# Interplay between perturbative and non-perturbative effects with the ARES method

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**Parton Showers and Resummation - 2023**

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[arXiv:2303.01534v1]

## Problem specification:

Precision measurements of $\alpha_s$ using event shapes in $e^+e^-$ annihilation

| category | $\alpha_S(m_Z^2)$ | relative $\alpha_S(m_Z^2)$ uncertainty |
|---|---|---|
| $\tau$ decays and low $Q^2$ | $0.1178 \pm 0.0019$ | 1.6% |
| $Q\overline{Q}$ bound states | $0.1181 \pm 0.0037$ | 3.1% |
| PDF fits | $0.1162 \pm 0.0020$ | 1.7% |
| $e^+e^-$ jets & shapes | $0.1171 \pm 0.0031$ | 2.6% |
| electroweak | $0.1208 \pm 0.0028$ | 2.3% |
| hadron colliders | $0.1165 \pm 0.0028$ | 2.4% |
| lattice | $0.1182 \pm 0.0008$ | 0.7% |
| world average (without lattice) | $0.1176 \pm 0.0010$ | 0.9% |
| world average (with lattice) | $0.1179 \pm 0.0009$ | 0.8% |

Snowmass 2021 White Paper [arXiv:2203.08271]

## <u>Context:</u>

For the determinations of $\alpha_s$ using event shapes and jet rates, we can make use of high-precision perturbative calculations (fixed-order + resummation)

However, we must also take into account the presence of <u>non-perturbative effects due to hadronisation</u>

***Power corrections*** - non-perturbative (hadronisation) corrections to hadronic observables in $e^+e^-$ annihilation are suppressed by powers of $1/Q$

For inclusive quantities, such as the total cross section, these effects are small

Leading power corrections to final-state event-shape observables however are linear in $1/Q$
Manohar, Wise [arXiv:9406293]
Webber [arXiv:9408222]

Compared to event shapes, jet rates are known to be less sensitive to hadronisation corrections
Dokshitzer, Marchesini, Webber [arXiv:9512336]

**Context:**

In 2019, state-of-the-art extractions of the strong coupling based on N³LO+NNLL accurate predictions for the two-jet rate in the Durham clustering algorithm at $e^+e^-$ collisions,
as well as a simultaneous fit of the two- and three-jet rates (taking into account correlations between the two) were presented.

> Owing to the high accuracy of the predictions used, the perturbative uncertainty is considerably smaller than that due to hadronization. Our best determination at the $Z$ mass is $\alpha_s(M_Z) = 0.11881 \pm 0.00063(exp.) \boxed{\pm 0.00101(hadr.)} \pm 0.00045(ren.) \pm 0.00034(res.)$, which is in agreement with the latest world average and has a comparable total uncertainty.

Verbytskyi, Banfi, Kardos, Monni, Kluth, Somogyi, Szőr, Trócsányi, Tulipánt, Zanderighi [arXiv:1902.08158]

- Experimental uncertainty was comparable to the perturbative uncertainty
- Hadronisation uncertainty was the dominant source of uncertainty

# Modelling Hadronisation Corrections

Monte Carlo parton shower event generators

1. Run the MC event generator down to both the parton and hadron level

2. Compute the ratio between an observable distribution or moment at both levels

3. Apply this correction to the corresponding perturbative prediction

4. Estimate the uncertainty due to hadronisation by changing the event generator and/or the hadronisation model

Problems:

- Equivalence of parton level of a MC simulation with a fixed order calculation?

- Accuracy of MC tuning?

## Modelling Hadronisation Corrections

### Analytic Hadronisation Models

We consider the *dispersive model* and leading hadronisation corrections only, $\sim 1/Q$

- Introduce an IR cutoff $\mu_I$, where $\Lambda_{\text{QCD}} \leq \mu_I \ll Q$

- Replace $\alpha_s$ below this scale with an effective coupling $\alpha_{\text{eff}}(k)$ that is finite in the IR region down to $k \to 0$

$$\frac{1}{\mu_I} \int_0^{\mu_I} dk \, \alpha_{\text{eff}}(k) \equiv \alpha_0(\mu_I)$$

- Leading hadronisation corrections are modelled as the contribution of an **ultra-soft gluon**

- It transpires that the leading hadronisation corrections are proportional to $\alpha_0(\mu_I)$

**Set-up:**

We consider a generic recursive infrared and collinear (rIRC) safe observable in $e^+e^-$ annihilation

$$V(\{\tilde{p}\}, k_1, \ldots, k_n) \geq 0$$

- $\{\tilde{p}\} = \{\tilde{p}_1, \tilde{p}_2\}$ are the momenta of the hard quark-antiquark pair
- $k_1, \ldots, k_n$ are the subsequent emissions

We consider the following region

$$V(\{\tilde{p}\}, k_1, \ldots, k_n) \simeq v \text{ , where } \Lambda_{\mathrm{QCD}}/Q \ll v \ll 1$$

Therefore we can write

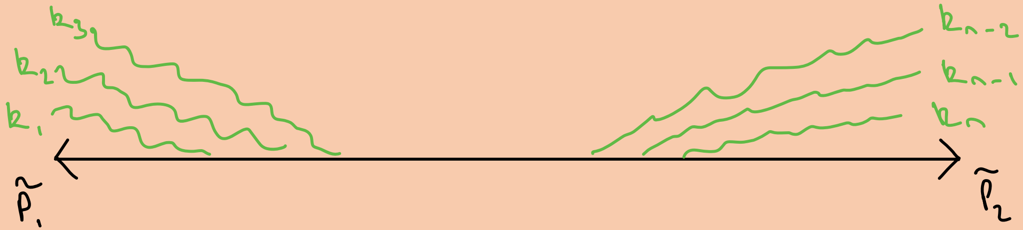
$$\Sigma(v) = \Sigma_{\mathrm{PT}}(v) + \delta\Sigma_{\mathrm{NP}}(v)$$

## Set-up:

$$\Sigma(v) = \boxed{\Sigma_{\mathrm{PT}}(v)} + \delta\Sigma_{\mathrm{NP}}(v)$$

$$\Sigma_{\mathrm{PT}}(v) = \int dZ[\{k_i\}]\,\Theta\big(v - V(\{\tilde{p}\},\{k_i\})\big)\ ,\ \text{with}\ \int dZ[\{k_i\}] = 1$$

The leading contribution comes from an ensemble of soft and collinear emissions widely separated in angle (**NLL accuracy**):



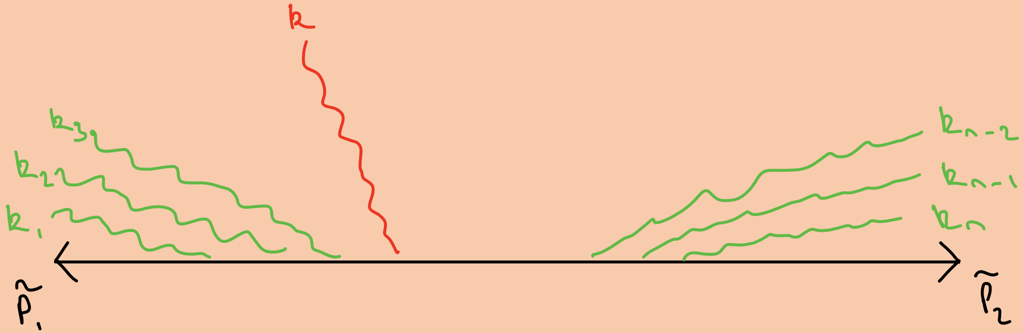

$$\boxed{\Sigma_{\mathrm{PT}}(v) = e^{-R(v)}\,\mathcal{F}(R')}$$

with $\mathcal{F}(R') = \int d\mathcal{Z}[\{R'_{\ell_i}, k_i\}]\,\Theta\left(1 - \frac{V_{\mathrm{sc}}(\{\tilde{p}\},\}k_i\})}{v}\right)$

**Set-up:**

$$\Sigma(v) = \Sigma_{\rm PT}(v) + \boxed{\delta\Sigma_{\rm NP}(v)}$$

$$\delta\Sigma_{\rm NP}(v) = \int [dk]\, \mathcal{M}^2_{\rm NP}(k) \int dZ[\{k_i\}] \left[\Theta\big(v - V(\{\tilde{p}\}, k, \{k_i\})\big) - \Theta\big(v - V(\{\tilde{p}\}, \{k_i\})\big)\right]$$




We define $\delta V_{\rm NP}(\{\tilde{p}\}, k, \{k_i\}) \equiv V(\{\tilde{p}\}, k, \{k_i\}) - V(\{\tilde{p}\}, \{k_i\}) \ll v$

Therefore we find $\delta\Sigma_{\rm NP}(v) \simeq -\langle \delta V_{\rm NP}\rangle \dfrac{d\Sigma_{\rm PT}}{dv}$

where, $$\langle \delta V_{\rm NP}\rangle \equiv \frac{\int [dk]\mathcal{M}^2_{\rm NP}(k) \int dZ[\{k_i\}]\, \delta V_{\rm NP}(\{\tilde{p}\}, k, \{k_i\})\, \delta\big(v - V(\{\tilde{p}\}, \{k_i\})\big)}{\int dZ[\{k_i\}]\, \delta\big(v - V(\{\tilde{p}\}, \{k_i\})\big)}$$

**Set-up:**

$$\Sigma(v) \simeq \Sigma_{\mathrm{PT}}(v) - \langle \delta V_{\mathrm{NP}} \rangle \frac{d\Sigma_{\mathrm{PT}}}{dv} \simeq \Sigma_{\mathrm{PT}}(v - \langle \delta V_{\mathrm{NP}} \rangle)$$

In principle, $\delta V_{\mathrm{NP}}$ is different for each observable and may depend non-trivially on unknown NP dynamics

Therefore we assume the following:

- $[dk]\mathcal{M}^2_{\mathrm{NP}}(k) = \sum_\ell \frac{d\kappa}{\kappa} M^2_{\mathrm{NP}}(\kappa)\, d\eta^{(\ell)} \frac{d\phi}{2\pi}$ , with $\kappa^2 \equiv 2\frac{(\tilde{p}_1 k)(\tilde{p}_2 k)}{(\tilde{p}_1 \tilde{p}_2)}$

- Accompanying PT emissions are soft, collinear and widely separated in angle (NLL accuracy)

- Consider event-shape variables (i.e. we exclude jet-resolution parameters) and those for which the NP correction is linear in $\kappa$

$$\delta V_{\mathrm{NP}}(\{\tilde{p}\}, k, \{k_i\}) = \frac{\kappa}{Q} h_V\left(\eta^{(\ell)}, \phi, \{\tilde{p}\}, \{k_i\}\right)$$

## Set-up:

Therefore
$$\langle \delta V_{\rm NP} \rangle = \frac{\langle \kappa \rangle_{\rm NP}}{Q} \langle h_V \rangle \mathcal{M}$$

$\mathcal{M}$ → Milan Factor

with
$$\langle \kappa \rangle_{\rm NP} = \int d\kappa\, M^2_{\rm NP}(\kappa)$$
$$= \frac{4C_F}{\pi^2} \mu_I \left( \alpha_0(\mu_I) - \alpha_s - 2\beta_0\, \alpha_s^2 \left( 1 + \ln \frac{Q}{\mu_I} + \frac{K}{4\pi\beta_0} \right) \right)$$

Dokshitzer, Webber [arXiv:9504219]

and
$$\langle h_V \rangle = \frac{1}{R' \mathcal{F}(R')} \sum_\ell \int d\eta^{(\ell)} \frac{d\phi}{2\pi} \int d\mathcal{Z}[\{R'_{\ell_i}, k_i\}]\, h_V(\eta^{(\ell)}, \phi, \{\tilde{p}\}, \{k_i\})\, \delta\left( 1 - \frac{V_{\rm sc}(\{\tilde{p}\}, \{k_i\})}{v} \right)$$

which is suitable for both analytic calculations and numerical determinations

**Interplay between PT and NP effects:**

Hadronisation corrections should be thought of as additional contributions to given perturbative configurations

⟶ an interplay between PT and NP effects

The interplay is particularly important for recoil-sensitive observables due to the recoil of a hard quark or anti-quark from multiple soft and collinear emissions

Our aim:

- **Devise a general, semi-numerical method to determine leading $1/Q$ hadronisation corrections, in the two-jet region, for a large class of event-shape variables, including the interplay with perturbative QCD radiation**

- The method follows the strategy of ARES

⟶ Monte Carlo simulation of an arbitrary number of soft and collinear emissions, accompanied by a *special* emission - in this case an ultra-soft gluon

**Observables considered:**

1. Thrust
2. C-parameter

$\langle h_V \rangle$ is a rigid shift

3. Heavy-Jet Mass

Trivial interplay between PT and NP radiation

4. Total and wide-jet broadening
5. Thrust major

No analytic treatment

Recoil-sensitive

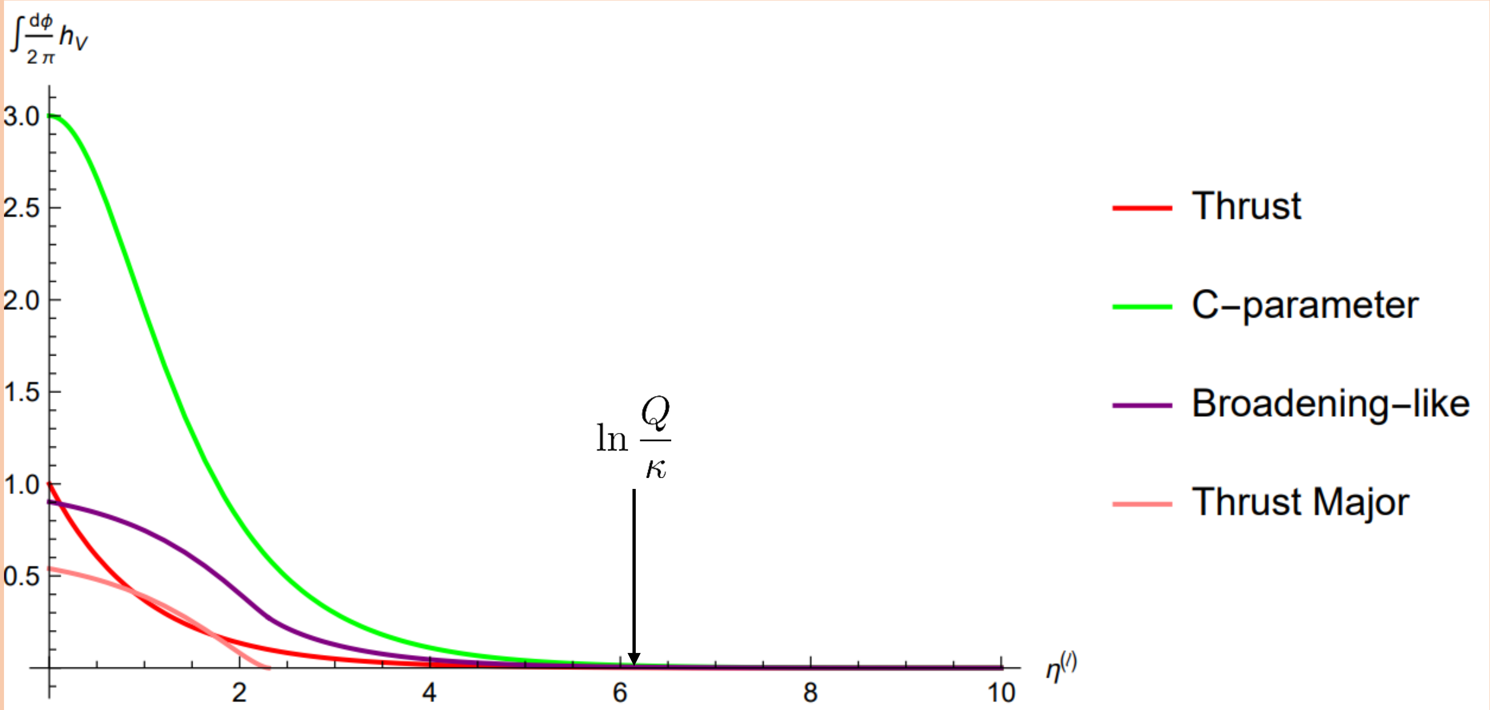
$\langle h_V \rangle$ known analytically

## Observables considered:

| Observable | $h_V$ |
|---|---|
| Thrust | $e^{-\eta^{(\ell)}}$ |
| C-parameter | $\frac{3}{\cosh \eta^{(\ell)}}$ |
| Heavy-Jet Mass | $e^{-\eta^{(\ell)}}\Theta(\rho_\ell - \rho_{\bar{\ell}})$ |
| Wide-Jet Broadening | $\frac{1}{2}\left[\sqrt{1 + 2e^{\eta^{(\ell)}}\frac{p_{t,\ell}}{Q}\cos\phi + e^{2\eta^{(\ell)}}\left(\frac{p_{t,\ell}}{Q}\right)^2} - e^{\eta^{(\ell)}}\frac{p_{t,\ell}}{Q}\right]\Theta(B_\ell - B_{\bar{\ell}})$ |
| Total Broadening | $\frac{1}{2}\left[\sqrt{1 + 2e^{\eta^{(\ell)}}\frac{p_{t,\ell}}{Q}\cos\phi + e^{2\eta^{(\ell)}}\left(\frac{p_{t,\ell}}{Q}\right)^2} - e^{\eta^{(\ell)}}\frac{p_{t,\ell}}{Q}\right]$ |
| Thrust Major | $\left\lvert\sin\phi + e^{\eta^{(\ell)}}\frac{\lvert p_{y,\ell}\rvert}{Q}\right\rvert - e^{\eta^{(\ell)}}\frac{\lvert p_{y,\ell}\rvert}{Q}$ |

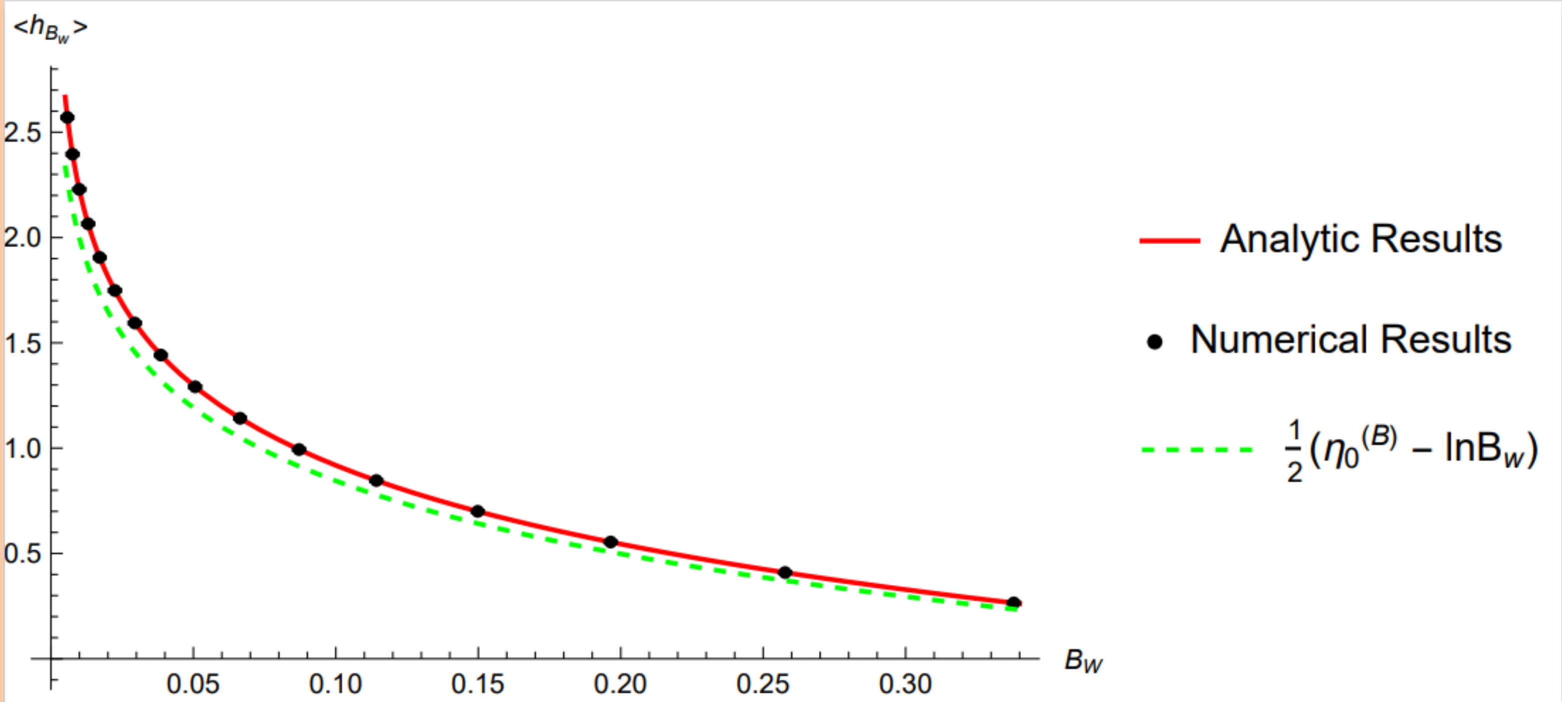
## Observables considered:

Illustration: $\frac{p_{t,\ell}}{Q}\,,\ \frac{p_{y,\ell}}{Q} \sim 0.1$ and $\kappa \sim \Lambda_{\rm QCD}$

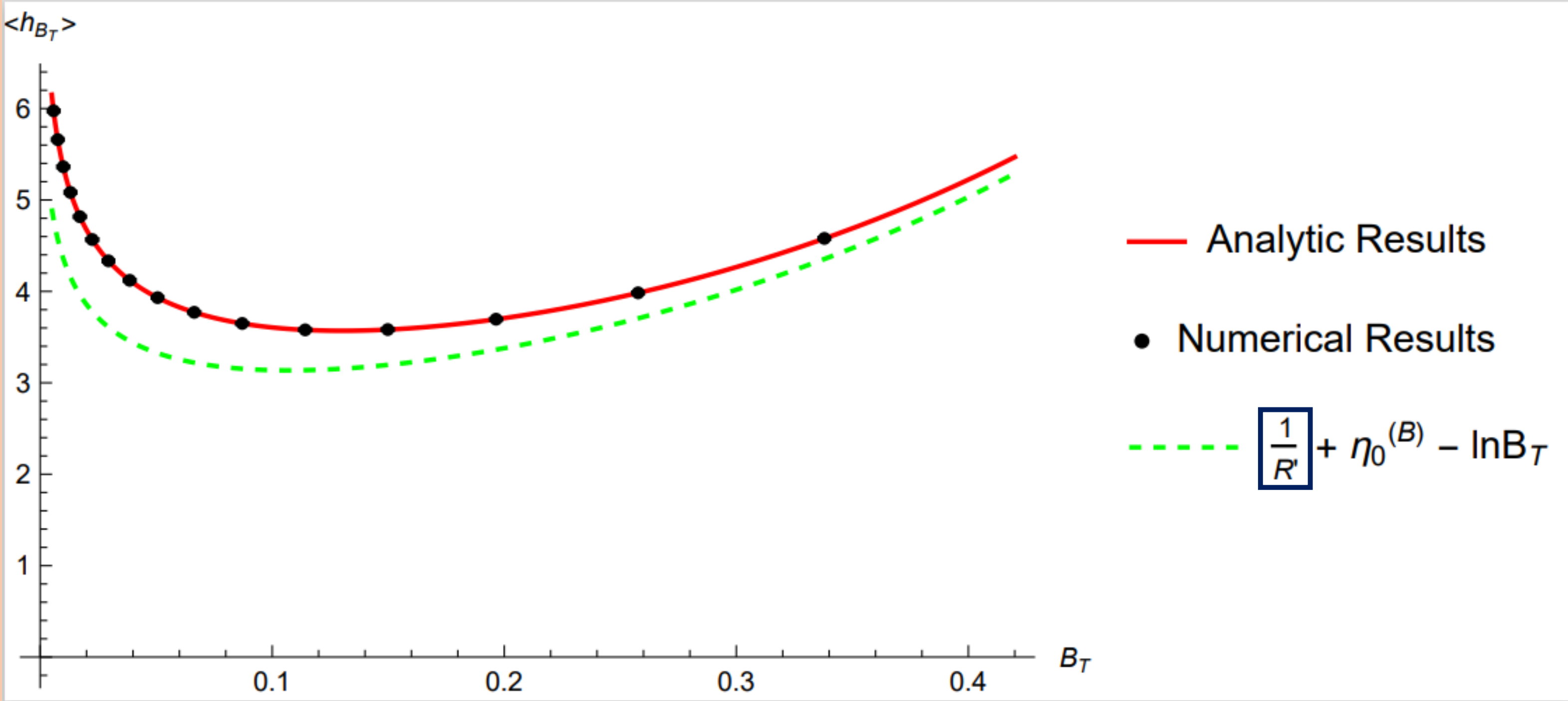


| Observable | $\langle h_V \rangle$ |
|---|---|
| Thrust | $2$ |
| C-parameter | $3\pi$ |
| Heavy-Jet Mass | $1$ |
| Wide-Jet Broadening | $\frac{1}{R'\,\mathcal{F}_{B_W}(R')}\int d\mathcal{Z}[\{R'_{\ell_i},k_i\}]\,\delta\!\left(1-\frac{B_{W,\mathrm{sc}}(\{\tilde{p}\},\{k_i\})}{B_W}\right)\times$ $\times\left[\int\prod_{\ell} d^2p_{t,1}\,\delta^{(2)}\!\left(\vec{p}_{t,1}+\sum_{i\in\mathcal{H}_1}\vec{k}_{t,i}\right)\boxed{\frac{1}{2}\ln\frac{Qe^{\eta_0^{(B)}}}{p_{t,1}}\,\Theta(B_1-B_2)+1\leftrightarrow 2}\right]$ |
| Total Broadening | $\frac{1}{R'\,\mathcal{F}_{B_T}(R')}\int d\mathcal{Z}[\{R'_{\ell_i},k_i\}]\,\delta\!\left(1-\frac{B_{T,\mathrm{sc}}(\{\tilde{p}\},\{k_i\})}{B_T}\right)\times$ $\times\int\prod_{\ell} d^2p_{t,\ell}\,\delta^{(2)}\!\left(\vec{p}_{t,\ell}+\sum_{i\in\mathcal{H}_\ell}\vec{k}_{t,i}\right)\boxed{\frac{1}{2}\sum_{\ell}\ln\frac{Qe^{\eta_0^{(B)}}}{p_{t,\ell}}}\,,\quad \eta_0^{(B)}=-0.6137056$ |
| Thrust Major | $\frac{1}{R'\,\mathcal{F}_{T_M}(R')}\,\frac{2}{\pi}\int d\mathcal{Z}[\{R'_{\ell_i},k_i\}]\,\delta\!\left(1-\frac{T_{M,\mathrm{sc}}(\{\tilde{p}\},\{k_i\})}{T_M}\right)\times$ $\times\int_{-\infty}^{\infty}\prod_{\ell} dp_{y,\ell}\,\delta\!\left(p_{y,\ell}+\sum_{i\in\mathcal{H}_\ell}k_{y,i}\right)\boxed{\sum_{\ell}\ln\frac{2Qe^{-2}}{\lvert p_{y,\ell}\rvert}}$ |

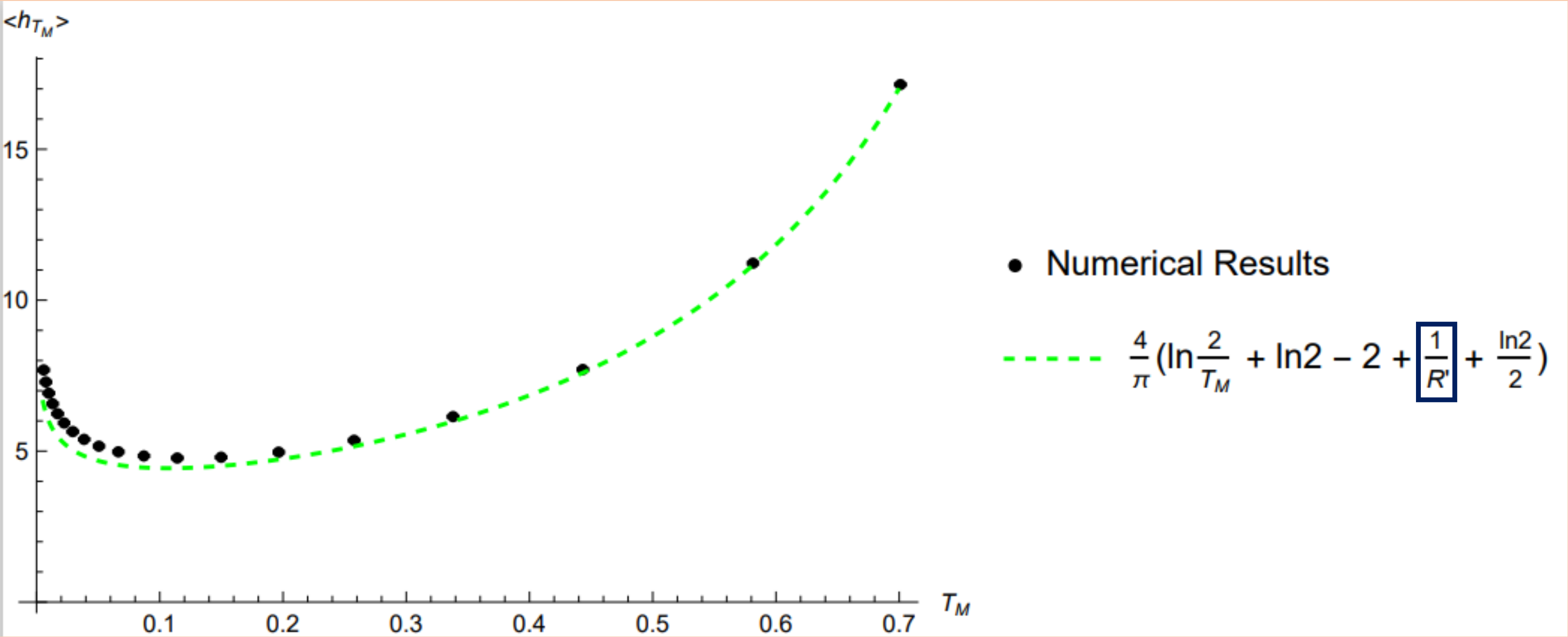
# Wide-Jet Broadening:



## Total Broadening:



## Thrust Major:



**Treating the divergence:**

The $1/R'$ divergence is an *unphysical behaviour*

If untreated it would considerably limit the available range to fit the non-perturbative moments

Numerical origins

As $R' \to 0$, one hemisphere contains a small number of emissions with $k_t \sim B_T Q \, , T_M Q$

Emissions in the other ("*empty*") hemisphere fall below the cutoff of the Monte Carlo integration

Problem for the Monte Carlo implementation, which assumes all transverse momenta of perturbative emissions to be of the same order

We find a zero value of $p_{t,\ell} \, , \, p_{y,\ell}$ in the *empty* hemisphere thus the calculation of $\ln(Q/p_{t,\ell}) \, , \, \ln(Q/p_{y,\ell})$ would give a floating point exception

⟶ To obtain finite numerical predictions we are forced to decrease the cutoff $\varepsilon$ more and more as $R' \to 0$

## Treating the divergence:

### Mathematical origins

When devising the measure $d\mathcal{Z}[\{R'_{\ell_i}, k_i\}]$, we have neglected all higher derivatives of the radiator beyond $R'$

In the region $R' \to 0$, $R'$ is now much smaller than the higher derivatives of $R(v)$ which can no longer be neglected

⟶ In particular, the second derivative $R''$ regularises this divergence

### Possible solution:

Try computing $\langle h_V \rangle$ from scratch, but retaining the higher derivatives of the radiator

- This *improved* evaluation may be carried out analytically for $\langle h_{B_T} \rangle$
  Dokshitzer, Marchesini, Salam [arXiv:9812487]

- Limited scope for observables which do not admit an analytic treatment (e.g thrust major)

**Treating the divergence:**

Our solution:

- Devise a procedure to compute $\langle h_V \rangle$, that is suitable for all observables and gives a finite result for $R' \to 0$
- Add and subtract to/from $\langle h_V \rangle$ a **counterterm** that displays the appropriate $1/R'$ behaviour
- Counterterm designed to cancel the divergence of $\langle h_V \rangle$ for $R' \to 0$ at the *integrand* level
- Counterterm should be simple enough to compute fully analytically for all values of $V$
- Thus when it is added back, the *improved* evaluation may be carried out analytically
- This procedure ensures that $\langle h_V \rangle$ is finite for all values of $R'$

**Improved evaluation of counterterm:**

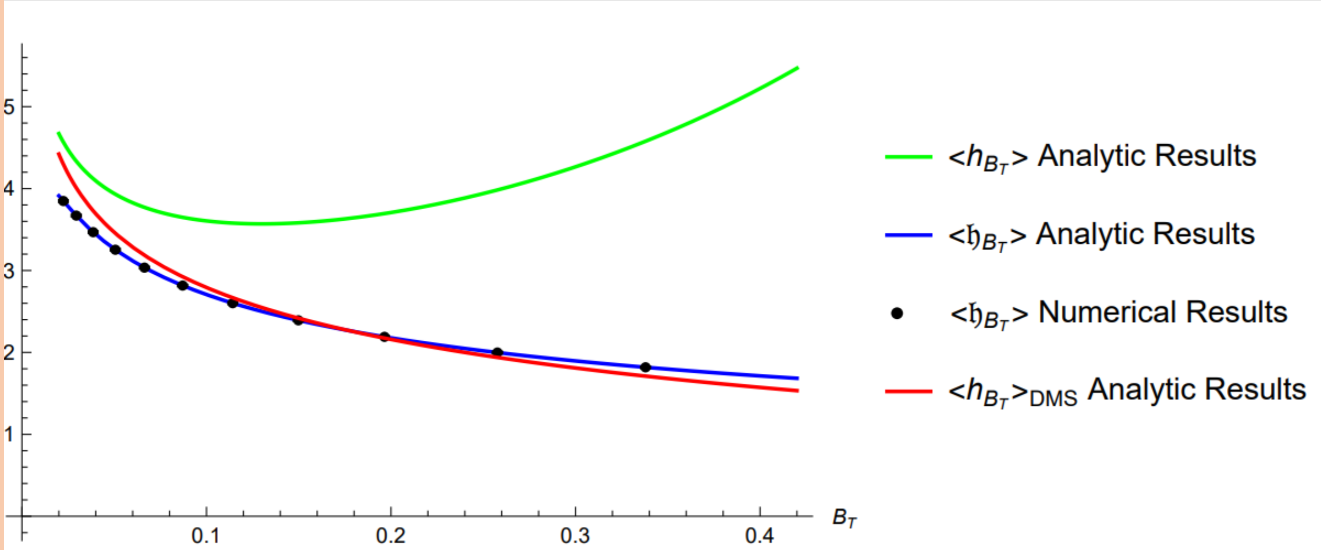
When adding back the counterterm, we perform the *improved* evaluation analytically:

- In the measure $d\mathcal{Z}[\{R'_{\ell_i}, k_i\}]$ we now retain higher derivatives of the radiator, $R''$ and $R^{(3)}$
- The second derivative regularises the $1/R'$ divergence, which now behaves as $1/\sqrt{\alpha_s}$
- The product of such a contribution with a finite correction of order $\alpha_s$ , which is beyond our nominal accuracy, gives a $\sqrt{\alpha_s}$ contribution
- Therefore in the *improved* evaluation of our counterterm we can account for all contributions up to order $\sqrt{\alpha_s}$
- In order to do so, we also need to upgrade the evaluation to take into account hard-collinear real and virtual corrections

## Total broadening:

The final expression for the shift of the total broadening is then

$$\langle \delta B_T \rangle = \frac{\langle \kappa \rangle_{\rm NP}}{Q} \langle \mathfrak{h}_{B_T} \rangle \qquad \text{where} \qquad \langle \mathfrak{h}_{B_T} \rangle \equiv \langle h_{B_T} \rangle - \langle h_{B_T}^{\rm c.t.} \rangle + \langle h_{B_T}^{\rm c.t.} \rangle_{\rm imp.}$$

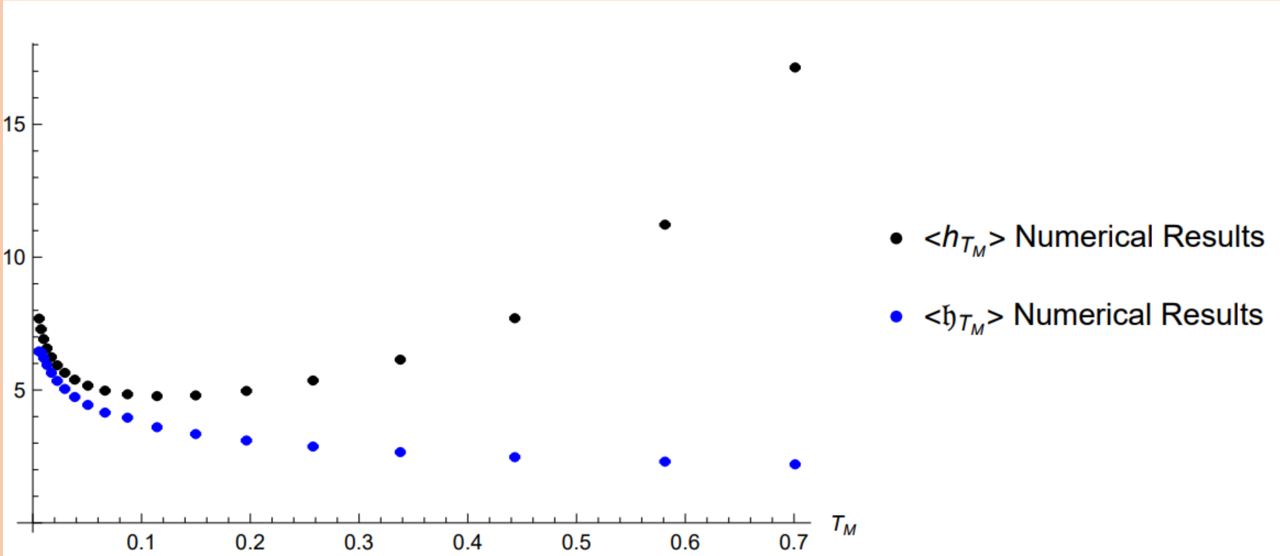



Dokshitzer, Marchesini, Salam [arXiv:9812487]

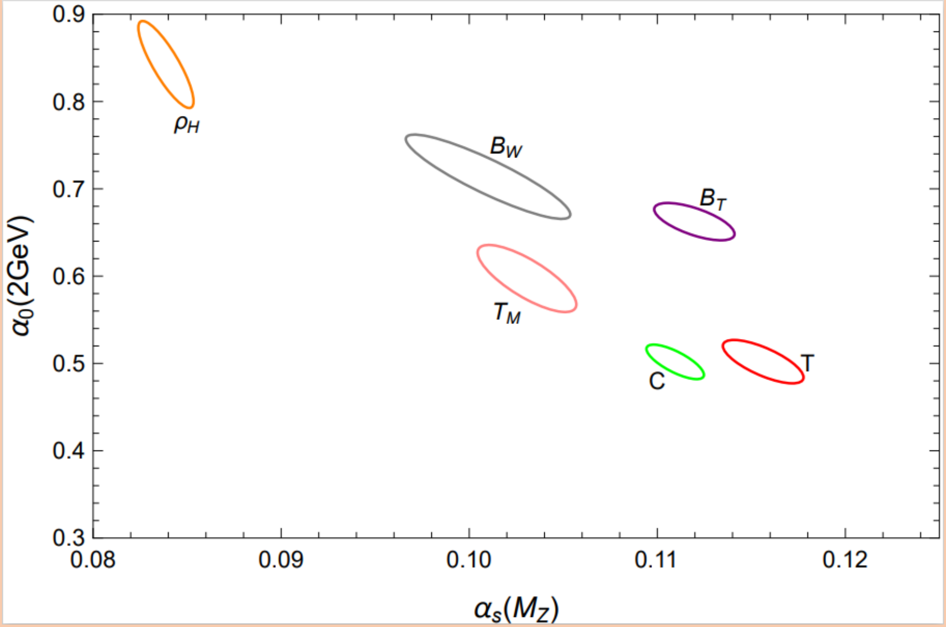
## Thrust major:

The final expression for the shift of the thrust major is then

$$\langle \delta T_M \rangle = \frac{\langle \kappa \rangle_{\rm NP}}{Q} \langle \mathfrak{h}_{T_M} \rangle \qquad \text{where} \qquad \langle \mathfrak{h}_{T_M} \rangle \equiv \langle h_{T_M} \rangle - \langle h^{\rm c.t.}_{T_M} \rangle + \langle h^{\rm c.t.}_{T_M} \rangle_{\rm imp.}$$



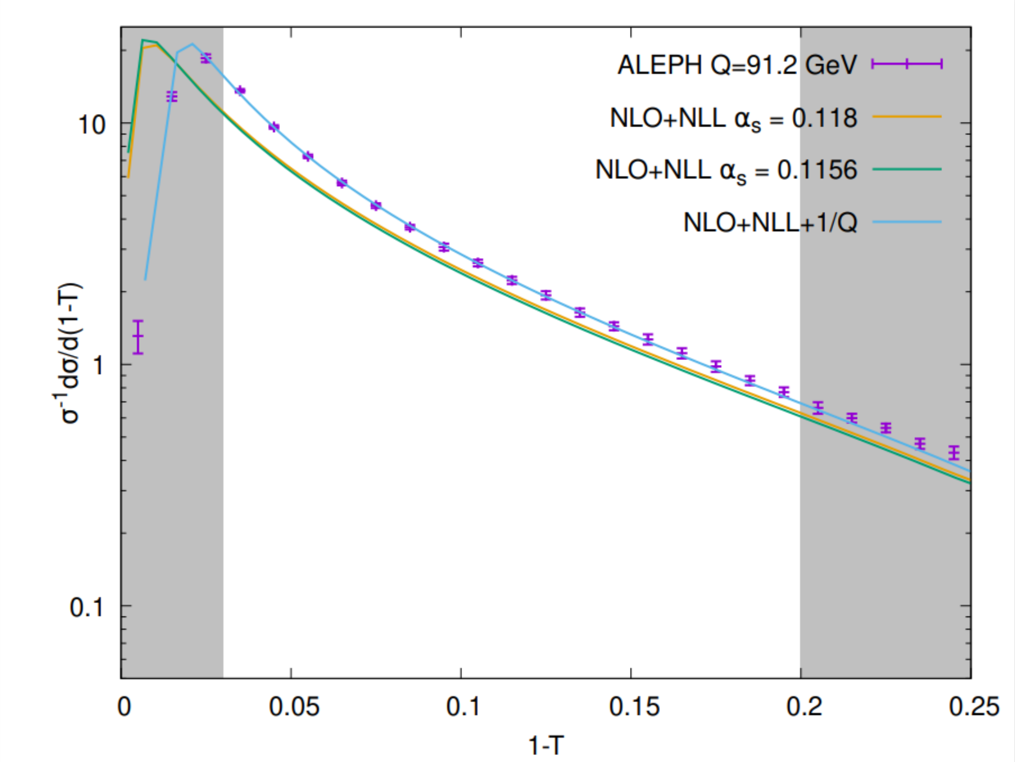
## Phenomenology - distributions:

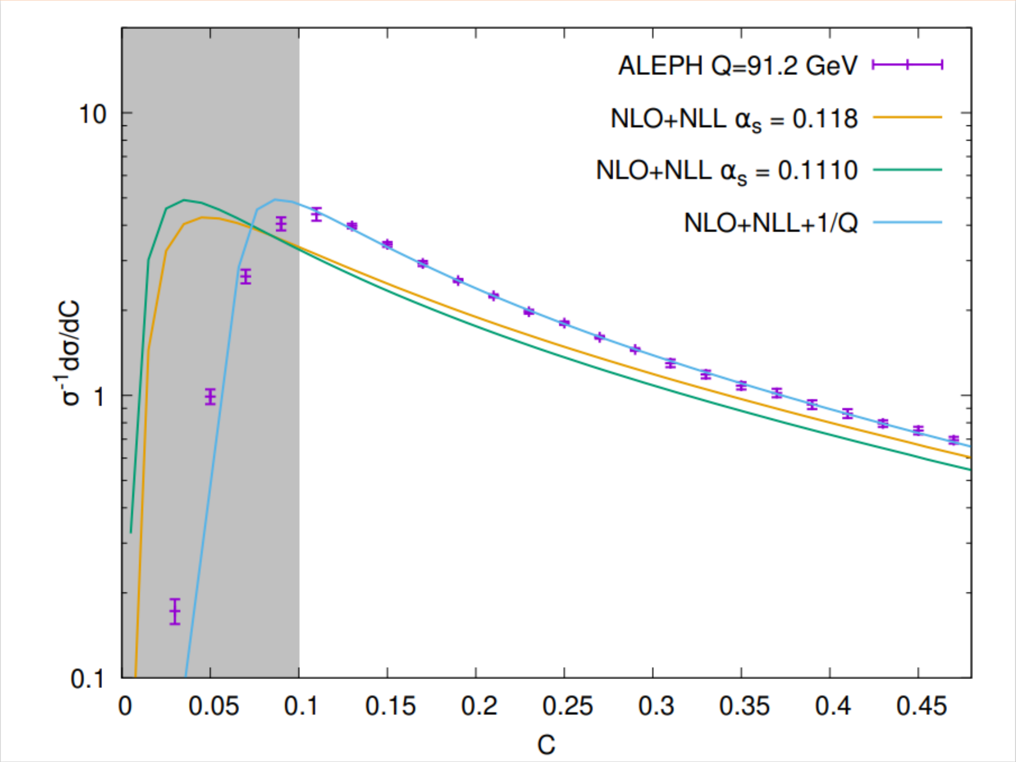



| Observable | $\alpha_s(M_Z)$ | $\alpha_0(2\text{GeV})$ | $\chi^2$ / d.o.f |
|---|---|---|---|
| $1-T$ | 0.1156 ± 0.0009 | 0.5020 ± 0.0102 | 54.9 / (56 - 2) |
| $C$ | 0.1110 ± 0.0006 | 0.5018 ± 0.0081 | 56.0 / (69 - 2) |
| $\rho_H$ | 0.0839 ± 0.0006 | 0.8424 ± 0.0203 | 137.7 / (61 - 2) |
| $B_W$ | 0.1010 ± 0.0018 | 0.7138 ± 0.0197 | 52.1 / (61 - 2) |
| $B_T$ | 0.1120 ± 0.0009 | 0.6624 ± 0.0087 | 77.4 / (72 - 2) |
| $T_M$ | 0.1031 ± 0.0011 | 0.5973 ± 0.0157 | 45.6 / (51 - 2) |

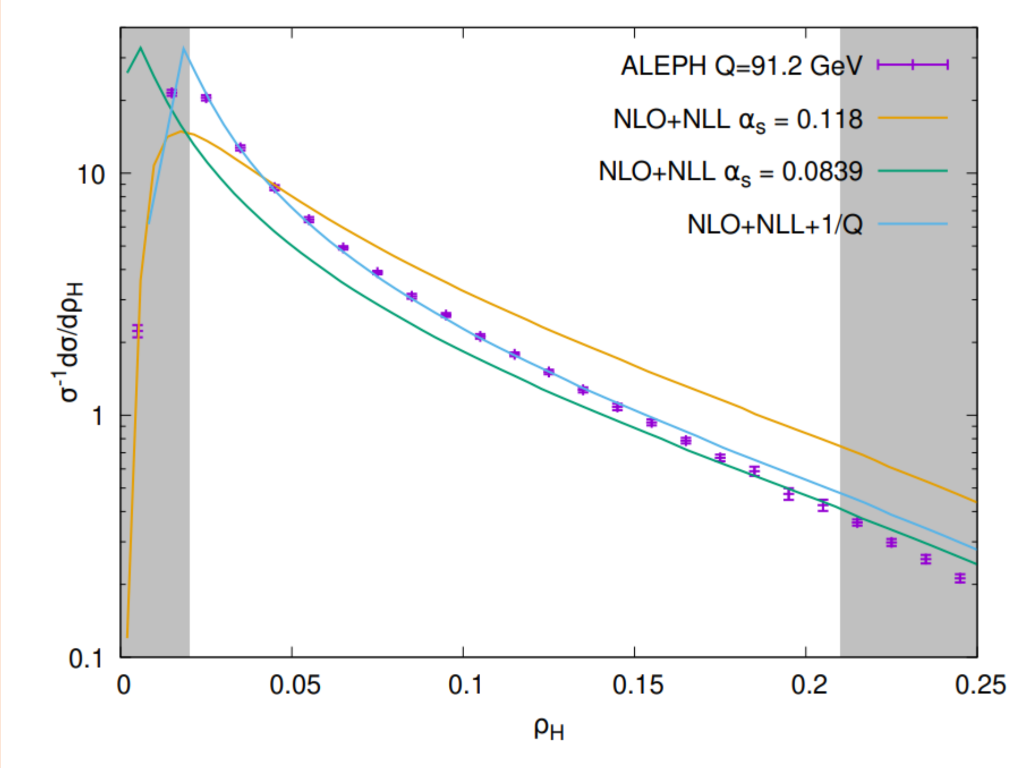
95% confidence level contours for the fitted values of $\alpha_s$ and $\alpha_0$ to experimental data (covering centre-of-mass energies between 91.2 GeV and 209 GeV) from ALEPH Collaboration [*Eur. Phys. J. C* **35** (2004) 457]

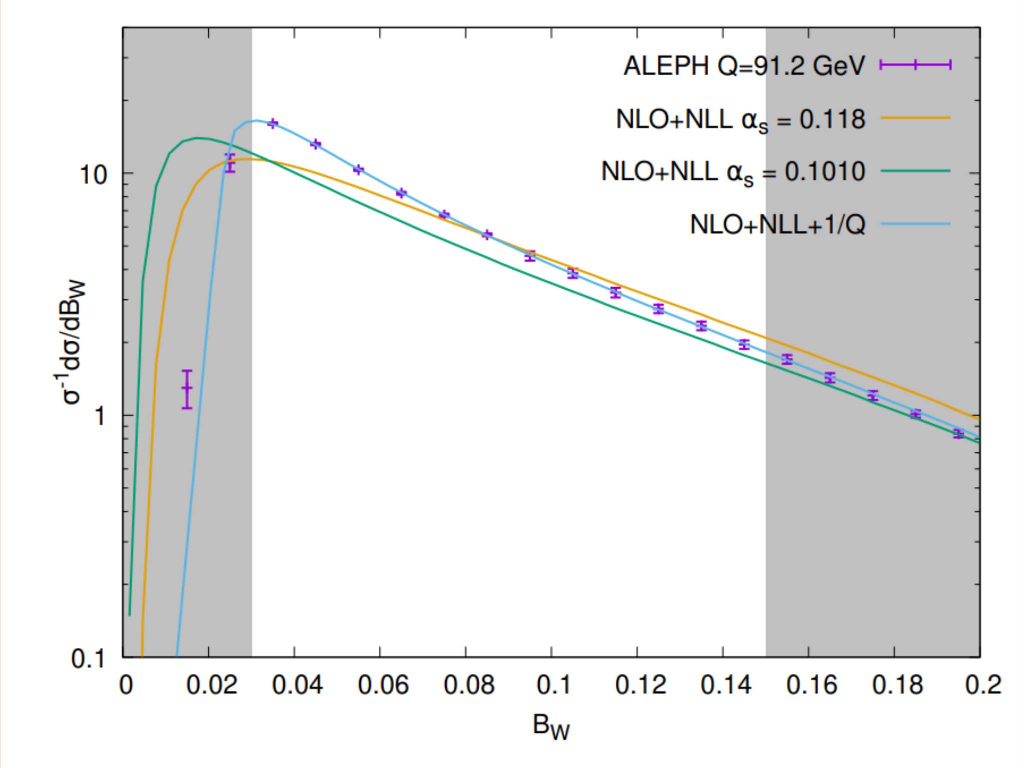
# Phenomenology - distributions:



## Phenomenology - distributions:



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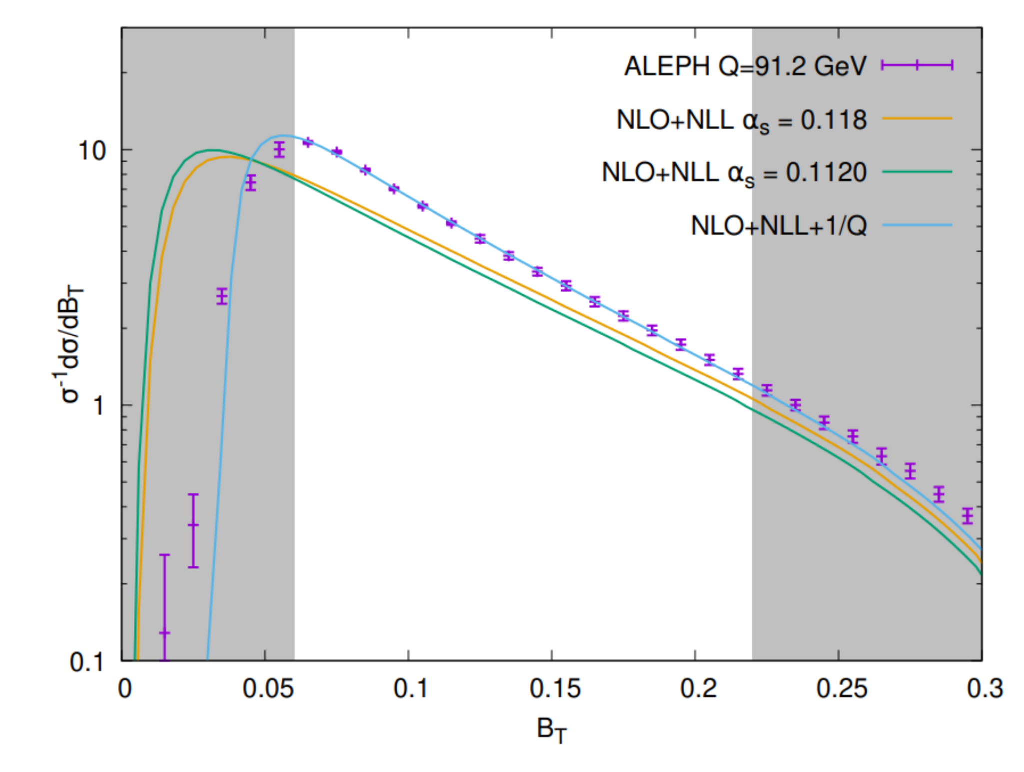
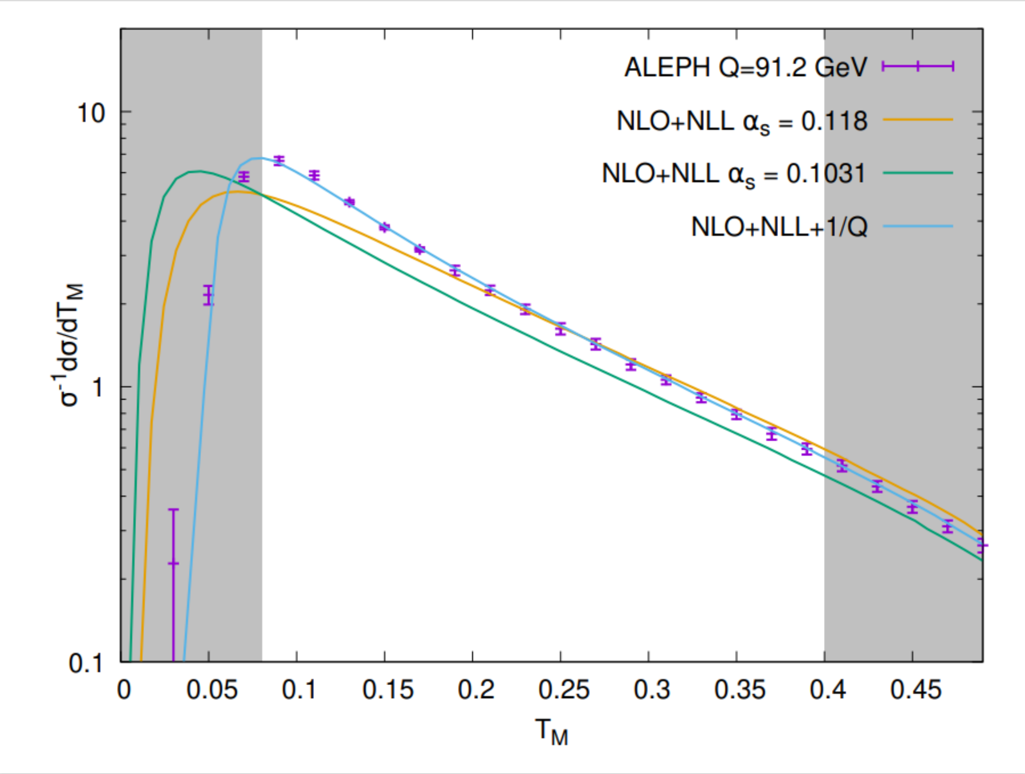
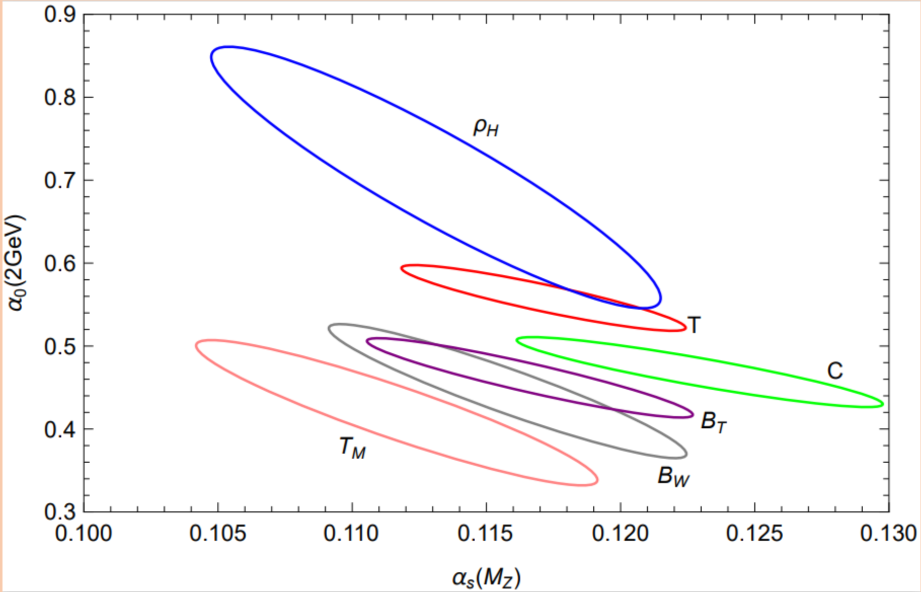


## Phenomenology - mean values:




| Observable | $\alpha_s(M_Z)$ | $\alpha_0(2\text{GeV})$ | $\chi^2$ / d.o.f |
|---|---|---|---|
| $1-T$ | 0.1171 ± 0.0022 | 0.558 ± 0.016 | 57.3 / (35 - 2) |
| $C$ | 0.1230 ± 0.0028 | 0.469 ± 0.017 | 15.3 / (20 - 2) |
| $\rho_H$ | 0.1131 ± 0.0034 | 0.703 ± 0.064 | 15.4 / (20 - 2) |
| $B_W$ | 0.1158 ± 0.0027 | 0.446 ± 0.033 | 10.8 / (20 - 2) |
| $B_T$ | 0.1166 ± 0.0025 | 0.462 ± 0.020 | 7.6 / (20 - 2) |
| $T_M$ | 0.1117 ± 0.0031 | 0.420 ± 0.036 | 9.2 / (13 - 2) |

95% confidence level contours for the fitted values of $\alpha_s$ and $\alpha_0$ to experimental mean value data used in ALEPH Collaboration [*Eur. Phys. J. C* **35** (2004) 457]

**Conclusions and next steps:**

- Presented a general method to compute the leading non-perturbative corrections to event-shape distributions in the two-jet region
- Validated our method by reproducing the leading hadronisation corrections to all known event-shape distributions and mean values
- Computed the leading hadronisation correction to the thrust major, for the first time
- To do so, we have devised a local subtraction procedure approach for dealing with the problem of unphysical divergences occurring for large values of recoil-sensitive event shapes
- Performed new simultaneous fits of $\alpha_s$ and $\alpha_0$ (with data obtained/used by the ALEPH collaboration) and have obtained consistent results
- Performed a similar fit for the thrust major distributions and mean values

## Conclusions and next steps:

There are several avenues for possible future extensions:

- Extend our calculation to the three-jet region
  Caola, Ferrario Ravasio, Limatola, Melnikov, Nason, Ozcelik [arXiv: 2204.02247]
  Nason, Zanderighi [arXiv:2301.03607]

- Extend our procedure to the two-jet rate, which is particularly important for precise $\alpha_s$ determinations

Thank you for listening

# Additional Slides

**Thrust and C-parameter:**

$$h_V(\eta^{(\ell)}, \phi, \{\tilde{p}\}, \{k_i\}) = h_V(\eta^{(\ell)}, \phi)$$

Therefore we find that,

$$\langle h_V \rangle = \sum_\ell \int d\eta^{(\ell)} \, \frac{d\phi}{2\pi} \, h_V(\eta^{(\ell)}, \phi)$$

For these observables,

$$h_{1-T}(\eta^{(\ell)}, \phi) = e^{-\eta^{(\ell)}} \quad , \qquad h_C(\eta^{(\ell)}, \phi) = \frac{3}{\cosh \eta^{(\ell)}}$$

Therefore,

$$\langle h_{1-T} \rangle = 2 \quad , \qquad \langle h_C \rangle = 3\pi$$

## Heavy-jet mass:

A non-zero NP correction arises only when the ultra-soft emission is in the heavier hemisphere

$$h_{\rho_H}(\eta^{(1)}, \phi, \{\tilde{p}\}, k_1, \ldots, k_n) = \Theta(\rho_1 - \rho_2)\, e^{-\eta^{(1)}}$$
$$h_{\rho_H}(\eta^{(2)}, \phi, \{\tilde{p}\}, k_1, \ldots, k_n) = \Theta(\rho_2 - \rho_1)\, e^{-\eta^{(2)}}$$

This gives,

$$\langle h_{\rho_H} \rangle = \frac{1}{R' \mathcal{F}_{\rho_H}(R')} \int_0^\infty d\eta^{(1)}\, e^{-\eta^{(1)}} \int d\mathcal{Z}[\{R'_{\ell_i}, k_i\}] \times$$
$$\times \left[\Theta\Big(\rho_1(\{\tilde{p}\}, \{k_i\}) - \rho_2(\{\tilde{p}\}, \{k_i\})\Big)\, \delta\left(1 - \frac{\rho_1(\{\tilde{p}\}, \{k_i\})}{\rho_H}\right)\right] + 1 \leftrightarrow 2$$

which implies,

$$\langle h_{\rho_H} \rangle = 1$$

## Total and wide-jet broadening:

Consider an ultra-soft emission $k$ in the hemisphere containing leg $\ell$

$$2B_\ell(\{\tilde{p}\}, k, \{k_i\})Q = k_t + \sum_{i\in\mathcal{H}_\ell} k_{ti} + \tilde{p}_{t,\ell}$$

thus,

$$2\delta B_\ell(\{\tilde{p}\}, k, \{k_i\})Q = k_t + \left|\vec{p}_{t,\ell} - \vec{k}_t\right| - p_{t,\ell}$$

Now,

$$k_t = \left|\vec{\kappa} + z^{(\ell)}\vec{p}_t\right| = \kappa\sqrt{1 + 2e^{\eta^{(\ell)}}\frac{p_{t,\ell}}{Q}\cos\phi + \left(e^{\eta^{(\ell)}}\frac{p_{t,\ell}}{Q}\right)^2}$$

$$\left|\vec{p}_{t,\ell} - \vec{k}_t\right| - p_{t,\ell} = \left|(1 - z^{(\ell)})\vec{p}_{t,\ell} - \vec{\kappa}\right| - p_{t,\ell} \simeq -z^{(\ell)}p_{t,\ell} - \kappa\cos\phi = -\kappa\, e^{\eta^{(\ell)}}\frac{p_{t,\ell}}{Q}$$

We use that

$$\int_0^\infty d\eta \int \frac{d\phi}{2\pi}\left[\sqrt{1 + 2e^\eta\frac{p}{Q}\cos\phi + \left(e^\eta\frac{p}{Q}\right)^2} - e^\eta\frac{p}{Q}\right] = \ln\frac{Q}{p} + \eta_0^{(B)} + \mathcal{O}\left(\frac{p}{Q}\right)$$

$\eta_0^{(B)}$ = - 0.6137056

## Total and wide-jet broadening:

We therefore find that

$$\langle h_{B_T} \rangle = \frac{1}{R' \, \mathcal{F}_{B_T}(R')} \int d\mathcal{Z}[\{R'_{\ell_i}, k_i\}] \, \delta\left(1 - \frac{B_{T,\mathrm{sc}}(\{\tilde{p}\}, \{k_i\})}{B_T}\right) \times$$
$$\times \int \prod_{\ell} d^2 p_{t,\ell} \, \delta^{(2)}\left(\vec{p}_{t,\ell} + \sum_{i \in \mathcal{H}_\ell} \vec{k}_{t,i}\right) \frac{1}{2} \sum_{\ell} \ln \frac{Q e^{\eta_0^{(B)}}}{p_{t,\ell}}$$

$$\langle h_{B_W} \rangle = \frac{1}{R' \, \mathcal{F}_{B_W}(R')} \int d\mathcal{Z}[\{R'_{\ell_i}, k_i\}] \, \delta\left(1 - \frac{B_{W,\mathrm{sc}}(\{\tilde{p}\}, \{k_i\})}{B_W}\right) \times$$
$$\times \left[\int \prod_{\ell} d^2 p_{t,1} \, \delta^{(2)}\left(\vec{p}_{t,1} + \sum_{i \in \mathcal{H}_1} \vec{k}_{t,i}\right) \frac{1}{2} \ln \frac{Q e^{\eta_0^{(B)}}}{p_{t,1}} \Theta(B_1 - B_2) + 1 \leftrightarrow 2\right]$$

## Thrust major:

Soft and collinear PT emissions determine the thrust major axis $\vec{n}_M$ (sets the y-axis by convention)

Consider an ultra-soft emission $k$ in the hemisphere containing leg $\ell$

$$T_M(\{\tilde{p}\}, k, k_1, \ldots, k_n)Q = \sum_i^n |k_{yi}| + |k_y| + |\tilde{p}_{y,1}| + |\tilde{p}_{y,2}|$$

Therefore,

$$\delta T_M(\{\tilde{p}\}, k, \{k_i\})Q = |k_y| + |\tilde{p}_{y,\ell}| - |p_{y,\ell}|$$

Now,

$$|k_y| = \left|\kappa_y + z^{(\ell)}|p_{y,\ell}|\right| = \kappa \left|\sin\phi + e^{\eta^{(\ell)}} \frac{|p_{y,\ell}|}{Q}\right|$$

$$|\tilde{p}_{y,\ell}| - |p_{y,\ell}| = \left|(1 - z^{(\ell)})|p_{y,\ell}| - \kappa_y\right| - |p_{y,\ell}| = \sqrt{\left[(1 - z^{(\ell)})|p_{y,\ell}| - \kappa_y\right]^2} - |p_{y,\ell}|$$

$$\simeq -z^{(\ell)}|p_{y,\ell}| - \kappa\sin\phi = -\kappa\, e^{\eta^{(\ell)}} \frac{|p_{y,\ell}|}{Q}$$

**Thrust major:**

We use that

$$\int_0^\infty d\eta \int_0^\pi \frac{d\phi}{\pi} \left( \left| \sin\phi + e^\eta \frac{p}{Q} \right| - e^\eta \frac{p}{Q} \right) = \frac{2}{\pi} \left( \ln\frac{Q}{p} + \ln 2 - 2 \right) + \mathcal{O}\left(\frac{p}{Q}\right)$$

and find

$$\langle h_{T_M} \rangle = \frac{1}{R' \, \mathcal{F}_{T_M}(R')} \frac{2}{\pi} \int d\mathcal{Z}[\{R'_{\ell_i}, k_i\}] \, \delta\!\left(1 - \frac{T_{M,\mathrm{sc}}(\{\tilde{p}\},\{k_i\})}{T_M}\right) \times$$
$$\times \int_{-\infty}^{\infty} \prod_\ell dp_{y,\ell} \, \delta\!\left(p_{y,\ell} + \sum_{i\in\mathcal{H}_\ell} k_{y,i}\right) \sum_\ell \ln\frac{2Qe^{-2}}{|p_{y,\ell}|}$$

## **Counterterm for Total broadening:**

In the limit $R' \to 0$ one emission, emitted in hemisphere $\mathcal{H}_\ell$, has a value of transverse momentum that is much larger than all other emissions and determines $B_T$

We denote by $\mathcal{H}_{\bar{\ell}}$ , the hemisphere that does not contain this emission

A good counterterm might therefore be

$$\langle h_{B_T} \rangle = \frac{1}{R' \mathcal{F}_{B_T}(R')} \int d^2 \vec{p}_{t,\bar{\ell}} \int d\mathcal{Z}[\{R'_{\ell_i}, k_i\}]\, \delta\left(1 - \frac{\max_i\{k_{ti}\}}{B_T Q}\right) \frac{1}{2} \ln \frac{Q}{p_{t,\bar{\ell}}} \times$$
$$\times\, \delta^{(2)}\left(\vec{p}_{t,\bar{\ell}} + \sum_{i \in \mathcal{H}_{\bar{\ell}}} \vec{k}_{t,i}\right) \Theta\left(\max_i\{k_{ti}\} - \frac{1}{2} p_{t,\bar{\ell}} - \frac{1}{2} \sum_{i \in \mathcal{H}_{\bar{\ell}}} k_{t,i}\right)$$

where $\left|\vec{k}_{t,\max}\right| = \max_i\{k_{ti}\}$

## Counterterm for Thrust major:

Similarly, in the limit $R' \to 0$ one emission, emitted in hemisphere $\mathcal{H}_\ell$, has a value of transverse momentum that is much larger than all other emissions.

This emission determines $T_M$ and sets the thrust-major axis

A good counterterm might therefore be

$$\langle h_{T_M} \rangle = \frac{1}{R' \mathcal{F}_{T_M}(R')} \int dp_{y,\bar{\ell}} \int d\mathcal{Z}[\{R'_{\ell_i}, k_i\}] \, \delta\left(1 - \frac{2 \max_i \{k_{ti}\}}{T_M Q}\right) \frac{2}{\pi} \ln \frac{Q}{|p_{y,\bar{\ell}}|} \times$$
$$\times \, \delta\left(p_{y,\bar{\ell}} + \sum_{i \in \mathcal{H}_{\bar{\ell}}} k_{y,i}\right) \Theta\left(\max_i \{k_{ti}\} - |p_{y,\bar{\ell}}| - \sum_{i \in \mathcal{H}_{\bar{\ell}}} k_{t,i}\right)$$

where $\left|\vec{k}_{t,\max}\right| = \max_i \{k_{ti}\}$ and the y-direction is along $\vec{k}_{t,\max}$

## Mean values:

With the formalism we have developed, we may also compute the leading non-perturbative corrections to the mean values of event shapes.

$$\langle v \rangle = \int_0^{v_{\max}} dv\, v \frac{d}{dv} \Sigma(v) = \langle v \rangle_{\mathrm{PT}} + \langle v \rangle_{\mathrm{NP}}$$

where

$$\langle v \rangle_{\mathrm{PT}} \equiv \int_0^{v_{\max}} dv\, v \frac{d}{dv} \Sigma_{\mathrm{PT}}(v)$$

Can be computed as an expansion in powers of $\alpha_s$

$$\langle v \rangle_{\mathrm{NP}} \equiv \int_0^{v_{\max}} dv\, v \frac{d}{dv} \delta\Sigma_{\mathrm{NP}}(v)$$

We find

$$\begin{aligned}\langle v \rangle_{\mathrm{NP}} &= v\, \delta\Sigma_{\mathrm{NP}}(v) \Big|_0^{v_{\max}} - \int_0^{v_{\max}} dv\, \delta\Sigma_{\mathrm{NP}}(v) \\ &\simeq \int [dk]\, M_{\mathrm{NP}}^2(k) \int dZ[\{k_i\}]\, \delta V_{\mathrm{NP}}(\{\tilde{p}\}, k, \{k_i\})\, \Theta\Big(v_{\max} - V(\{\tilde{p}\}, \{k_i\})\Big)\end{aligned}$$

## Mean values:

Therefore

$$\langle v\rangle_{\rm NP} \simeq \mathcal{M}\frac{\langle\kappa\rangle_{\rm NP}}{Q}\sum_\ell \int d\eta^{(\ell)}\,\frac{d\phi}{2\pi}\int dZ_{\rm sc}[\{k_i\}]\,h_V\left(\eta^{(\ell)},\phi,\{\tilde p\},\{k_i\}\right)\Theta\Big(v_{\rm max}-V_{\rm sc}(\{\tilde p\},\{k_i\})\Big)$$

1. **Thrust, C-parameter, Heavy-jet mass** $\langle v\rangle_{\rm NP} = \mathcal{M}\frac{\langle\kappa\rangle_{\rm NP}}{Q}\langle h_V\rangle$

2. **Jet broadenings and thrust major**

$$\langle B_W\rangle_{\rm NP} \simeq \mathcal{M}\frac{\langle\kappa\rangle_{\rm NP}}{Q}\frac{1}{2}\left[\eta_0^{(B)} + \frac{\pi}{2\sqrt{2C_F\,\alpha_s(Q)}} + \frac{3}{4} - \frac{\pi\beta_0}{3C_F}\right] + \mathcal{O}(\sqrt{\alpha_s})$$

$$\langle B_T\rangle_{\rm NP} \simeq \mathcal{M}\frac{\langle\kappa\rangle_{\rm NP}}{Q}\left[\eta_0^{(B)} + \frac{\pi}{2\sqrt{C_F\,\alpha_s(Q)}} + \frac{3}{4} - \frac{2\pi\beta_0}{3C_F}\right] + \mathcal{O}(\sqrt{\alpha_s})$$

Results agree up to terms of order $\sqrt{\alpha_s}$ with Dokshitzer, Marchesini, Salam [arXiv:9812487]

$$\langle T_M\rangle_{\rm NP} \simeq \mathcal{M}\frac{\langle\kappa\rangle_{\rm NP}}{Q}\frac{4}{\pi}\left[2\ln 2 - 2 + \frac{\pi}{2\sqrt{C_F\,\alpha_s(Q)}} + \frac{3}{4} - \frac{2\pi\beta_0}{3C_F} + \frac{\ln 2}{2}\right] + \mathcal{O}(\sqrt{\alpha_s})$$