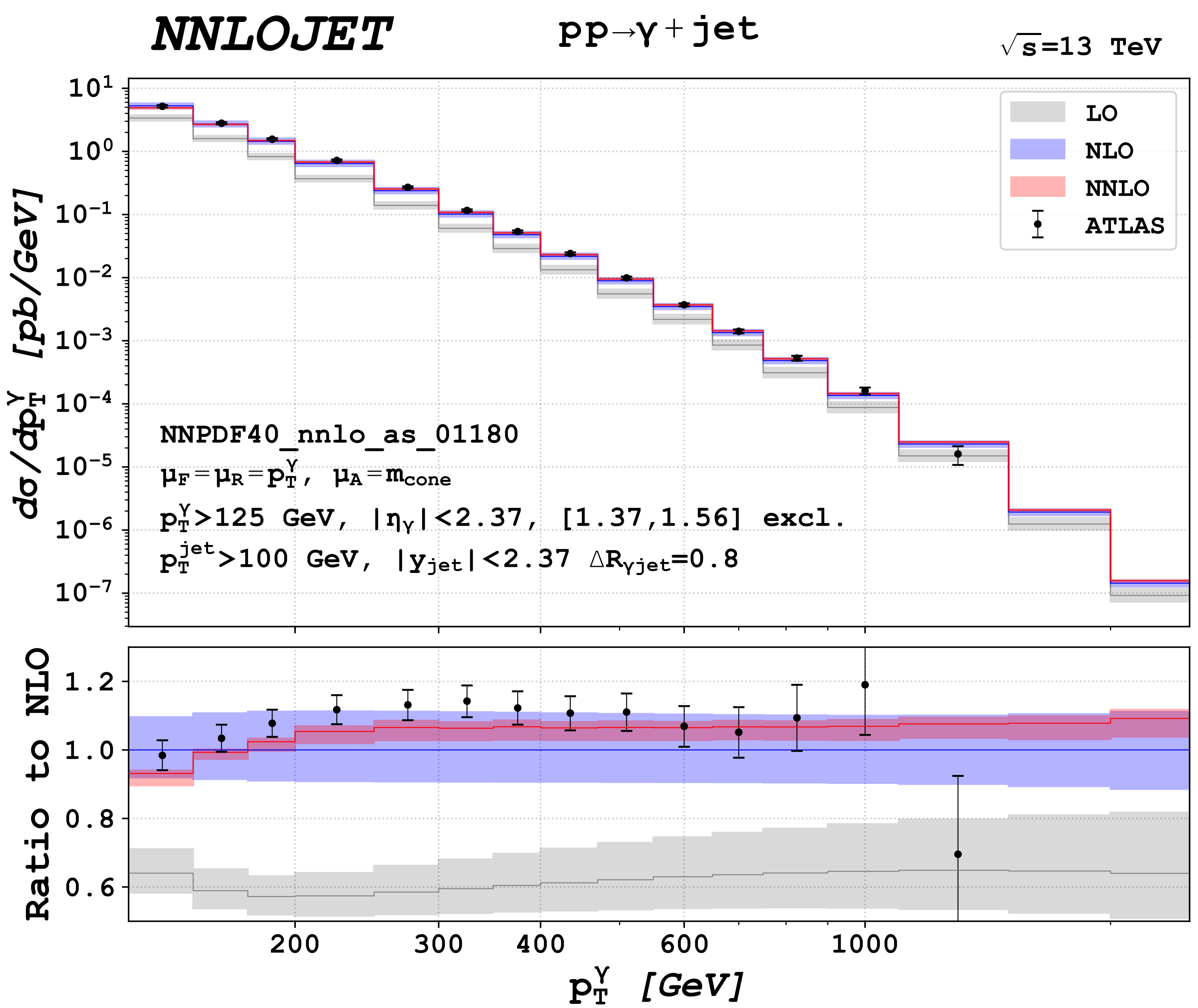
Non-perturbative fragmentation function

Status of fixed order predictions

- NLO predictions
    - Jetphox (Catani et al. '99) , Diphox (Binoth et al. '99)
    - MCFM
    - MG5_ aMC@NLO but restricted to Frixone-cone

  Have verified (thanks to Alex Huss!) that different codes produce compatible reference cross sections.

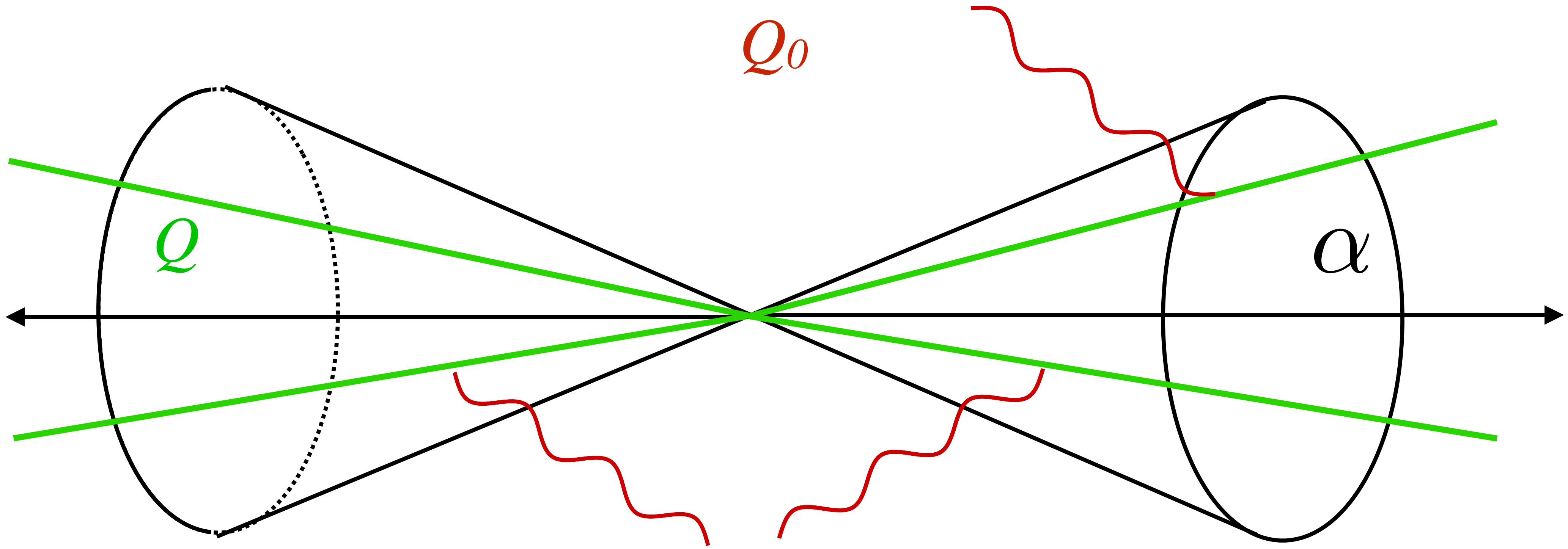
- NNLO predictions
    - Prompt photon Campbell et al. '17, Chen et al. '19
    - Isolation with hybrid-cone Gehrmann et. al. '21



New: first NNLO results with fixed-cone isolation Chen, Gehrmann, Glover, Höfer, Huss, Schürmann ’22

Factorization for jet process

(Becher, Neubert, Rothen, Shao '15)



$$r = \tan^2(\alpha/2)$$

- The cross sections are factorized as

$$\sigma(Q, Q_0) = \sum_{m=m_0}^{\infty} \langle \mathcal{H}_m(\{\underline{n}\}, Q, \mu) \otimes \mathcal{S}_m(\{\underline{n}\}, Q_0, \mu) \rangle$$

Color trace

Integration over direction {n}

(Becher, Neubert, Rothen, Shao '15)

$$\sigma(Q, Q_0) = \sum_{m=m_0}^{\infty} \langle \mathcal{H}_m(\{\underline{n}\}, Q, \mu) \otimes \mathcal{S}_m(\{\underline{n}\}, Q_0, \mu) \rangle$$

Color trace

Integration over direction {n}

Hard function with fixed direction $\{\underline{n}\} = \{n_1, \ldots, n_m\}$

$$\mathcal{H}_m(\{\underline{n}\}, Q, \epsilon) = \frac{1}{2Q^2} \sum_{\text{spins}} \prod_{i=1}^{m} \int \frac{dE_i \, E_i^{d-3}}{\tilde{c}^{\epsilon} \, (2\pi)^2} \, |\mathcal{M}_m(\{\underline{p}\})\rangle\langle\mathcal{M}_m(\{\underline{p}\})| \times (2\pi)^d \, \delta\Big(Q - \sum_{i=1}^{m} E_i\Big) \, \delta^{(d-1)}(\vec{p}_{\text{tot}}) \, \Theta_{\text{in}}(\{\underline{n}\})$$

Soft function along directions $\{\underline{n}\} = \{n_1, \ldots, n_m\}$

$$\mathcal{S}_m(\{\underline{n}\}, Q_0, \epsilon) = \sum\!\!\!\!\!\!\int_{X_s} \langle 0| \, \mathbf{S}_1^{\dagger}(n_1) \, \ldots \, \mathbf{S}_m^{\dagger}(n_m) \, |X_s\rangle\langle X_s| \, \mathbf{S}_1(n_1) \, \ldots \, \mathbf{S}_m(n_m) \, |0\rangle \, \theta(Q_0 - 2E_{\text{out}})$$

Parton shower

RG equations at LO

$$t = \frac{1}{2\beta_0} \ln \frac{\alpha_s(\mu_h)}{\alpha_s(\mu_s)}$$

$$\frac{d}{dt}\mathcal{H}_m(t) = \mathcal{H}_m(t)\mathbf{V}_m + \mathcal{H}_{m-1}(t)\mathbf{R}_{m-1}$$

Solutions of RG equations

$$\mathcal{H}_m(t) = \mathcal{H}_m(t_0)e^{(t-t_0)\mathbf{V}_m} + \int_{t_0}^{t} dt'\mathcal{H}_{m-1}(t')\mathbf{R}_{m-1}e^{(t-t')\mathbf{V}_m}$$

Parton shower for hard function

(Balsiger, Becher, Shao '18),
(Dasgupta, Salam '02),
(Banfi, Marchesini, Smye '02),