Resummation for e^+e^- Event Shapes to N4LL

Gherardo Vita

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Based on:

"Collinear expansion for color singlet cross sections"
M.Ebert, B.Mistlberger, **GV**
[2006.03055]

"The Four-Loop Rapidity Anomalous Dimension and Event Shapes to Fourth Logarithmic Order"
C.Duhr, B.Mistlberger, **GV**
[2205.02242]

"Soft Integrals and Soft Anomalous Dimensions at N3LO and Beyond"
C.Duhr, B.Mistlberger, **GV**
[2205.04493]

Introduction

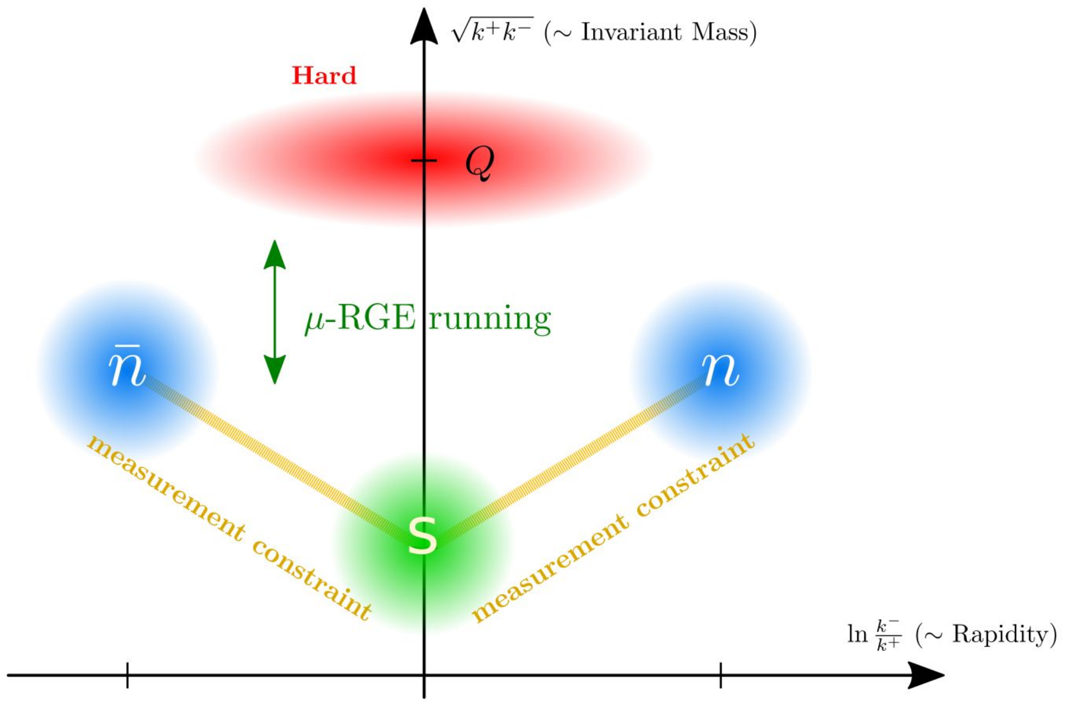
- Given the very nice resummation overview by Thomas I will focus on:
  - **General aspects of resummation using Soft and Collinear Effective Theor**
    - Modes, Factorization, Resummation from RGE, Boundaries
  - **Pushing Resummation to N4LL at $e^+ e^-$**
    - The Energy - Energy Correlation
    - Boundaries: from N3LO q_T beam Functions to EEC Jet functions
    - Anomalous dimensions: The Rapidity Anomalous dimension to 4 loops
    - The EEC in the back-to-back limit at N4LL
  - **What about N4LL resummation for other observables?**

Resummation in SCET: Modes

- For a large class of standard observables (thrust, C-Parameter), the singular limit takes

$$d\sigma \sim H(Q,\mu)\, J_n(\tau,\mu) \otimes J_{\bar{n}}(\tau,\mu) \otimes S(\tau,\mu)$$

Hard — Collinear — Soft



Resummation in SCET

- For a variety of simple observables, the singular limit takes the form

$$d\sigma \sim H(Q,\mu) J_n(\tau,\mu) \otimes J_{\bar{n}}(\tau,\mu) \otimes S(\tau,\mu)$$

Hard — Collinear — Soft

- Each of these ingredients is a **gauge invariant cross section level object** (not an amplitude) defined in terms of fields with fixed momentum scaling
- We can calculate each object separately with SCET feynman rules (derived from collinear and soft lagrangians)
- In calculating these objects, at each order one gets explicit $\log\left(\frac{Q^2\tau^p}{\mu^2}\right)$
- These logs have UV nature in the EFT

  => resum them using counterterms as done in standard QFT for running coupling!

Resummation in SCET: RGEs for H,J, and S

- For a variety of simple observables, the singular limit takes the form

$$\mathrm{d}\sigma \sim H(Q,\mu) J_n(\tau,\mu) \otimes J_{\bar{n}}(\tau,\mu) \otimes S(\tau,\mu)$$

$$\frac{\mathrm{d}}{\mathrm{d}\ln\mu} H(Q^2,\mu) = \gamma_H(Q^2,\mu) H(Q^2,\mu)$$

$$\frac{\mathrm{d}}{\mathrm{d}\ln\mu} J(Q^2\tau,\mu) = \gamma_J(Q^2\tau,\mu) \underset{\tau}{\otimes} J(Q^2\tau,\mu)$$

$$\frac{\mathrm{d}}{\mathrm{d}\ln\mu} S(Q^2\tau^2,\mu) = \gamma_S(Q^2\tau^2,\mu) \underset{\tau}{\otimes} S(Q^2\tau^2,\mu)$$

Factorized objects obey
RG Equations

Resummation in SCET: RGE Solution

- For a variety of simple observables, the singular limit takes the form

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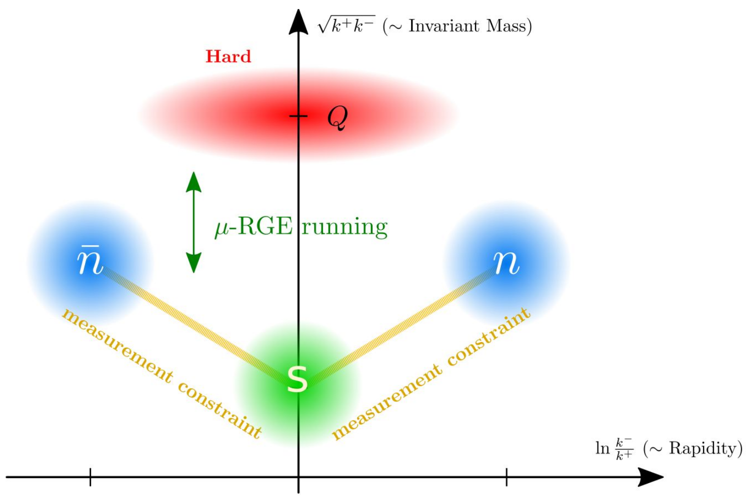

$$H^{\mathrm{res}}(Q^2,\mu) = H\left(Q^2/\mu_H^2;\, \alpha_s(\mu_H)\right) \mathcal{U}_H(Q^2,\mu_H,\mu)$$

$$J^{\mathrm{res}}(Q^2\tau,\mu) = J\left(Q^2\tau/\mu_J^2;\, \alpha_s(\mu_J)\right) \underset{\tau}{\otimes} \mathcal{U}_J(Q^2\tau,\mu_J,\mu)$$

$$S^{\mathrm{res}}(Q^2\tau^2,\mu) = S\left(Q^2\tau^2/\mu_S^2;\, \alpha_s(\mu_S)\right) \underset{\tau}{\otimes} \mathcal{U}_S(Q^2\tau^2,\mu_S,\mu)$$

Solution in terms of
Boundary
and

Resummation in SCET: RGE Solution

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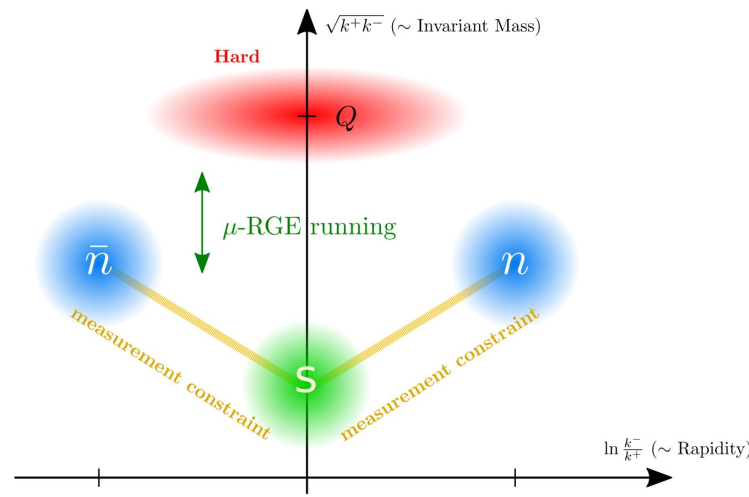

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Solution in terms of **Boundary** and **Evolution Factor**

Resummation in SCET: Boundary Scales

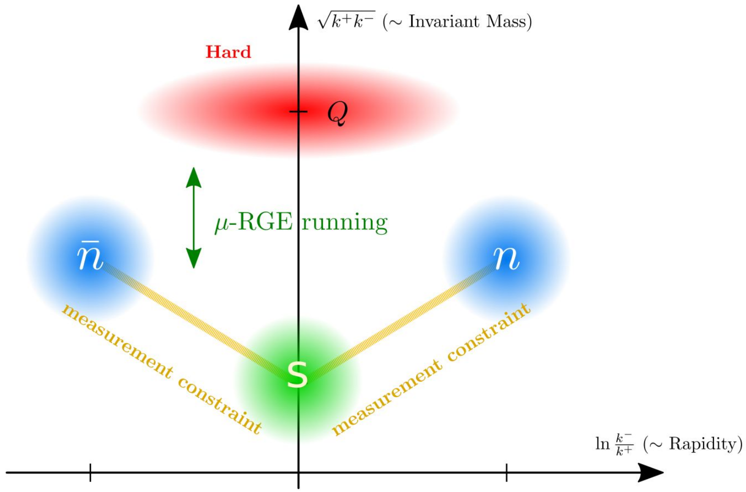
- For a variety of simple observables, the singular limit takes the form

$$d\sigma \sim H(Q,\mu) J_n(\tau,\mu) \otimes J_{\bar{n}}(\tau,\mu) \otimes S(\tau,\mu)$$

"Canonical" choice of scales for the boundaries is the one that minimizes all the logs in them

$$\mu_H^2 \sim Q^2\,, \quad \mu_J^2 \sim Q^2\tau\,, \quad \mu_S^2 \sim Q^2\tau^2$$

Resummation uncertainties usually estimated by varying around these canonical choices



$$H^{\rm res}(Q^2,\mu) = H\left(Q^2/\mu_H^2;\, \alpha_s(\mu_H)\right) \mathcal{U}_H(Q^2,\mu_H,\mu)$$

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Solution in terms of **Boundary** and **Evolution Factor**

Resummation in SCET: Resummation Accuracy

- For a variety of simple observables, the singular limit takes the form

Accuracy	$H,\ J,\ S$	$\gamma^{\text{non-cusp}}_{H,J,S}(\alpha_s)$	$\beta(\alpha_s)$	$\Gamma_{\text{cusp}}(\alpha_s)$
LL	Tree level	–	1-loop	1-loop
NLL	Tree level	1-loop	2-loop	2-loop
NLL′	1-loop	1-loop	2-loop	2-loop
NNLL	1-loop	2-loop	3-loop	3-loop
NNLL′	2-loop	2-loop	3-loop	3-loop
N^3LL	2-loop	3-loop	4-loop	4-loop
N^3LL′	3-loop	3-loop	4-loop	4-loop
N^4LL	3-loop	4-loop	5-loop	5-loop

$\ldots, \mu) \otimes S(\tau, \mu)$

Resummation accuracy is determined by **perturbative accuracy** of ingredients entering resummed cross section

$$H^{\text{res}}(Q^2,\mu) = H\left(Q^2/\mu_H^2;\, \alpha_s(\mu_H)\right) \mathcal{U}_H(Q^2,\mu_H,\mu)$$

$$J^{\text{res}}(Q^2\tau,\mu) = J\left(Q^2\tau/\mu_J^2;\, \alpha_s(\mu_J)\right) \underset{\tau}{\otimes} \mathcal{U}_J(Q^2\tau,\mu_J,\mu)$$

$$S^{\text{res}}(Q^2\tau^2,\mu) = S\left(Q^2\tau^2/\mu_S^2;\, \alpha_s(\mu_S)\right) \underset{\tau}{\otimes} \mathcal{U}_S(Q^2\tau^2,\mu_S,\mu)$$

Solution in terms of **Boundary** and **Evolution Factor**

Resummation in SCET: Resummation Accuracy

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NLL′	1-loop	1-loop	2-loop	2-loop
NNLL	1-loop	2-loop	3-loop	3-loop
NNLL′	2-loop	2-loop	3-loop	
N^{3}LL	2-loop	3-loop	4-loop	
N^{3}LL′	3-loop	3-loop	4-loop	
N^{4}LL	3-loop	4-loop	5-loop	

Resummation accuracy is determined by **perturbative** ...nts ...oss

Primed orders:
include **boundaries** at same order as anomalous dimensions
=> **better accuracy**.
e.g.
<u>NNLL' is *better* than NNLL</u>
(also tames issues between scale setting/log accuracy in direct vs conjugate space)

$$H^{\text{res}}(Q^2,\mu) = H\left(Q^2/\mu_H^2;\, \alpha_s(\mu_H)\right) \mathcal{U}_H(Q\ldots$$

$$J^{\text{res}}(Q^2\tau,\mu) = J\left(Q^2\tau/\mu_J^2;\, \alpha_s(\mu_J)\right) \underset{\tau}{\otimes} \mathcal{U}_J(Q^2\tau, \mu_J, \mu)$$

$$S^{\text{res}}(Q^2\tau^2,\mu) = S\left(Q^2\tau^2/\mu_S^2;\, \alpha_s(\mu_S)\right) \underset{\tau}{\otimes} \mathcal{U}_S(Q^2\tau^2, \mu_S, \mu)$$

...s of **Boundary** and **Evolution Factor**

Resummation in e^+e^- at N4LL

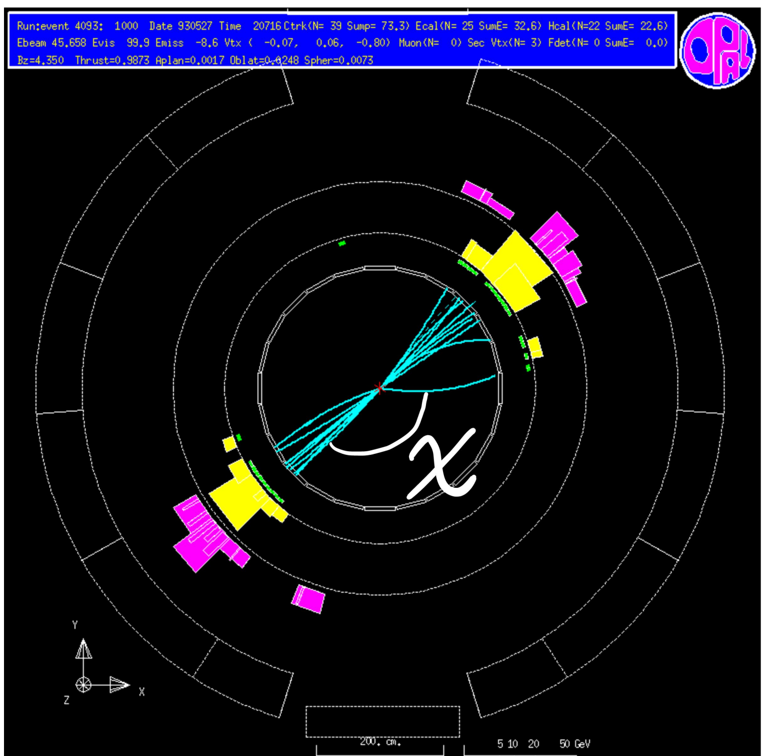
“The Four-Loop Rapidity Anomalous Dimension and Event Shapes to Fourth Logarithmic Order”
C.Duhr, B.Mistlberger, **GV**
[2205.02242]

Energy-Energy Correlation

- One of the oldest IRC safe observables proposed to study QCD radiation is the Energy-Energy Correlation (EEC)

[Basham, Brown, Ellis, Love, PRL **41**, 1585 **(1978)**]

$$\mathrm{EEC}(\chi) = \frac{\mathrm{d}\sigma}{\mathrm{d}\chi} = \sum_{i,j} \int \mathrm{d}\sigma_{e^+e^- \to ij+X} \frac{E_i E_j}{Q^2} \, \delta(\cos\theta_{ij} - \cos\chi)$$

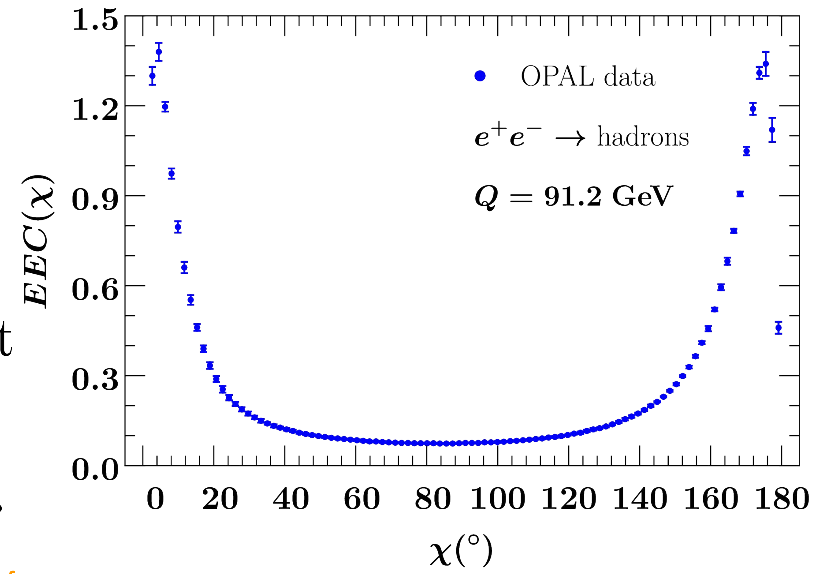


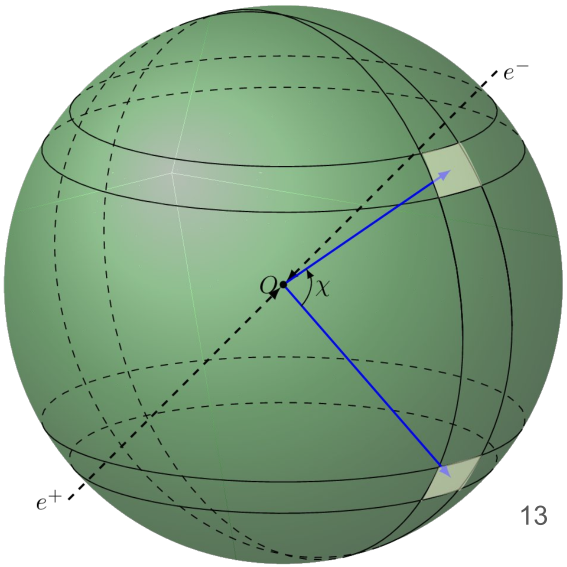
- Measure of the **angle** χ between pairs of color charged particles, **weighted by energy**

Energy-Energy Correlation: Motivations

Interest in EEC for a variety of reasons:

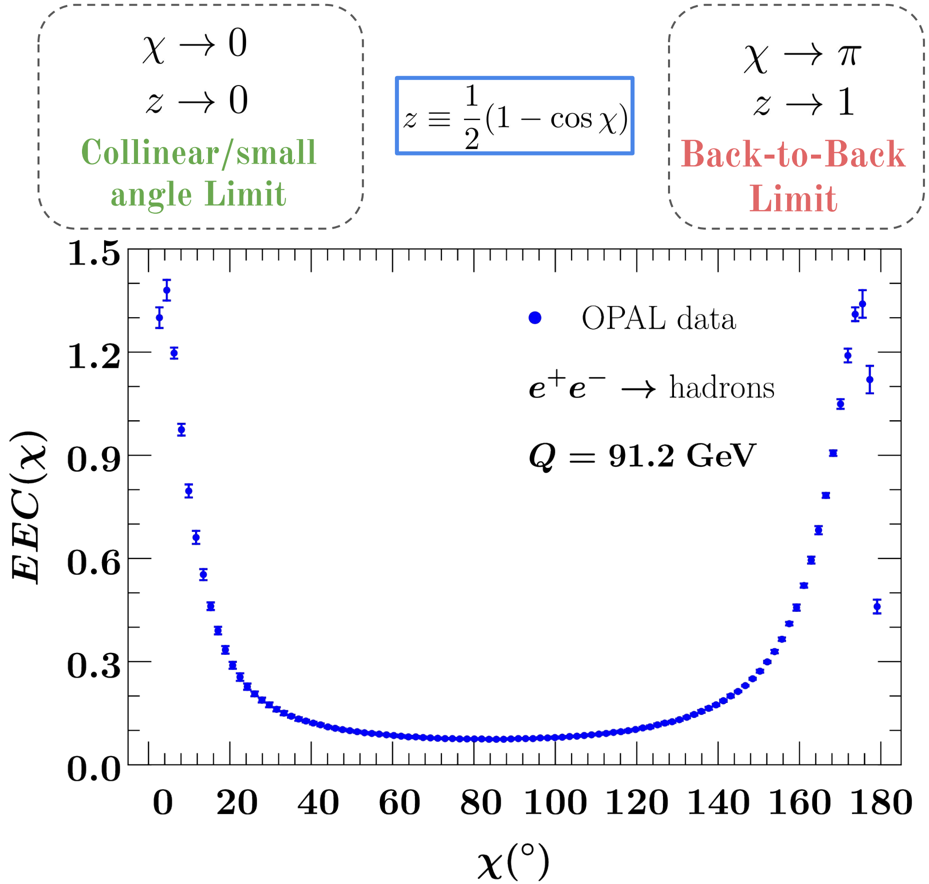
- Very well measured event shape at electron-positron colliders **GOOD DATA!**
- Arguably the event shape with the simplest analytic structure **GOOD THEORY!**
- Can be expressed as a four point correlator in terms of energy flow operators [Maldacena, Hofman; Korchemsky;]
- Substantial recent progress in understanding it both in QCD and CFTs. [Moult, Dixon, Zhu; Korchemsky; Chicherin, Henn, Sokatchev, Yan; Simmons Duffin, Kologlu, Kravchuk, Zhiboedov; Moult, Vita, Yan; Luo, Shtabovenko, Yang;...]
- Natural playground for connections between $\mathcal{N}=4$ and QCD
- Allows precise extraction of α_s [OPAL Collaboration; Kardos, Kluth, Somogyi, Tulipant, Verbytskyi; ...]
- ...





Energy-Energy Correlation: End Points

- It has singular structure and logarithmic enhancement at both end points



- We can derive factorization theorems at both ends in SCET for resummation

[Moult, Zhu] [Moult, Dixon, Zhu] [Ebert, Mistlberger, **GV**]

Energy-Energy Correlation: Collinear limit

- The two limits have very different structure (no symmetry between them)

$\chi \to 0$

$z \to 0$

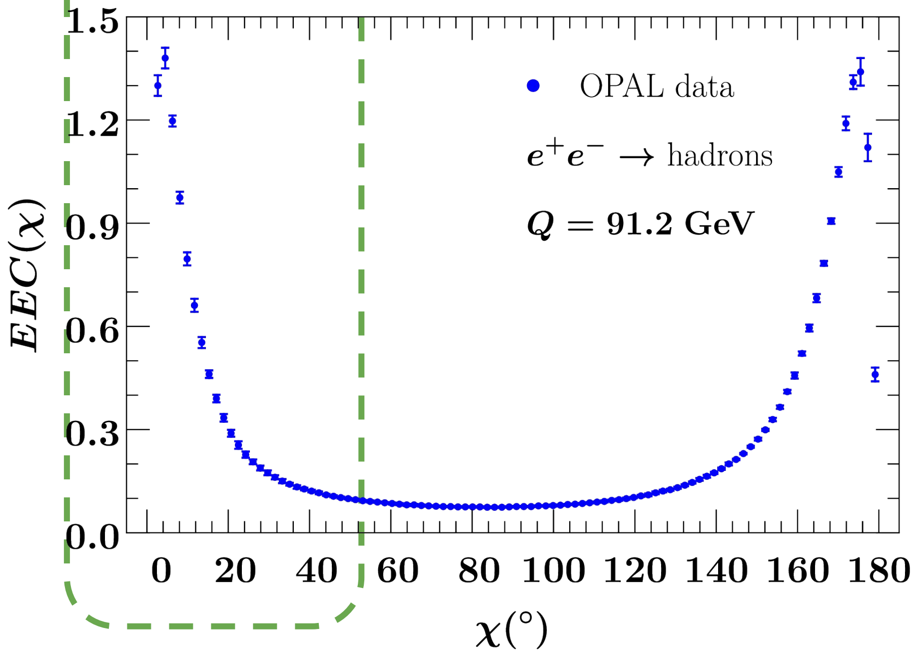
Collinear/small angle Limit

$z \equiv \frac{1}{2}(1-\cos\chi)$

$\chi \to \pi$

$z \to 1$

Back-to-Back Limit



Collinear/small angle Limit

- **Single logarithmic** series

$$\frac{d\sigma}{dz} \overset{z\to 0}{\sim} \sum_{L=1}^{\infty} \sum_{m=0}^{L-1} \left(\frac{\alpha_s}{4\pi}\right)^L c_{L,m} \frac{\log^m z}{z}$$

- Contact terms $\sim \delta(z)$

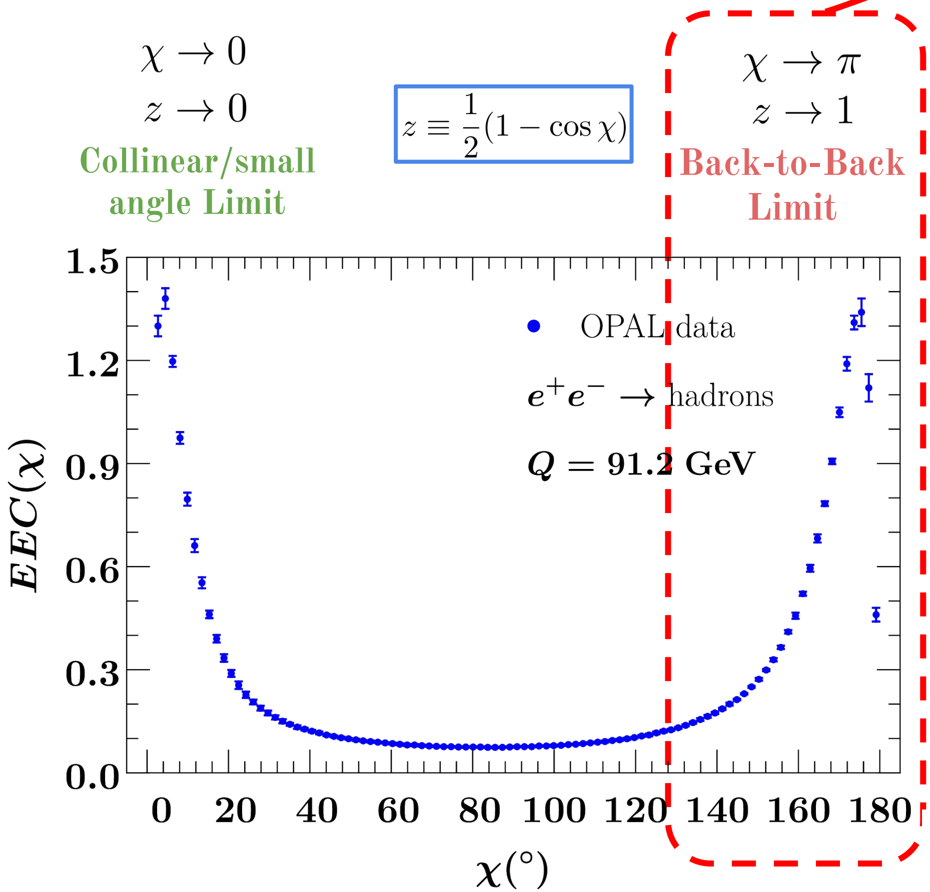
- Simple (time-like) collinear factorization in terms of Hard and single Jet function [Konishi Ukawa, Veneziano] [Moult, Dixon, Zhu]

$$\Sigma(z, \ln\frac{Q^2}{\mu^2}, \mu) = \int_0^1 dx\, x^2 \vec{J}(\ln\frac{zx^2Q^2}{\mu^2}, \mu) \cdot \vec{H}(x, \frac{Q^2}{\mu^2}, \mu)$$

- No rapidity divergences

Energy-Energy Correlation: Back-to-Back Limit

- The two limits have very different structure (no symmetry between them)



Back-to-Back Limit

- **Double logarithmic** series

$$\frac{d\sigma}{dz} \overset{z\to 1}{\sim} \sum_{L=1}^{\infty} \sum_{m=0}^{2L-1} \left(\frac{\alpha_s}{4\pi}\right)^L d_{L,m} \frac{\log^m(1-z)}{(1-z)}$$

- **Contact terms** $\sim \; \delta(1-z)$

- **TMD Factorization** related to di-hadron production at small q_T in electron-positron annihilation

[Collins, Soper]

$$\lim_{z\to 1}\frac{d\sigma}{dz} = \int_0^1 dz_1 dz_2 \frac{z_1 z_2}{2} \int d^2\vec{q}_T\, \delta\left(1 - z - \frac{q_T^2}{Q^2}\right) \lim_{q_T\to 0} \sum_{h_1,h_2} \frac{d\sigma_{e^+e^-\to h_1 h_2}}{dz_1 dz_2 d^2\vec{q}_T}$$

- Sensitive to rapidity divergences

EEC in the back to back limit to N4LL

- Back-to-back region of EEC obeys TMD-like fact. thm and resummation ("crossed version of q_T")
- In pure rapidity renormalization it takes the following form

$$\frac{d\sigma}{dz} = \frac{\hat{\sigma}_0}{2} \overbrace{H_{q\bar{q}}(Q,\mu)}^{\text{Hard Function}} \int \frac{d^2\vec{b}_T\, d^2\vec{q}_T}{(2\pi)^2} e^{i\vec{q}_T\cdot\vec{b}_T} \overbrace{\delta\left(1 - z - \frac{q_T^2}{Q^2}\right)}^{1-z\equiv(\cos\frac{\chi}{2})^2\approx\frac{q_T^2}{Q^2}} \underbrace{\mathcal{J}_q\left(b_T,\mu,\frac{Qb_T}{\upsilon}\right)\mathcal{J}_{\bar{q}}\left(b_T,\mu,Qb_T\upsilon\right)}_{\text{Pure Rapidity EEC Jet Functions}}$$

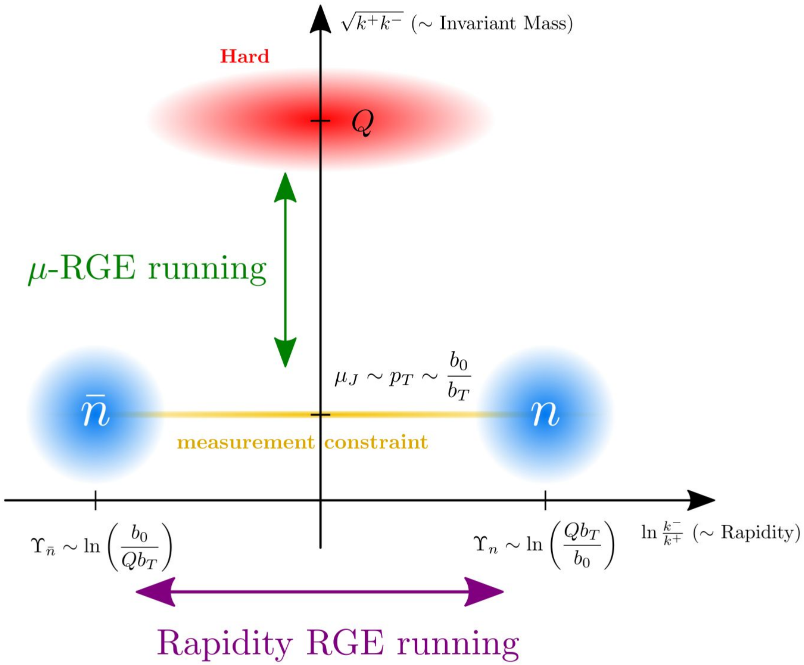
Standard RGE

$$\mu\frac{d}{d\mu}\ln H_{q\bar{q}}(Q,\mu) = \gamma_H^q(Q,\mu)\,,$$

$$\mu\frac{d}{d\mu}\ln \mathcal{J}_q\left(b_T,\mu,\frac{Qb_T}{\upsilon}\right) = \gamma_{\mathcal{J}_q}(\mu,\upsilon\mu/Q)$$

Rapidity RGE

$$\upsilon\frac{d}{d\upsilon}\ln \mathcal{J}_q\left(b_T,\mu,\frac{Qb_T}{\upsilon}\right) = -\frac{1}{2}\gamma_r^q(b_T,\mu)$$



Logarithmic Accuracy for Resummed Predictions

- **Resummation accuracy** is determined by perturbative accuracy of ingredients entering resummed cross section

- For **N4LL resummation:**
  - 3 Loop Hard Function
    [Gehrmann, Glover, Huber, Ikizlerli, Studerus '10]
  - **3 Loop EEC Jet Function**
  - 4 Loop Collinear Anom. Dim.
    [von Manteuffel, Panzer, Schabinger '20]
  - **4 Loop Rapidity Anomalous Dimension**
  - 5 Loop Beta function
    [Baikov, Chetyrkin, Kuhn '16]
  - 5 Loop Cusp (approx)
    [Herzog, Moch, Ruijl,Ueda, Vermaseren, Vogt '18]

Resummed cross section to all orders (at LP)

$$\frac{\mathrm{d}\sigma}{\mathrm{d}z} = \frac{\hat{\sigma}_0}{8}\int_0^\infty \mathrm{d}(b_T Q)^2 J_0\left(b_T Q\sqrt{1-z}\right) H_{q\bar{q}}(Q,\mu_H)$$
$$\times \mathcal{J}_q\left(b_T,\mu_J,\frac{Qb_T}{\upsilon_n}\right)\mathcal{J}_{\bar{q}}\left(b_T,\mu_J,Qb_T\upsilon_{\bar{n}}\right)\left(\frac{\upsilon_n}{\upsilon_{\bar{n}}}\right)^{\frac{1}{2}\gamma_r^q(b_T,\mu_J)}$$
$$\times \exp\left[4\int_{\mu_J}^{\mu_H}\frac{\mathrm{d}\mu'}{\mu'}\Gamma^q_{\mathrm{cusp}}[\alpha_s(\mu')]\ln\frac{\mu'}{Q} - \gamma_H^q[\alpha_s(\mu')]\right]$$

Accuracy	H, $\mathcal{J}$	$\gamma_H^q(\alpha_s)$	$\gamma_r^q(\alpha_s)$	$\beta(\alpha_s)$	$\Gamma_{\mathrm{cusp}}(\alpha_s)$
LL	Tree level	–	–	1-loop	1-loop
NLL	Tree level	1-loop	1-loop	2-loop	2-loop
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NNLL	1-loop	2-loop	2-loop	3-loop	3-loop
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$\mathrm{N^3LL}$	2-loop	3-loop	3-loop	4-loop	4-loop
$\mathrm{N^3LL'}$	3-loop	3-loop	3-loop	4-loop	4-loop
$\mathrm{N^4LL}$	3-loop	4-loop	4-loop	5-loop	5-loop

Boundaries: from N3LO BF to EEC Jet Functions

Calculation of the q_T beam functions to N3LO

"TMD PDFs at N3LO"
M.Ebert, B.Mistlberger, **GV** [2006.05329]

Calculation of the TMDFF to N3LO via analytic continuation

"TMD Fragmentation Functions at N3LO"
M.Ebert, B.Mistlberger, **GV** [2012.07853]

Calculation of the EEC Jet Functions to N3LO

"The EEC in the back-to-back limit at N3LO and N3LL'"
M.Ebert, B.Mistlberger, **GV** [2012.07859]

Boundaries: from N3LO BF to EEC Jet Functions

Calculation of the q_T beam functions to N3LO

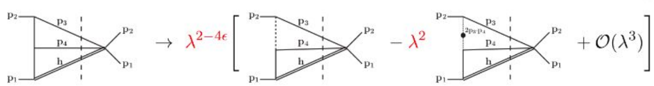
"TMD PDFs at N3LO"

M.Ebert, B.Mistlberger, **GV** [2006.05329]

A 1 million integral calculation in 1 slide

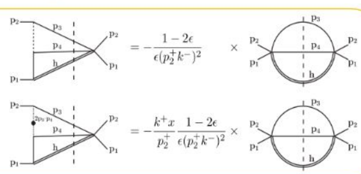
- We calculated the **collinear expansion of the partonic cross section** for DY and Higgs @N3LO **differential in (Q_T, τ, z)**
  - Collinear Expansion at the XS level

    "Collinear expansion for color singlet cross sections" [Ebert, Mistlberger, GV]

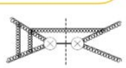


  - Reduction to basis of **Master Integrals** via Integration By Parts (IBPs)

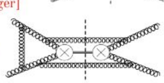
Expanded diagrams admit (simplified) **IBPs** identities



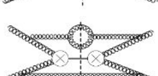
  - RVV: known in full kinematics [Duhr, Gehrmann] [Duhr, Gehrmann, Jaquier] [Dulat, Mistlberger]



  - RRV: **170** Master Integrals



  - RRR: **320** Master Integrals



- Derived system of Differential Equations for the Master Integrals
- System has 2 non trivial scales with algebraic dependence on the variables (not something you can solve algorithmically with CANONICA, Fuchsia, etc...)
- Algebraic sectors: constructed dlog integrand basis via calculation of **leading singularities** of candidate integrals on maximal cut surface
- Boundaries from soft integrals [Anastasiou, Duhr, Dulat, Mistlberger] and constraints on singular behavior
- Result has large non-rational alphabet

$$\mathcal{A} = \{1-x, x, z-x(z-1), x(z-1)+1, x(z-1)+2, x(z-1)-z+2, (z-1)x-x+1, x(z-1)^2+2(z-1)+1,$$
$$(x(z-1)^2+4x(z-1)+4)z+(z+1)\sqrt{x^2(z-1)^3+5x^2(z-1)^2+8x(z-1)+4}\sqrt{z},$$
$$z\sqrt{x(z-1)+1}+\sqrt{z}\sqrt{x(z-1)^2-3x(z-1)+z}, \sqrt{x(z-1)+1}\sqrt{z}\sqrt{x(z-1)^2-3x(z-1)+z}-(x(z-1)-1)z,$$
$$(x(z-1)+1)z+\sqrt{x(z-1)+1}\sqrt{z}\sqrt{x(z-1)^2-3x(z-1)+z}, ((z-1)x-2x+1)z+\sqrt{x(z-1)+1}\sqrt{z}\sqrt{x(z-1)^2-3x(z-1)+z},$$

Boundaries: from N3LO BF to EEC Jet Functions

Calculation of the q_T beam functions to N3LO

“TMD PDFs at N3LO”
M.Ebert, B.Mistlberger, **GV** [2006.05329]

↓

Calculation of the TMDFF to N3LO via analytic continuation

“TMD Fragmentation Functions at N3LO”
M.Ebert, B.Mistlberger, **GV** [2012.07853]

↓

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“The EEC in the back-to-back limit at N3LO and N3LL’”
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“The EEC in

M.Eb

Analytic Continuation

- Crossing symmetry for SIDIS (and similarly for decay processes)

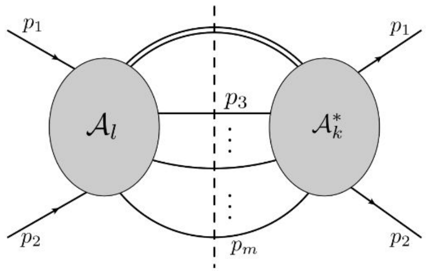
$$\underbrace{p(p_1) + h(q) \to p(-p_2) + X(-k)}_{\text{Semi-Inclusive DIS}} \quad \longleftrightarrow \quad \underbrace{p(p_1) + p(p_2) \to h(-q) + X(-k)}_{\text{DY/Higgs production in pp}}$$

- Variables we are differential in change sign under crossing!
- Therefore, crossing means analytically continuing cross section in those variables
- Mandelstam equipped with prescription $\quad (p_i + p_j)^2 \quad \to \quad (p_i + p_j)^2 + i0$
- Partonic cross section branch cut structure can be tracked by organizing it as

Difficult, but universal part (no branch cuts at 0)

$$\frac{d\eta_{ij}^{(m+l+k)}}{dQ^2 dw_1 dw_2 dx} = (sw_1w_2)^{-m\epsilon} \times \left\{ \sum_{i_1,i_2=0}^{l} \sum_{j_1,j_2=0}^{k} \frac{d\eta_{ij}^{(m+l+k,i_1,i_2,j_1,j_2)}}{dQ^2 dw_1 dw_2 dx} \times \Re\left\{ \left[(-s)^{(i_1+i_2-l)\epsilon}(sw_1)^{-i_1\epsilon}(sw_2)^{-i_2\epsilon}\right]\left[(-s)^{(j_1+j_2-k)\epsilon}(sw_1)^{-j_1\epsilon}(sw_2)^{-j_2\epsilon}\right]^* \right\}\right\}$$

Mandelstam variables, easy to analytically continue



Simplest l-loop example:

$$\underbrace{\Re\left[(-s - i0)^{-l\epsilon}\right]}_{\text{DIS}} \quad \longleftrightarrow \quad \underbrace{\cos(l\pi\epsilon)\, \Re\left[(s + i0)^{-l\epsilon}\right]}_{\text{production}}$$

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Boundaries: from N3LO BF to EEC Jet Functions

$$
\begin{aligned}
\frac{1}{C_F}\frac{\mathrm{d}\bar\sigma^{(3)}}{\mathrm{d}z} = &-4C_F^2\mathcal{L}_5(\bar z) \\
&+\mathcal{L}_4(\bar z)\Big[-30C_F^2-\frac{220}{9}C_FC_A+\frac{40}{9}C_Fn_f\Big] \\
&+\mathcal{L}_3(\bar z)\Big[C_F^2(-16\zeta_2-104)+\frac{88}{9}C_Fn_f+C_FC_A\Big(-16\zeta_2-\frac{388}{9}\Big) \\
&\qquad -\frac{242}{9}C_A^2+\frac{88}{9}C_An_f-\frac{8}{9}n_f^2\Big] \\
&+\mathcal{L}_2(\bar z)\Big[C_F^2(-144\zeta_2-16\zeta_3-189)+C_FC_A\Big(-\frac{592}{3}\zeta_2-72\zeta_3+\frac{244}{3}\Big) \\
&\qquad +C_Fn_f\Big(\frac{88}{3}\zeta_2-\frac{40}{3}\Big)+C_A^2\Big(\frac{2471}{27}-\frac{88}{3}\zeta_2\Big) \\
&\qquad +C_An_f\Big(\frac{16}{3}\zeta_2-\frac{760}{27}\Big)+\frac{44n_f^2}{27}\Big] \\
&+\mathcal{L}_1(\bar z)\Big[C_F^2\Big(-\frac{542}{3}-412\zeta_2+224\zeta_3-192\zeta_4\Big) \\
&\qquad +C_FC_A\Big(-\frac{2900}{9}\zeta_2-\frac{1688}{3}\zeta_3 \ldots +\frac{3797}{\ldots}\Big) \\
&\qquad +C_Fn_f\Big(\frac{536}{9}\zeta_2+\frac{32}{3}\zeta_3-\frac{479}{9}\ldots \\
&\qquad +C_An_f\Big(\frac{448}{9}\zeta_2+16\zeta_3-\frac{380}{81}\ldots \\
&+\mathcal{L}_0(\bar z)\Big[C_F^2\Big(64\zeta_3\zeta_2-402\zeta_2+332\zeta_3 \ldots \\
&\qquad +C_FC_A\Big(-128\zeta_3\zeta_2+\frac{212}{3}\zeta_2-\ldots\zeta_3-\ldots \\
&\qquad +C_Fn_f\Big(-\frac{20}{3}\zeta_2-\frac{296}{9}\zeta_3+\frac{244}{3}\zeta_4-\frac{623}{18}\Big) \\
&\qquad +C_A^2\Big(\frac{4420}{9}\zeta_2-\frac{560}{9}\zeta_3-\frac{326}{3}\zeta_4-40\zeta_5-\frac{4241}{27}\Big) \\
&\qquad +C_An_f\Big(-\frac{1508}{9}\zeta_2+\frac{184}{9}\zeta_3+\frac{56}{3}\zeta_4+\frac{1414}{27}\Big) \\
&\qquad +n_f^2\Big(\frac{112}{9}\zeta_2+\frac{16}{9}\zeta_3-\frac{98}{27}\Big)\Big] \\
&+\delta(\bar z)\Big[C_F^2\Big(-\frac{337}{3}-\frac{1049}{3}\zeta_2+\frac{530}{3}\zeta_3+512\zeta_2\zeta_3-64\zeta_3^2-1396\zeta_4+\frac{3136}{3}\zeta_5-672\zeta_6\Big) \\
&\quad +C_FC_A\Big(\frac{10169}{27}+\frac{2729}{3}\zeta_2-\frac{22070}{9}\zeta_3+\frac{2176\zeta_4}{9}+528\zeta_5+22\zeta_6-288\zeta_2\zeta_3+64\zeta_3^2\Big) \\
&\quad +C_Fn_f\Big(-\frac{148}{27}-\frac{985\zeta_2}{9}+\frac{3340\zeta_3}{9}+\frac{58\zeta_4}{9}-\frac{368\zeta_5}{3}-\frac{224}{3}\zeta_2\zeta_3\Big) \\
&\quad +C_A^2\Big(-\frac{55504}{81}-\frac{3968}{81}\zeta_2+\frac{39337}{27}\zeta_3+\frac{3815}{18}\zeta_4-\frac{2720}{3}\zeta_5-\frac{700}{3}\zeta_2\zeta_3+59\zeta_6-56\zeta_3^2\Big) \\
&\quad +C_An_f\Big(\frac{15626}{81}-\frac{3326}{81}\zeta_2-\frac{3788}{27}\zeta_3-\frac{290}{9}\zeta_4+80\zeta_5+72\zeta_2\zeta_3\Big) \\
&\quad +n_f^2\Big(-\frac{1048}{81}+\frac{616}{81}\zeta_2-\frac{464}{27}\zeta_3-\frac{16}{9}\zeta_4\Big) \\
&\quad +N_{F,V}\frac{d_{abc}d^{abc}}{N_r}\Big(2+5\zeta_2+\frac{7}{3}\zeta_3-\frac{\zeta_4}{2}-\frac{40}{3}\zeta_5\Big)\Big].
\end{aligned}
$$

beam functions to N3LO

DFs at N3LO"

rger, **GV** [2006.05329]

$$J_q\Big(b_T,\mu,\frac{\nu}{Q}\Big)=\int_0^1 \mathrm{d}\zeta\,\zeta\sum_i \tilde{\mathcal{K}}_{qi}\Big(\zeta,\vec b_T,\mu,\frac{\nu}{\omega_b}\Big)$$

Matching Kernels for TMD Fragmentation Function

rger, GV [2012...

C Jet Functions to N3LO

back limit at N3LO and N3LL'"

rger, **GV** [2012.07859]

Logarithmic Accuracy for Resummed Predictions

- **Resummation accuracy** is determined by perturbative accuracy of ingredients entering resummed cross section

- For **N4LL resummation:**
  - 3 Loop Hard Function
    [Gehrmann, Glover, Huber, Ikizlerli, Studerus '10]
  - **3 Loop EEC Jet Function** ✓
  - 4 Loop Collinear Anom. Dim.
    [von Manteuffel, Panzer, Schabinger '20]
  - **4 Loop Rapidity Anomalous Dimension**
  - 5 Loop Beta function
    [Baikov, Chetyrkin, Kuhn '16]
  - 5 Loop Cusp (approx)
    [Herzog, Moch, Ruijl,Ueda, Vermaseren, Vogt '18]

Resummed cross section to all orders (at LP)

$$\frac{d\sigma}{dz} = \frac{\hat{\sigma}_0}{8}\int_0^\infty d(b_T Q)^2 J_0(b_T Q\sqrt{1-z}) H_{q\bar{q}}(Q,\mu_H) \times \mathcal{J}_q\left(b_T,\mu_J,\frac{Qb_T}{\upsilon_n}\right)\mathcal{J}_{\bar{q}}\left(b_T,\mu_J,Qb_T\upsilon_{\bar{n}}\right)\left(\frac{\upsilon_n}{\upsilon_{\bar{n}}}\right)^{\frac{1}{2}\gamma_r^q(b_T,\mu_J)} \times \exp\left[4\int_{\mu_J}^{\mu_H}\frac{d\mu'}{\mu'}\Gamma^q_{\text{cusp}}[\alpha_s(\mu')]\ln\frac{\mu'}{Q} - \gamma_H^q[\alpha_s(\mu')]\right]$$

Accuracy	H, $\mathcal{J}$	$\gamma_H^q(\alpha_s)$	$\gamma_r^q(\alpha_s)$	$\beta(\alpha_s)$	$\Gamma_{\text{cusp}}(\alpha_s)$
LL	Tree level	–	–	1-loop	1-loop
NLL	Tree level	1-loop	1-loop	2-loop	2-loop
NLL′	1-loop	1-loop	1-loop	2-loop	2-loop
NNLL	1-loop	2-loop	2-loop	3-loop	3-loop
NNLL′	2-loop	2-loop	2-loop	3-loop	3-loop
N^3LL	2-loop	3-loop	3-loop	4-loop	4-loop
$\text{N}^3\text{LL}'$	3-loop	3-loop	3-loop	4-loop	4-loop
N^4LL	3-loop	4-loop	4-loop	5-loop	5-loop

The Rapidity Anomalous dimension

- Key ingredients for the resummation of large logarithms for transverse observables is the **rapidity anomalous dimension**. It appears in many contexts under different names: *Collins Soper Kernel*, *Anomaly Exponent*, piece of *B coefficient* in Sudakov Exponent, *TMD anomalous dimension*, etc...

In short: if you want to do anything involving transverse momentum logs beyond NLL, you need this ingredient.

The Rapidity Anomalous dimension

- Key ingredients for the resummation of large logarithms for transverse observables is the **rapidity anomalous dimension**. It appears in many contexts under different names: *Collins Soper Kernel*, *Anomaly Exponent*, piece of *B coefficient* in Sudakov Exponent, *TMD anomalous dimension*, etc…

In short: if you want to do anything involving transverse momentum logs beyond NLL, you need this ingredient.

- RAD can be decomposed in a term directly related to the cusp anomalous dimension and a non cusp term which contains the information intrinsic to the rapidity

$$\gamma_r^i(b_T,\mu) = -4\int_{\mu_0}^{\mu}\frac{\mathrm{d}\mu'}{\mu'}\Gamma^i_{\text{cusp}}[\alpha_s(\mu')] + \gamma_r^i(\mu_0,b_T)$$

- Non cusp term vanishes at LO and NLO.
- NNLO: known for a long time. [Davies, Webber, Stirling '85] [de Florian, Grazzini '00]
- N3LO: determined in 2016 via bootstrap methods [Li, Zhu '16]
- **N4LO**: C.Duhr, B.Mistlberger, **GV** [2205.02242] This talk (see also [Moult, Zhu, Zhu '22])

Rapidity Anomalous Dimension to Four Loops

- The calculation of the **Rapidity anomalous dimension** to 4 loops by brute force would require calculation of some **differential object** (e.g. p_T soft function) **to 4 loops**

- This is beyond the current technology for fixed order calculations (more difficult than 4 loop splitting functions)

- Anomalous dimensions known at 4 loops:
  - **Hard/Collinear** Anomalous Dimension to 4 loops [von Manteuffel, Panzer, Schabinger - 2002.04617]

$$\mu^2 \frac{\mathrm{d}}{\mathrm{d}\mu^2} H_{ij}^B(\mu^2) = \gamma_H^r(\alpha_S(\mu^2), \mu^2) H_{ij}^B(\mu^2),$$

$$\gamma_H^r(\alpha_S(\mu^2), \mu^2) = \Gamma_{\mathrm{cusp}}^r(\alpha_S(\mu)) \ln \frac{Q^2}{\mu^2} + \frac{1}{2}\gamma_H(\alpha_S(\mu^2))$$

Hard anomalous dimension (2 x collinear anomalous dimension of form factors)

  - **Virtual** Anomalous Dimension to 4 loops [Das, Moch, Vogt - 1912.12920]

$$\mu^2 \frac{\mathrm{d}}{\mathrm{d}\mu^2} f_i^{\mathrm{th}}(z, \mu^2) = \gamma_f^r(z, \alpha_S(\mu^2)) \otimes_z f_i^{\mathrm{th}}(z, \mu^2),$$

$$\gamma_f^r(z, \alpha_S(\mu^2)) = \Gamma_{\mathrm{cusp}}^r(\alpha_S(\mu^2)) \left[\frac{1}{1-z}\right]_+ + \frac{1}{2}\gamma_f^r(\alpha_S(\mu^2))\delta(1-z)$$

DGLAP at threshold

Rapidity Anomalous Dimension to Four Loops

- There is a **Rapidity**/**Threshold** **correspondence** for conformal theories, which holds at the critical dimension of QCD [Vladimirov - 1610.05791]

$$\gamma_r^i[\alpha_s, \epsilon^*] + \gamma_{\rm th}^i[\alpha_s, \epsilon^*] = 0$$

$$\beta[\alpha_s, \epsilon] = -2\alpha_s \left[\epsilon + \frac{\alpha_s}{4\pi}\beta_0 + \left(\frac{\alpha_s}{4\pi}\right)^2 \beta_1 + \dots\right] \qquad \beta[\alpha_s, \epsilon^*] = 0$$

$$\epsilon^* = -\left[\left(\frac{\alpha_s}{4\pi}\right)\beta_0 + \left(\frac{\alpha_s}{4\pi}\right)^2 \beta_1 + \dots\right]$$

Critical dimension of QCD

- **Threshold anomalous dimension** is part of RGE of soft function

$$\mu \frac{\mathrm{d}}{\mathrm{d}\mu} \ln S_i(\vec{b}_T, \mu, \nu) = 4\Gamma_{\rm cusp}^i[\alpha_s(\mu)] \ln \mu/\nu + \gamma_{\rm th}^i[\alpha_s]$$

$$\nu \frac{\mathrm{d}}{\mathrm{d}\nu} \ln S_i(\vec{b}_T, \mu, \nu) = -4 \int_{b_0/b_T}^{\mu} \frac{\mathrm{d}\mu'}{\mu'} \Gamma_{\rm cusp}^i[\alpha_s(\mu')] + \gamma_r^i[\alpha_s]$$

- Via SCET I consistency relations, relate **Threshold** to **Virtual** and **Collinear** anomalous dimensions

$$\gamma_{\rm thr.}^r(\alpha_S(\mu^2)) = -2\gamma_f^r(\alpha_S(\mu^2)) - \gamma_H^r(\alpha_S(\mu^2))$$

Rapidity Anomalous Dimension to Four Loops

- Difference between threshold and rapidity anomalous dimension comes from **higher orders in dimensional regularization evaluated at critical point!**

$$\gamma_r^{\mathrm{N4LO}} \sim \gamma_{\mathrm{th}}^{\mathrm{N4LO}} + \gamma_r^{\mathrm{N3LO}}[\epsilon = \epsilon^*] \qquad \epsilon^* = -\left[\left(\frac{\alpha_s}{4\pi}\right)\beta_0 + \left(\frac{\alpha_s}{4\pi}\right)^2\beta_1 + \dots\right]$$

- To obtain these terms it is necessary to calculate the **TMD Soft Function at N3LO** to **higher orders in dimensional regularization**

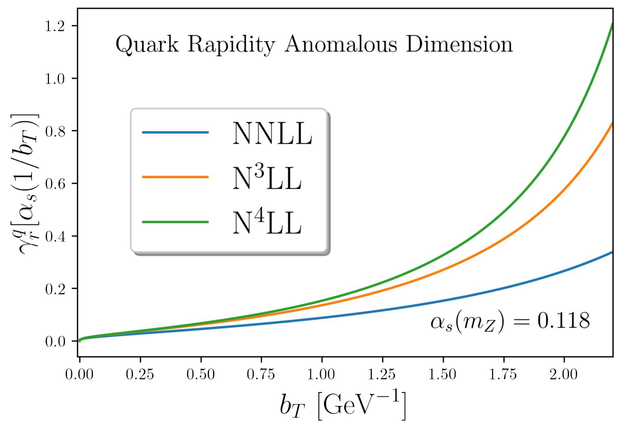
- We obtained this in

“Soft Integrals and Soft Anomalous Dimensions at N3LO and Beyond”
C.Duhr, B.Mistlberger, **GV** [2205.04493]

- Key point: Use method of differential equations and fix boundaries by relations between differential and inclusive threshold integrals

Rapidity Anomalous Dimension to Four Loops

- Obtained results at N4LO
- Quark and gluon related by **generalized casimir scaling**
- Well behaved series (stable coefficients) (see also [Moult, Zhu, Zhu])



$$\gamma_r^q(n_f=5) = 0.53929\alpha_s^2 + 0.68947\alpha_s^3 + \left(0.61753 \pm 5\cdot 10^{-5}\right)\alpha_s^4$$

$$\gamma_r^g(n_f=5) = 1.21341\alpha_s^2 + 1.55130\alpha_s^3 + \left(1.6041 \pm 5\cdot 10^{-4}\right)\alpha_s^4$$

- 4 coefficients are not known analytically but only numerically (very well)

$$\begin{aligned}
\gamma^i_{r,4} = {} & C_A^3 C_R \Big(-\frac{21164}{9}\zeta_3^2 - \frac{26104}{9}\zeta_2\zeta_3 + \frac{4228}{3}\zeta_4\zeta_3 + \frac{2752}{3}\zeta_2\zeta_5 \\
& + \frac{1201744\zeta_3}{81} + \frac{778166\zeta_2}{243} + \frac{8288\zeta_4}{9} - \frac{181924\zeta_5}{27} \\
& - \frac{63580\zeta_6}{27} + \frac{11071\zeta_7}{3} - \frac{28290079}{2187} - \frac{b^4_{q,C^4_{AF}}}{6}\Big) \\
& + C_A C_R n_f^2 \Big(\frac{224}{9}\zeta_3\zeta_2 + \frac{6752\zeta_2}{243} - \frac{22256\zeta_3}{81} + \frac{160\zeta_4}{9} + \frac{1472\zeta_5}{9} \\
& - \frac{898033}{2916}\Big) + C_R n_f^3 \Big(\frac{160\zeta_3}{9} - \frac{16\zeta_4}{9} + \frac{10432}{2187}\Big) \\
& + C_R C_A^2 n_f \Big(-\frac{8584}{9}\zeta_3^2 + \frac{2080}{3}\zeta_2\zeta_3 - \frac{247652\zeta_3}{81} - \frac{182134\zeta_2}{243} \\
& + \frac{43624\zeta_4}{27} - \frac{17936\zeta_5}{27} + \frac{1582\zeta_6}{27} + \frac{10761379}{2916} \\
& - \frac{b^4_{q,C^4_{FF}}}{12} - 2b^4_{q,n_f C_F^2 C_A} - b^4_{q,n_f C_F^3}\Big) \\
& + C_R C_F n_f^2 \Big(\frac{6928\zeta_3}{27} + \frac{160\zeta_4}{3} + 32\zeta_5 - \frac{110059}{243}\Big) \\
& + \frac{C^4_{AR}}{d_R}\Big(\frac{6688\zeta_3^2}{3} + 3584\zeta_2\zeta_3 + 736\zeta_4\zeta_3 + \frac{15616\zeta_3}{9} - \frac{224\zeta_4}{3} \\
& + \frac{4352\zeta_2}{3} - 2048\zeta_2\zeta_5 + \frac{3680\zeta_5}{9} - \frac{6952\zeta_6}{9} - 6968\zeta_7 \\
& - 384 + 4b_{4,d4AF}\Big) \\
& + \frac{C^4_{FR}}{d_R} n_f \Big(-\frac{2432}{3}\zeta_3^2 - 256\zeta_2\zeta_3 + \frac{10624\zeta_3}{9} - \frac{9088\zeta_2}{3} \\
& + \frac{1600\zeta_4}{3} + \frac{43520\zeta_5}{9} - \frac{2368\zeta_6}{9} + 768 + 4b^4_{q,C^4_{FF}}\Big) \\
& + C_A C_F C_R n_f \Big(4b_{4,n_f C_F^2 C_A} + \frac{6800\zeta_3^2}{3} - \frac{8864}{9}\zeta_2\zeta_3 - \frac{1892\zeta_3}{9} \\
& + \frac{5122\zeta_2}{27} - \frac{122216\zeta_4}{27} + \frac{21904\zeta_5}{9} - 1436\zeta_6 + \frac{2149049}{486}\Big) \\
& + C_F^2 C_R n_f \Big(4b^4_{q,n_f C_F^3} - 736\zeta_3^2 + \frac{1024}{3}\zeta_2\zeta_3 + \frac{2240\zeta_3}{9} - 648\zeta_2 \\
& + 668\zeta_4 - \frac{7744\zeta_5}{3} + \frac{29336\zeta_6}{9} - \frac{27949}{54}\Big)
\end{aligned}$$

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Resummed cross section to all orders (at LP)

$$\frac{d\sigma}{dz} = \frac{\hat{\sigma}_0}{8}\int_0^\infty d(b_T Q)^2\, J_0(b_T Q\sqrt{1-z})\, H_{q\bar{q}}(Q,\mu_H)$$
$$\times \mathcal{J}_q\left(b_T,\mu_J,\frac{Qb_T}{\upsilon_n}\right)\mathcal{J}_{\bar{q}}\left(b_T,\mu_J,Qb_T\upsilon_{\bar{n}}\right)\left(\frac{\upsilon_n}{\upsilon_{\bar{n}}}\right)^{\frac{1}{2}\gamma_r^q(b_T,\mu_J)}$$
$$\times \exp\left[4\int_{\mu_J}^{\mu_H}\frac{d\mu'}{\mu'}\Gamma^q_{\rm cusp}[\alpha_s(\mu')]\ln\frac{\mu'}{Q} - \gamma_H^q[\alpha_s(\mu')]\right]$$

Accuracy	H, $\mathcal{J}$	$\gamma_H^q(\alpha_s)$	$\gamma_r^q(\alpha_s)$	$\beta(\alpha_s)$	$\Gamma_{\rm cusp}(\alpha_s)$
LL	Tree level	–	–	1-loop	1-loop
NLL	Tree level	1-loop	1-loop	2-loop	2-loop
NLL′	1-loop	1-loop	1-loop	2-loop	2-loop
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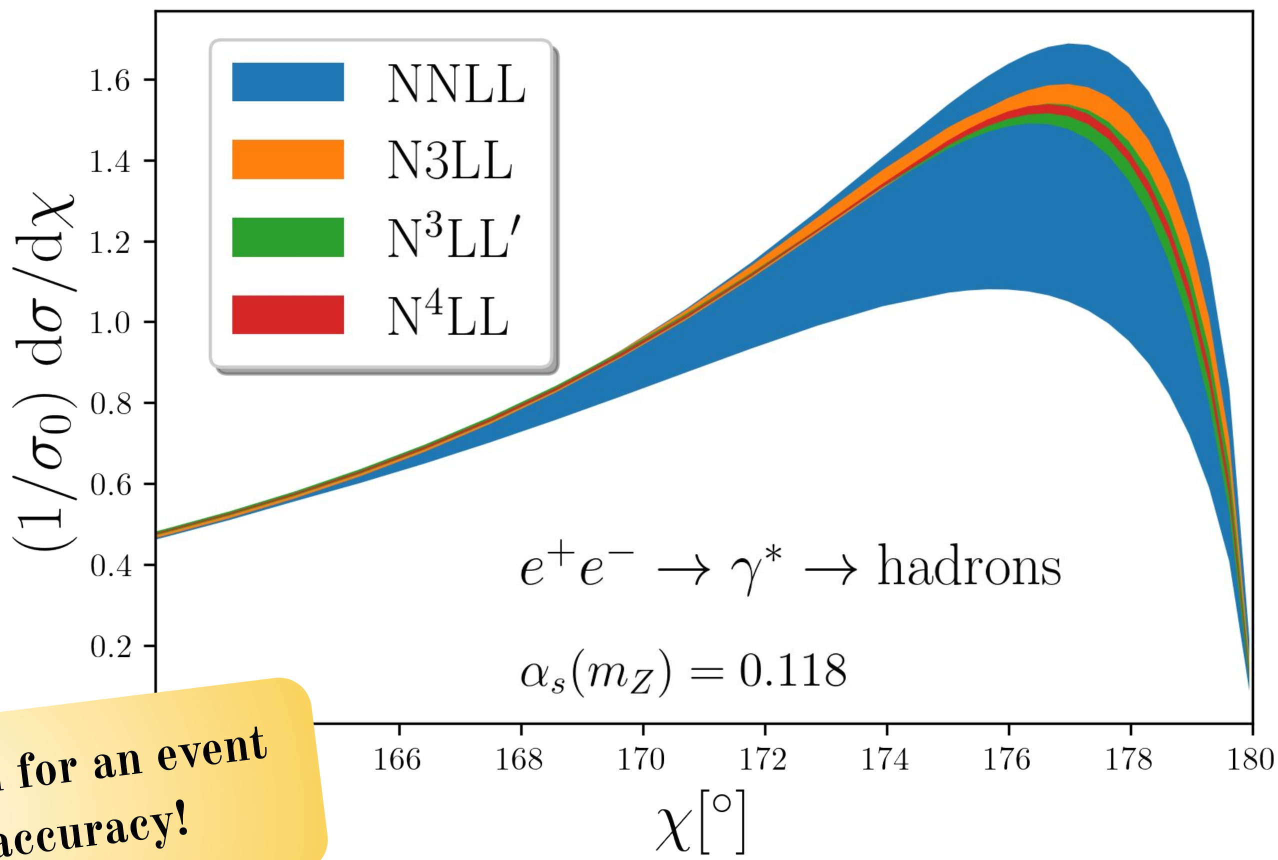
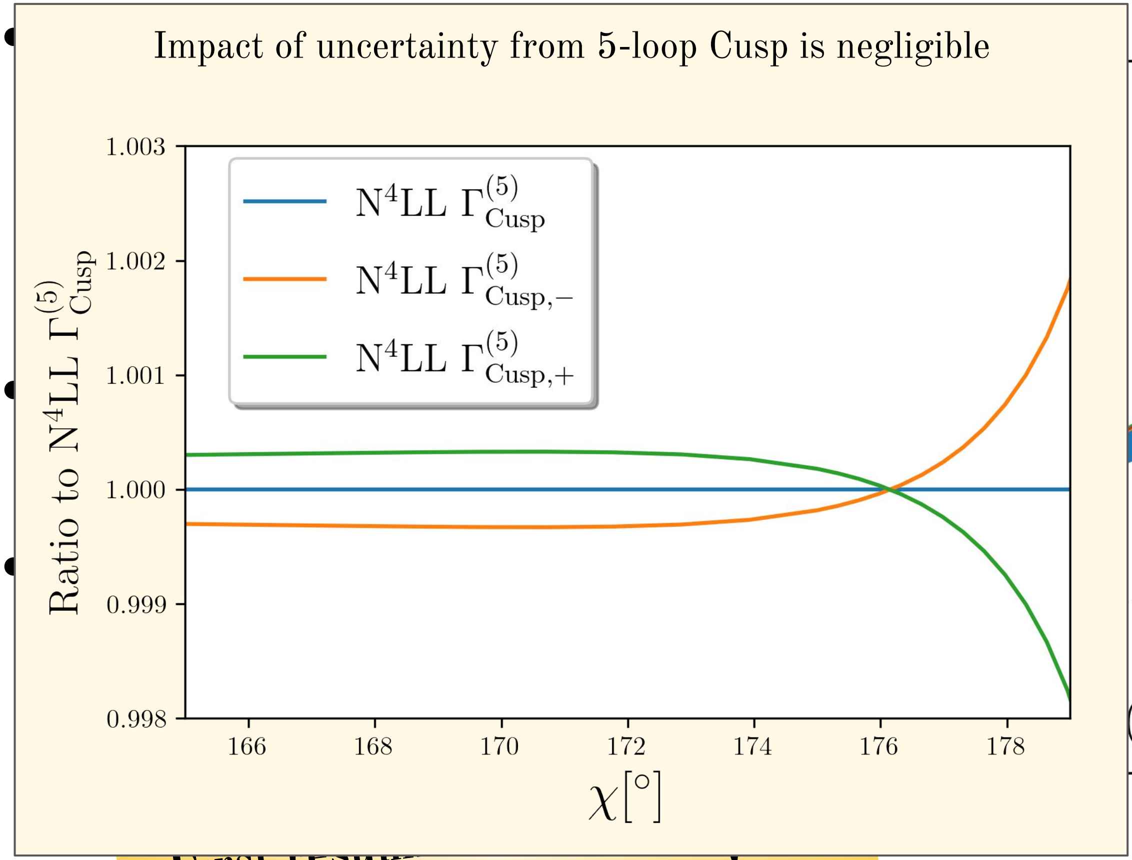
EEC in the back to back limit to N4LL

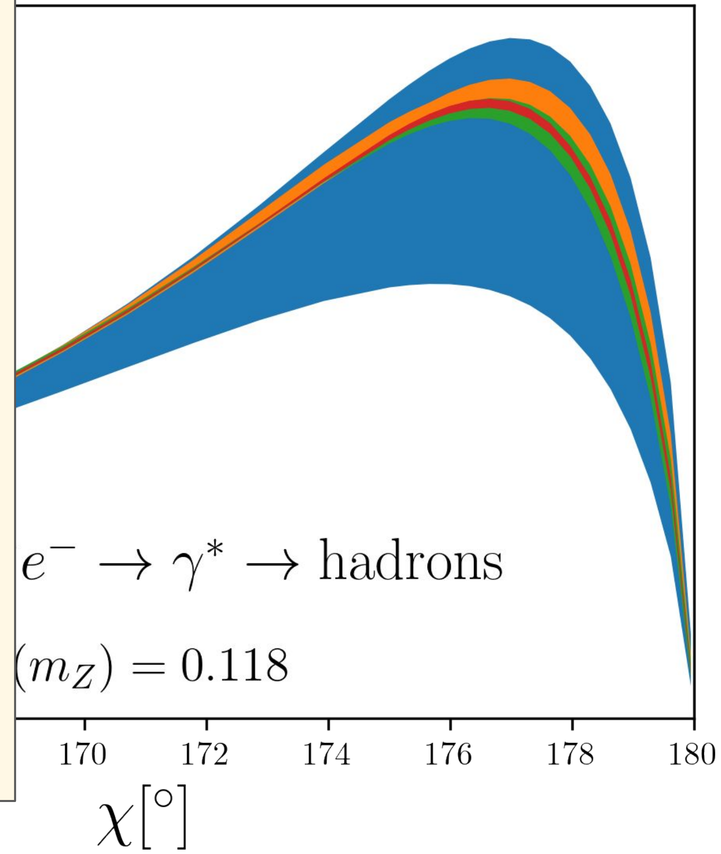
- Implemented the resummation of this event shape at **N4LL** in new numerical framework: **pySCET**
- Nice convergence of perturbative result
- Uncertainties obtained by 15 point scale variation in SCET



First resummation for an event shape at this accuracy!

EEC in the back to back limit to N4LL





First

shape at this accuracy!

What else is resummable to full N4LL?

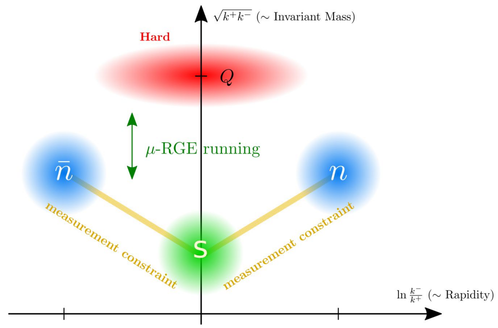
What else is resummable to full N4LL?

As explained before, for N4LL one needs both N4LO Anomalous dimensions and N3LO Boundaries. The EEC is the only one where all the ingredients have been calculated. We are not far away for some color singlet observables, but not there yet.

Obs	**Boundaries at N3LO**	**Anomalous Dim. at N4LO**
Thrust	N3LO Thrust Soft function unknown	All anomalous dimensions known
C-param	N3LO C-parameter Soft function unknown	All anomalous dimensions known
q_T	N3LO Hard, Soft and Beam Functions known	Missing 4 loop DGLAP
$\mathcal{T}_0$	N3LO 0-Jettiness Soft function unknown	Missing 4 loop DGLAP

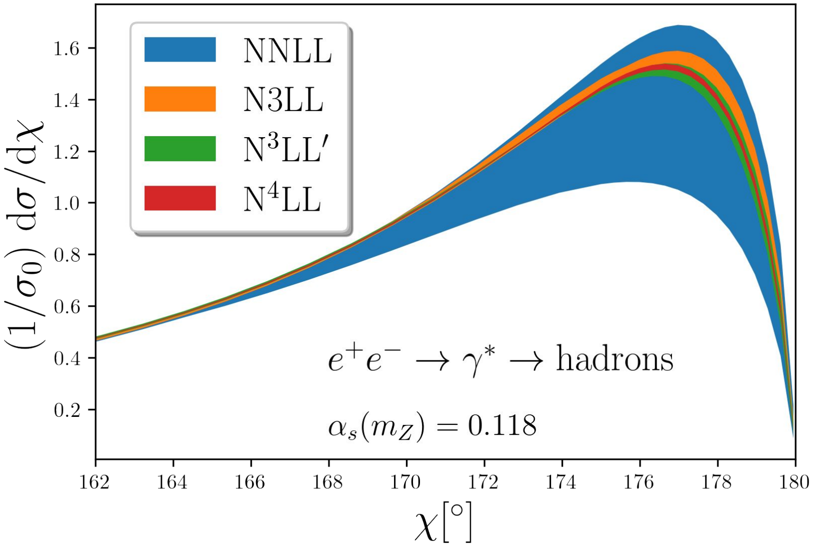
Conclusion

- ➢ Introduced the ingredients for resummation in SCET
- ➢ Discussed the calculation of N3LO boundary terms and N4LO anomalous dimensions
- ➢ Presented results for the Resummation at N4LL on event shapes



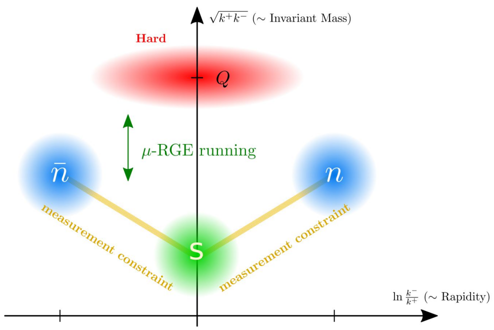
$$
\begin{aligned}
\frac{d\sigma}{dz} &= \frac{\hat{\sigma}_0}{8}\int_0^\infty d(b_TQ)^2\, J_0\big(b_TQ\sqrt{1-z}\big)H_{q\bar{q}}(Q,\mu_H) \\
&\times \mathcal{J}_q\Big(b_T,\mu_J,\frac{Qb_T}{\upsilon_n}\Big)\mathcal{J}_{\bar{q}}\Big(b_T,\mu_J,Qb_T\upsilon_{\bar{n}}\Big)\Big(\frac{\upsilon_n}{\upsilon_{\bar{n}}}\Big)^{\frac{1}{2}\gamma_r^q(b_T,\mu_J)} \\
&\times \exp\Big[4\int_{\mu_J}^{\mu_H}\frac{d\mu'}{\mu'}\Gamma^q_{\text{cusp}}[\alpha_s(\mu')]\ln\frac{\mu'}{Q} - \gamma_H^q[\alpha_s(\mu')]\Big]
\end{aligned}
$$

$$
\begin{aligned}
\gamma^i_{r,4} = C_A^3C_R\Big(&-\frac{21164}{9}\zeta_3^2 - \frac{26104}{9}\zeta_2\zeta_3 + \frac{4228}{3}\zeta_4\zeta_3 + \frac{2752}{3}\zeta_2\zeta_5 \\
&+ \frac{1201744\zeta_3}{81} + \frac{778166\zeta_2}{243} + \frac{8288\zeta_4}{9} - \frac{181924\zeta_5}{27} \\
&- \frac{63580\zeta_6}{27} + \frac{11071\zeta_7}{3} - \frac{28290079}{2187} - \frac{b^4_{q,C^4_{AF}}}{6}\Big) \\
+ C_AC_Rn_f^2\Big(&\frac{224}{9}\zeta_3\zeta_2 + \frac{6752\zeta_2}{243} - \frac{22256\zeta_3}{81} + \frac{160\zeta_4}{9} + \frac{1472\zeta_5}{9} \\
&- \frac{898033}{2916}\Big) + C_Rn_f^3\Big(\frac{160\zeta_3}{9} - \frac{16\zeta_4}{9} + \frac{10432}{2187}\Big) \\
+ C_RC_A^2n_f\Big(&-\frac{8584}{9}\zeta_3^2 + \frac{2080}{3}\zeta_2\zeta_3 - \frac{247652\zeta_3}{81} - \frac{182134\zeta_2}{243} \\
&+ \frac{43624\zeta_4}{27} - \frac{17936\zeta_5}{27} + \frac{1582\zeta_6}{27} + \frac{10761379}{2916} \\
&- \frac{b^4_{q,C^4_{FF}}}{12} - 2b^4_{q,n_fC_F^2C_A} - b^4_{q,n_fC_F^3}\Big) \\
+ C_RC_Fn_f^2\Big(&\frac{6928\zeta_3}{27} + \frac{160\zeta_4}{3} + 32\zeta_5 - \frac{110059}{243}\Big) \\
+ \frac{C^4_{AR}}{d_R}\Big(&\frac{6688\zeta_3^2}{3} + 3584\zeta_2\zeta_3 + 736\zeta_4\zeta_3 + \frac{15616\zeta_3}{9} - \frac{224\zeta_4}{3} \\
&+ \frac{4352\zeta_2}{3} - 2048\zeta_2\zeta_5 + \frac{3680\zeta_5}{9} - \frac{6952\zeta_6}{9} - 6968\zeta_7 \\
&- 384 + 4b_{4,d4AF}\Big)
\end{aligned}
$$



Conclusion

➢ Introduced the ingredients for resummation in SCET

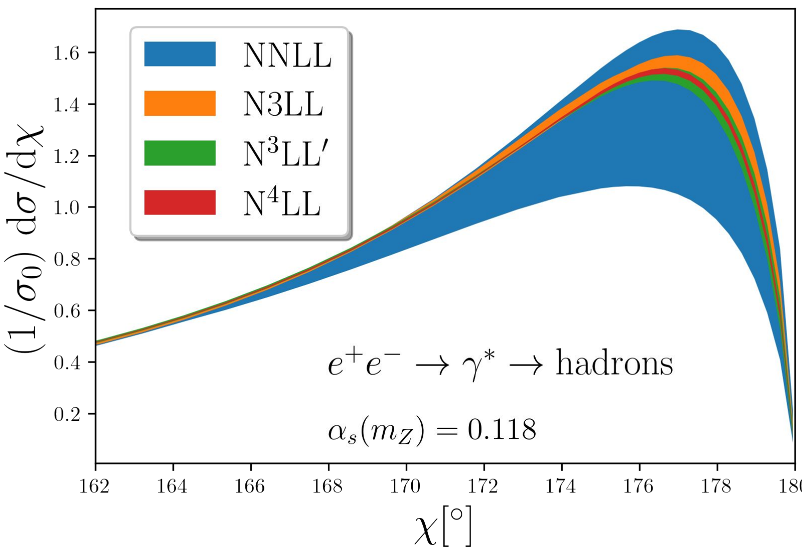


$$\gamma^i_{r,4} = C_A^3 C_R \left(-\frac{21164}{9}\zeta_3^2 - \frac{26104}{9}\zeta_2\zeta_3 + \frac{4228}{3}\zeta_4\zeta_3 + \frac{2752}{3}\zeta_2\zeta_5 + \frac{1201744\zeta_3}{81} + \frac{778166\zeta_2}{243} + \frac{8288\zeta_4}{9} - \frac{181924\zeta_5}{27} - \frac{63580\zeta_6}{27} + \frac{11071\zeta_7}{3} - \frac{28290079}{2187} - \frac{b^4_{q,C^4_{AF}}}{6}\right)$$
$$+ C_A C_R n_f^2 \left(\frac{224}{9}\zeta_3\zeta_2 + \frac{6752\zeta_2}{243} - \frac{22256\zeta_3}{81} + \frac{160\zeta_4}{9} + \frac{1472\zeta_5}{9} - \frac{898033}{2916}\right) + C_R n_f^3 \left(\frac{160\zeta_3}{9} - \frac{16\zeta_4}{9} + \frac{10432}{2187}\right)$$
$$+ C_R C_A^2 n_f \left(-\frac{8584}{9}\zeta_3^2 + \frac{2080}{3}\zeta_2\zeta_3 - \frac{247652\zeta_3}{81} - \frac{182134\zeta_2}{243} + \frac{43624\zeta_4}{27} - \frac{17936\zeta_5}{27} + \frac{1582\zeta_6}{27} + \frac{10761379}{2916} - \frac{b^4_{q,C^4_{FF}}}{12} - 2b^4_{q,n_f C_F^2 C_A} - b^4_{q,n_f C_F^3}\right)$$
$$+ C_R C_F n_f^2 \left(\frac{6928\zeta_3}{27} + \frac{160\zeta_4}{3} + 32\zeta_5 - \frac{110059}{243}\right)$$
$$+ \frac{C^4_{AR}}{d_R}\left(\frac{6688\zeta_3^2}{3} + 3584\zeta_2\zeta_3 + 736\zeta_4\zeta_3 + \frac{15616\zeta_3}{9} - \frac{224\zeta_4}{3} + \frac{4352\zeta_2}{3} - 2048\zeta_2\zeta_5 + \frac{3680\zeta_5}{9} - \frac{6952\zeta_6}{9} - 6968\zeta_7 - 384 + 4b_{4,d4AF}\right)$$

➢ Discussed the calculation of N3LO boundary terms and N4LO anomalous dimensions

$$\frac{\mathrm{d}\sigma}{\mathrm{d}z} = \frac{\hat{\sigma}_0}{8}\int_0^\infty \mathrm{d}(b_T Q)^2\, J_0\!\left(b_T Q\sqrt{1-z}\right) H_{q\bar{q}}(Q,\mu_H)$$
$$\times\, \mathcal{J}_q\!\left(b_T,\mu_J,\frac{Qb_T}{\upsilon_n}\right)\mathcal{J}_{\bar{q}}\!\left(b_T,\mu_J,Qb_T\upsilon_{\bar{n}}\right)\left(\frac{\upsilon_n}{\upsilon_{\bar{n}}}\right)^{\frac{1}{2}\gamma_r^q(b_T,\mu_J)}$$
$$\times\, \exp\left[4\int_{\mu_J}^{\mu_H}\frac{\mathrm{d}\mu'}{\mu'}\Gamma^q_{\rm cusp}[\alpha_s(\mu')]\ln\frac{\mu'}{Q} - \gamma^q_H[\alpha_s(\mu')]\right]$$

➢ Presented results for the Resummation at N4LL on event shapes



Thank you!

Backup

$$\frac{\mathrm{d}\sigma_{pp\to H+X}}{\mathrm{d}Q\mathrm{d}Y\mathrm{d}q_T} \sim f_a\left(\frac{x_a}{z_a}, \mu_F\right) f_b\left(\frac{x_b}{z_b}, \mu_F\right) \hat{\sigma}_{ab\to H+X}(z_a, z_b, q_T, \mu)$$

$$\hat{\sigma}_{ab\to H+X}(z_a, z_b, q_T, \mu) \sim \overbrace{\int_0^\infty \mathrm{d}(b_T Q)^2\, J_0(b_T q_T)}^{\text{Fourier Transform}} \overbrace{H_{gg}(Q,\mu)}^{\text{Hard F}} \overbrace{\mathcal{I}_{ga}\Big(z_a, b_T, \mu, \tfrac{Qe^{-Y} b_T}{\upsilon}\Big) \mathcal{I}_{gb}\Big(z_b, b_T, \mu, Qe^{Y} b_T \upsilon_{\bar{n}}\Big)}^{\text{Collinear Matching Functions}}$$

$$\mu\frac{\mathrm{d}}{\mathrm{d}\mu}\mathcal{I}_{ij}(z, b_T, \mu, \nu) = \sum_k \int_z^1 \frac{\mathrm{d}z'}{z'} \left[\gamma_B^i(\mu,\nu)\delta_{kj}\delta(1-z') - \boxed{P_{kj}(z',\mu)}\right] \mathcal{I}_{ik}\Big(\frac{z}{z'}, b_T, \mu, \nu\Big)$$

$$B_i(x, b_T, \mu, \nu) = \sum_j \int_x^1 \frac{\mathrm{d}z}{z} \mathcal{I}_{ij}(z, b_T, \mu, \nu) f_j\left(\frac{x}{z}, \mu\right) + \mathcal{O}(\Lambda_{\mathrm{QCD}} b_T)$$

$$\mu\frac{\mathrm{d}}{\mathrm{d}\mu} B_i(x, b_T, \mu, \nu) = \gamma_B^i(\mu,\nu)\, B_i(x, b_T, \mu, \nu)$$

$$\mu\frac{\mathrm{d}}{\mathrm{d}\mu} f_i(z, \mu) = \sum_j \int_z^1 \frac{\mathrm{d}z'}{z'} P_{ij}(z', \mu)\, f_j\Big(\frac{z}{z'}, \mu\Big)$$

$$\frac{\mathrm{d}\sigma^{\mathrm{res}}}{\mathrm{d}Q\mathrm{d}Y\mathrm{d}q_T} \sim \hat{\sigma}_0 \overbrace{\int_0^\infty \mathrm{d}(b_T Q)^2\, J_0\big(b_T q_T\big)}^{\text{Fourier Transform}} \overbrace{H_{gg}(Q,\mu_H)}^{\text{Hard F}} \overbrace{B_g\Big(x_1, b_T, \mu_B, \tfrac{Q e^{-Y} b_T}{\upsilon_n}\Big) B_g\Big(b_T, \mu_B, Q e^{Y} b_T \upsilon_{\bar{n}}\Big)}^{\text{Beam Functions}}$$

$$\times \exp\Bigg[\underbrace{\int_{\mu_H}^{\mu} \frac{\mathrm{d}\mu'}{\mu'} \gamma_H^g(Q,\mu')}_{\text{RG evolution of Hard function}} + \underbrace{2\int_{\mu_J}^{\mu} \frac{\mathrm{d}\mu'}{\mu'} \tilde{\gamma}_B^g(\mu', Q/\mu)}_{\mu\text{-RG evolution of Beam functions}}\Bigg] \underbrace{\left(\frac{\upsilon_n}{\upsilon_{\bar{n}}}\right)^{\frac{1}{2}\gamma_r^g(b_T,\mu_B)}}_{\text{Rapidity RG evolution}}$$

EEC in Pure Rapidity Renormalization

[GV, to appear]

- Back-to-back region of EEC obeys TMD-like fact. thm and resummation ("crossed version of q_T")
- In pure rapidity renormalization it takes the following form

$$\frac{d\sigma}{dz} = \frac{\hat{\sigma}_0}{2} \overbrace{H_{q\bar{q}}(Q,\mu)}^{\text{Hard Function}} \int \frac{d^2\vec{b}_T\, d^2\vec{q}_T}{(2\pi)^2} e^{i\vec{q}_T\cdot\vec{b}_T}\, \delta\overbrace{\left(1 - z - \frac{q_T^2}{Q^2}\right)}^{1-z\equiv(\cos\frac{\chi}{2})^2\approx\frac{q_T^2}{Q^2}} \underbrace{\mathcal{J}_q\left(b_T,\mu,\frac{Qb_T}{\upsilon}\right)\mathcal{J}_{\bar{q}}\left(b_T,\mu,Qb_T\upsilon\right)}_{\text{Pure Rapidity EEC Jet Functions}}$$

- Soft Function corrections are scaleless => S=1 to all orders

- Rapidity divergences cancel between Jet Functions only, but **finite terms are identical**

- Similar form to time-like Collins-Soper TMD structure (no soft function, symmetry between collinear directions), but retain full control on rapidity scale at the matching kernel level (better handle for resummation uncertainties on RRGE)

EEC in the back to back limit to N4LL

- Back-to-back region of EEC obeys TMD-like fact. thm and resummation (“crossed version of q_T”)
- In pure rapidity renormalization it takes the following form

$$\frac{\mathrm{d}\sigma}{\mathrm{d}z} = \frac{\hat{\sigma}_0}{2} \overbrace{H_{q\bar{q}}(Q,\mu)}^{\text{Hard Function}} \int \frac{\mathrm{d}^2\vec{b}_T\, \mathrm{d}^2\vec{q}_T}{(2\pi)^2} e^{\mathrm{i}\vec{q}_T\cdot\vec{b}_T} \overbrace{\delta\left(1-z-\frac{q_T^2}{Q^2}\right)}^{1-z\equiv(\cos\frac{\chi}{2})^2\approx\frac{q_T^2}{Q^2}} \mathcal{J}_q\left(b_T,\mu,\frac{Qb_T}{\upsilon}\right)\mathcal{J}_{\bar{q}}\left(b_T,\mu,Qb_T\upsilon\right)$$

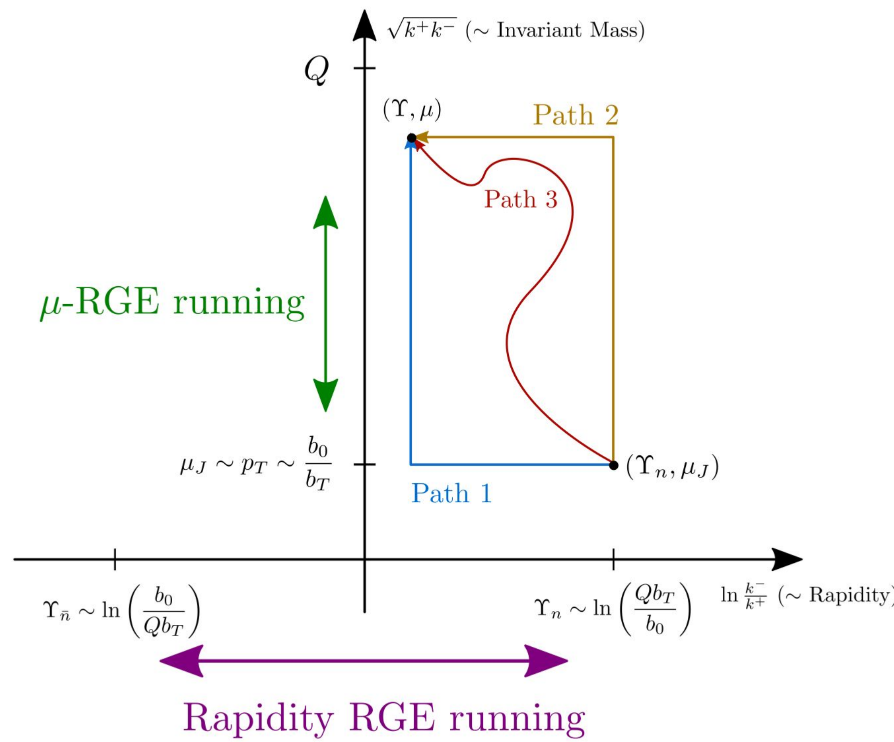
Standard RGE

$$\mu\frac{\mathrm{d}}{\mathrm{d}\mu}\ln H_{q\bar{q}}(Q,\mu) = \gamma_H^q(Q,\mu)\,,$$

$$\mu\frac{\mathrm{d}}{\mathrm{d}\mu}\ln\mathcal{J}_q\left(b_T,\mu,\frac{Qb_T}{\upsilon}\right) = \gamma_{\mathcal{J}_q}(\mu,\upsilon\mu/Q)$$

Rapidity RGE

$$\upsilon\frac{\mathrm{d}}{\mathrm{d}\upsilon}\ln\mathcal{J}_q\left(b_T,\mu,\frac{Qb_T}{\upsilon}\right) = -\frac{1}{2}\gamma_r^q(b_T,\mu)$$



Resummation of the EEC in the back-to-back limit

- Extending method of collinear expansion of cross sections to processes with final state color charged particles we were able to calculate EEC Jet Function at **N3LO**

$$\frac{\mathrm{d}\sigma}{\mathrm{d}z} = \frac{\hat{\sigma}_0}{2} \overbrace{H_{q\bar{q}}(Q,\mu)}^{\text{Hard Function}} \int \frac{\mathrm{d}^2\vec{b}_T\,\mathrm{d}^2\vec{q}_T}{(2\pi)^2} e^{\mathrm{i}\vec{q}_T\cdot\vec{b}_T}\,\delta\left(1 - z - \frac{q_T^2}{Q^2}\right) \underbrace{J_q\left(b_T,\mu,\frac{\nu}{Q}\right) J_{\bar{q}}\left(b_T,\mu,\frac{\nu}{Q}\right)}_{\text{EEC Jet Functions}} \overbrace{\tilde{S}_q(b_T,\mu,\nu)}^{\text{TMD Soft Function}}$$

- SCET allows to resum large logs appearing in this limit.
- Each function obeys renormalization group equations (RGEs)

Anomalous dimensions obtained by poles of calculation in the EFT (known in the literature, rechecked in our calculation)

$$\frac{\mathrm{d}}{\mathrm{d}\ln\mu}\ln H_i(Q,\mu) = \gamma_H^i(Q,\mu)\,,$$

$$\frac{\mathrm{d}}{\mathrm{d}\ln\mu}\ln J_i(b_T,\mu,\nu/Q) = \tilde{\gamma}_J^i(\mu,\nu/Q)\,,$$

$$\frac{\mathrm{d}}{\mathrm{d}\ln\mu}\ln \tilde{S}_i(b_T,\mu,\nu) = \tilde{\gamma}_S^i(\mu,\nu)\,,$$

$$\frac{\mathrm{d}}{\mathrm{d}\ln\nu}\ln J_i(b_T,\mu,\nu/Q) = -\frac{1}{2}\tilde{\gamma}_\nu^i(b_T,\mu)\,,$$

$$\frac{\mathrm{d}}{\mathrm{d}\ln\nu}\ln \tilde{S}_i(b_T,\mu,\nu) = \tilde{\gamma}_\nu^i(b_T,\mu)\,.$$

Rapidity Renormalization Group Equations

- Running of operators resum logs as for running coupling in standard QFT