# Soft photon emission and the LBK theorem

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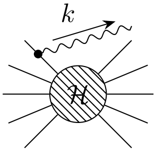
WWU MÜNSTER

Graduiertenkolleg 2149 Research Training Group

# Introduction

- Understanding the soft radiation is vital in order to have reliable theoretical calculations.




Leading Power (LP) $= \mathcal{O}\big(k^{-1}\big)$
Next-to-Leading Power (NLP) $= \mathcal{O}\big(k^{0}\big)$

- In the recent years great developments have been done in Next-to-Leading Power factorization and resummation, using both SCET and direct QCD approaches.
  - N. Bahjat-Abbas, D. Bonocore, J. Sinninghe Damsté, E. Laenen, L. Magnea, M. van Beekveld, L. Vernazza, C.D. White (2015-2021)
  - M. Beneke, A. Broggio, M. Garny, S. Jaskiewicz, R. Szafron, L. Vernazza, J. Wang (2016-2022)
  - I. Moult, I.W. Stewart, F.J. Tackmann, G. Vita, H.X. Zhu (2015-2020)
  - Z.L. Liu, M. Neubert, M. Schnubel, X. Wang (2021-2022)
  - A.H. Ajjath, P. Mukherjee, V. Ravindran, A. Sankar, S. Tiwari (2020-2022)
  + many others!

- In the context of QED, Next-to-Leading Power terms are also of great interest for describing single photon Bremsstrahlung and there have also been recent advances in this direction.
  - E. Laenen, J. Sinninghe Damsté, L. Vernazza, W. Waalewijn, L. Zoppi - *Phys.Rev.D* (2020)
  - T. Engel, A. Signer, Y. Ulrich - *JHEP* (2021)
  - D. Bonocore, A. Kulesza - *Phys.Rev.B* (2021)
  - P. Lebiedowicz, O. Nachtmann, A. Szczurek - *Phys.Rev.D* (2021)
  - T. Engel (2023)

# Next-to-Leading Power in QED

But $\mathrm{NLP}$ contributions seem to lead to different expressions for the same amplitudes. For example, for the process $\pi^-\pi^0 \to \pi^-\pi^0\gamma$
Original formulation by Low [F.E. Low - *Phys.Rev.* (1958)]

$$\frac{1}{e}\mathcal{A}^\mu = \mathcal{H}(s_L,t)\left[\frac{p_a^\mu}{p_a\cdot k} - \frac{p_1^\mu}{p_1\cdot k}\right] + \frac{\partial\mathcal{H}(s_L,t)}{\partial s_L}\left[p_b^\mu - \frac{p_b\cdot k}{p_a\cdot k}p_a^\mu + p_2^\mu - \frac{p_2\cdot k}{p_1\cdot k}p_1^\mu\right]$$

Another formulation of the same process [P. Lebiedowicz, O. Nachtmann, A. Szczurek - *Phys.Rev.D* (2021)]

$$\begin{aligned}\frac{1}{e}\mathcal{A}^\mu =& \mathcal{H}(s_L,t)\left[\frac{p_a^\mu}{p_a\cdot k} - \frac{p_1^\mu}{p_1\cdot k} - \frac{(p_1^\mu l_1^\nu - p_1^\nu l_1^\mu)k_\nu}{(p_1\cdot k)^2}\right] + 2\frac{\partial\mathcal{H}(s_L,t)}{\partial s_L}\left[p_b^\mu - \frac{p_b\cdot k}{p_a\cdot k}p_a^\mu\right] \\ &- 2\frac{\partial\mathcal{H}(s_L,t)}{\partial t}\left[(p_a - p_1)\cdot k - p_a\cdot l_1\right]\left[\frac{p_a^\mu}{p_a\cdot k} - \frac{p_1^\mu}{p_1\cdot k}\right]\end{aligned}$$

**Do $\mathrm{NLP}$ formulas give consistent results?**

# Anomalous soft photon problem

- Single soft photon emissions and soft dileptons can be detected experimentally, so we can compare these measurements with $\mathrm{LP}$ and $\mathrm{NLP}$ formulas.

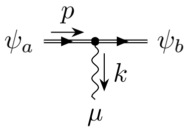
Ratio of observed soft photon over expected from $\mathrm{LP}$ soft bremsstrahlung. [C. Wong (2014)]

| Experiment | Collision Energy | Photon $k_T$ | Obs/Brem Ratio |
|---|---|---|---|
| $K^+p$, CERN, WA27, BEBC (1984) | 70 GeV | $k_T$ < 60 MeV | 4.0 $\pm$0.8 |
| $K^+p$, CERN, NA22, EHS (1993) | 250 GeV | $k_T$ < 40 MeV | 6.4 $\pm$1.6 |
| $\pi^+p$, CERN, NA22, EHS (1997) | 250 GeV | $k_T$ < 40 MeV | 6.9 $\pm$1.3 |
| $\pi^-p$, CERN, WA83, OMEGA (1997) | 280 GeV | $k_T$ < 10 MeV | 7.9 $\pm$1.4 |
| $\pi^+p$, CERN, WA91, OMEGA (2002) | 280 GeV | $k_T$ <20 MeV | 5.3 $\pm$0.9 |
| $pp$, CERN, WA102, OMEGA (2002) | 450 GeV | $k_T$ <20 MeV | 4.1 $\pm$0.8 |
| $e^+e^- \to$ hadrons, CERN, LEP, DELPHI with hadron production (2010) | $\sim$91 GeV(CM) | $k_T$ <60 MeV | 4.0 |
| $e^+e^- \to \mu^+\mu^-$, CERN, LEP, DELPHI with no hadron production (2008) | $\sim$91 GeV(CM) | $k_T$ <60 MeV | 1.0 |

- Excess of observed soft photons, but only for processes involving hadrons.
- Bremsstrahlung predictions are calculated using $\mathrm{LP}$ formulas.
- Future upgrades on the ALICE detector (ALICE 3 expected by $\sim$2035) will be able to measure ultra-soft photons, up to 1MeV.
- We want an efficient implementation of the $\mathrm{LBK}$ theorem for soft photon emission.

# Soft photon emission: Eikonal (LP) approximation

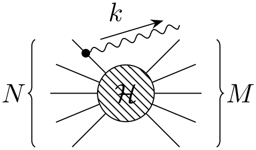
Eikonal approximation:




$$= \eta Q \frac{p^\mu}{p\cdot k}\delta_{ab}$$

$$\eta = \begin{cases} +1 & \text{for initial anti-fermions and final fermions} \\ -1 & \text{for final anti-fermions and initial fermions} \end{cases}$$

Emission of a soft photon from a general process $N \to M + \gamma$:




$$\mathcal{A}_{\mathrm{LP}}(p,k) = \left(\sum_i \eta_i Q_i \frac{p_i \cdot \varepsilon^*(k)}{p_i \cdot k}\right)\mathcal{H}(p)$$

The emission of a photon factorizes at LP;

$$\mathcal{A}_{\mathrm{LP}}(p,k) = \mathcal{S}_{\mathrm{LP}}(p,k)\mathcal{H}(p)$$

# Soft photon emission: Eikonal (LP) approximation

$$\mathcal{A}_{\mathrm{LP}}(p,k) = \left( \sum_i \eta_i Q_i \frac{p_i \cdot \varepsilon^*(k)}{p_i \cdot k} \right) \mathcal{H}(p)$$

**Is this approximation consistent?**

- Relation between $\mathcal{A}$ and $\mathcal{H}$
- It is impossible to impose momentum conservation in both;

$$\sum_j p_j = k \Longrightarrow \sum_j p_j \neq 0,$$

- When $\sum_j p_j \neq 0$ the amplitude $\mathcal{H}$ is not unambiguously defined.

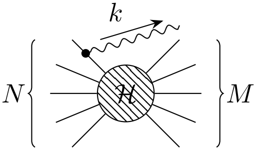
$$\mathcal{H}'(p) = \mathcal{H}(p) + \Delta(p)$$

  $\mathcal{H}'$ is obviously equivalent to $\mathcal{H}$ if $\Delta$ vanishes when four-momentum is conserved.

$$\sum_j p_j \to 0 \Longrightarrow \Delta(p) \to 0$$

**Yes!** The effects of using $\mathcal{H}'$ vs $\mathcal{H}$ are $\mathrm{NLP}$ effects.

# Soft photon emission: NLP (Low-Burnett-Kroll Theorem)




$$\mathcal{A}_j = \varepsilon^*_\mu(k)\bar{\mathcal{H}}_j(p_1, \ldots, p_j - k, \ldots, p_{N+M})\frac{i(\not{p}_j - \not{k} + m)}{(p_j - k)^2 - m^2}(-iQ\gamma^\mu)u(p_j)$$

$$\mathcal{A}_j = -Q\varepsilon^*_\mu(k)\left[\bar{\mathcal{H}}_j(p) - k^\nu\frac{\partial\bar{\mathcal{H}}_j(p)}{\partial p_j^\nu}\right]\frac{2p_j^\mu - ik_\lambda\sigma^{\mu\lambda}}{2k \cdot p_j}u(p_j) + \mathcal{O}(k)$$

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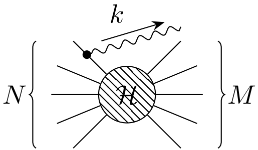


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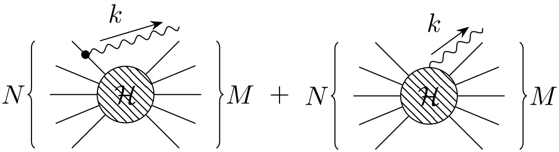
$$\mathcal{A}^\mu_{\text{ext}} = \sum_j \frac{\eta_j Q_j}{k \cdot p_j}\left[p^\mu_j \mathcal{H} + i\eta_j k_\nu \frac{\hat{\sigma}^{\mu\nu}_j}{2}\mathcal{H} - p^\mu_j k^\nu \hat{D}_{j\nu}\mathcal{H}\right] + \mathcal{O}(k)$$

- $\hat{\sigma}^{\mu\nu}_j\mathcal{H}$: Substitute $u(p_j) \to \sigma^{\mu\nu}u(p_j)$ or equivalent for anti-fermions and final-state fermions.
- $\hat{D}_{j\nu}\mathcal{H}$: Differentiate the amputated part of $\mathcal{H}$ with respect to $p^\nu_j$:

$$\bar{\mathcal{H}}_j u(p_j) \to \frac{\partial \bar{\mathcal{H}}_j}{\partial p^\nu_j}u(p_j)$$

# Soft photon emission: NLP (Low-Burnett-Kroll Theorem)




$$\mathcal{A} = \varepsilon^*_\mu \left(\mathcal{A}^\mu_{\text{ext}} + \mathcal{A}^\mu_{\text{ext}}\right) \Longrightarrow k_\mu \left(\mathcal{A}^\mu_{\text{ext}} + \mathcal{A}^\mu_{\text{ext}}\right) = 0$$

$$k_\mu \mathcal{A}^\mu_{\text{int}} = -k_\mu \mathcal{A}^\mu_{\text{ext}} = k_\mu \sum_j \eta_j Q_j g^{\mu\nu} \hat{D}_{j\nu} \mathcal{H} + \mathcal{O}(k)$$

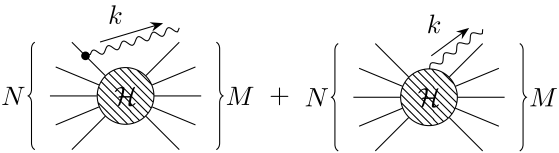
Considering only tree-level diagrams: [S.L. Adler, Y. Dothan - *Phys.Rev.* (1966)]

$$\mathcal{A}^\mu_{\text{int}} = \sum_j \eta_j Q_j g^{\mu\nu} \hat{D}_{j\nu} \mathcal{H} + \mathcal{O}(k)$$

$$\mathcal{A}_{\text{LP+NLP}}(p,k) = \sum_j \frac{\eta_j Q_j}{k \cdot p_j} \left[ p^\mu_j + i\eta_j k_\nu \frac{\hat{\sigma}^{\mu\nu}_j}{2} + (k \cdot p_j) G^{\mu\nu}_j \hat{D}_{j\nu} \right] \mathcal{H}$$

$$G^{\mu\nu}_i = g^{\mu\nu} - \frac{p^\mu_i k^\nu}{p_i \cdot k}$$

- $\hat{\sigma}^{\mu\nu}_j$ and $\hat{D}_{j\nu}$ make an efficient numerical implementation difficult.




The expression is simplified considering the unpolarized process and computing $\overline{|\mathcal{A}|}^2$. [T.H. Burnett, N.M. Kroll - *Phys.Rev.Lett.* (1967)]
This is because of the relation

$$ik_\nu \left[\sigma^{\mu\nu}, \not{p}_j \pm m\right] = -2(k \cdot p_j) G^{\mu\nu} \frac{\partial(\not{p}_j \pm m)}{\partial p_j^\nu}$$

which allows to combine all the $\mathrm{NLP}$ terms together;

$$\overline{|\mathcal{A}|}^2_{\mathrm{LP+NLP}} = -\sum_{i,j} \frac{(\eta_i Q_i p_i) \cdot (\eta_j Q_j p_j)}{(p_i \cdot k)(p_j \cdot k)} \left[1 + \frac{(p_j \cdot k) p_{i\mu}}{p_i \cdot p_j} G_j^{\mu\nu} \frac{\partial}{\partial p_j^\nu}\right] \overline{|\mathcal{H}|}^2$$

# Soft photon emission: $\mathrm{NLP}$ (Low-Burnett-Kroll Theorem)

As before, one cannot impose conservation of four-momentum in both $\mathcal{A}$ and $\mathcal{H}$.

$$\overline{|\mathcal{H}'|}^2(p) = \overline{|\mathcal{H}|}^2(p) + \Delta(p), \qquad \sum_j p_j \to 0 \Longrightarrow \Delta(p) \to 0$$

The theorem is consistent if, for any such function $\Delta(p)$,

$$\sum_{i,j} \frac{(\eta_i Q_i p_i)\cdot(\eta_j Q_j p_j)}{(p_i\cdot k)(p_j\cdot k)}\left[1 + \frac{(p_j\cdot k)p_{i\mu}}{p_i\cdot p_j} G_j^{\mu\nu}\frac{\partial}{\partial p_j^\nu}\right]\Delta(p) = \mathcal{O}(1)$$

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$\Delta(p)$ does not explicitly depend on $k$, only implicitly trough the relation $k = \sum_j p_j$. But it is possible to find a function $\tilde{\Delta}(p,k)$ that fulfils,

$$\tilde{\Delta}\left(p,k\right)\Big|_{k=\sum_j p_j} = \Delta(p), \qquad \tilde{\Delta} = k^\mu \frac{\partial \Delta}{\partial p_j^\mu} + \mathcal{O}(k)$$

$$\sum_{i,j} \frac{\eta_i Q_i \eta_j Q_j \cancel{(p_i\cdot p_j)}}{(p_i\cdot k)\cancel{(p_j\cdot k)}}\left[\cancel{k^\mu \frac{\partial \Delta}{\partial p_j^\mu}} + \frac{\cancel{(p_j\cdot k)}p_{i\mu}}{\cancel{p_i\cdot p_j}}\left(g^{\mu\nu} - \cancel{\frac{p_j^\mu k^\nu}{p_j\cdot k}}\right)\frac{\partial \Delta}{\partial p_j^\nu}\right] = 0$$

# Example: $\pi^-\pi^0 \to \pi^-\pi^0 + \gamma$

Original formulation of Low's Theorem: [F.E. Low - *Phys.Rev.* (1958)]

$$\frac{1}{e}\mathcal{A}^\mu = \mathcal{H}(s_L,t)\left[\frac{p_a^\mu}{p_a\cdot k} - \frac{p_1^\mu}{p_1\cdot k}\right] + \frac{\partial\mathcal{H}(s_L,t)}{\partial s_L}\left[p_b^\mu - \frac{p_b\cdot k}{p_a\cdot k}p_a^\mu + p_2^\mu - \frac{p_2\cdot k}{p_1\cdot k}p_1^\mu\right]$$

Another version of Low's Theorem found in recent literature: [P. Lebiedowicz, O. Nachtmann, A. Szczurek - *Phys.Rev.D* (2021)]

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$$s_L = p_a\cdot p_b + p_1\cdot p_2, \qquad t = (p_b - p_2)^2, \qquad l_1 = \mathcal{O}(k)$$

The two formulations seem incompatible with each other.

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$$s_L = p_a\cdot p_b + p_1\cdot p_2, \qquad t = (p_b - p_2)^2, \qquad l_1 = p_1 - p_1 = \mathcal{O}(k)$$

The two formulations seem incompatible with each other.

**But they use different definitions of $p_1$ and $p_2$;**

$$p_a + p_b = p_1 + p_2 + k, \qquad p_a + p_b = p_1 + p_2$$

# Shifted kinematics

Evaluate $\mathcal{H}$ using physical momenta so that $\mathcal{H}$ is uniquely defined.

[T.H. Burnett, N.M. Kroll - *Phys.Rev.Lett.* (1967)] [V. Del Duca, E. Laenen, L. Magnea, L. Vernazza, C.D. White - *JHEP* (2017)] [D. Bonocore, A. Kulesza - *Phys.Rev.B* (2021)]

The expression for LBK theorem looks like a first order expansion:

$$\overline{|\mathcal{A}|}^2_{\mathrm{LP+NLP}} = -\sum_{i,j} \frac{(\eta_i Q_i p_i)\cdot(\eta_j Q_j p_j)}{(p_i \cdot k)(p_j \cdot k)} \left[1 + \frac{(p_j \cdot k) p_{i\mu}}{p_i \cdot p_j} G_j^{\mu\nu} \frac{\partial}{\partial p_j^\nu}\right] \overline{|\mathcal{H}|}^2$$

$$\overline{|\mathcal{A}|}^2_{\mathrm{LP+NLP}} = -\left(\sum_{i,j} \frac{(\eta_i Q_i p_i)\cdot(\eta_j Q_j p_j)}{(p_i \cdot k)(p_j \cdot k)}\right) \overline{|\mathcal{H}(p+\delta p)|}^2$$

$$= -C\overline{|\mathcal{H}(p+\delta p)|}^2$$

$$\delta p_j^\nu = \eta_j Q_j C^{-1} \sum_i \left(\frac{\eta_i Q_i p_{i\mu}}{p_i \cdot k}\right) G_j^{\mu\nu}$$

$p_j + \delta p_j$ fulfil the conservation of momentum for $\mathcal{H}$;

$$\sum_j \delta p_j = -k \Longrightarrow \sum_j (p_j + \delta p_j) = 0$$

# Shifted kinematics: off-shell momenta

$$\overline{|\mathcal{A}|}^2_{\mathrm{LP+NLP}} = -C\overline{|\mathcal{H}(p+\delta p)|}^2$$

$$\delta p_j^{\nu} = \eta_j Q_j C^{-1} \sum_i \left( \frac{\eta_i Q_i p_{i\mu}}{p_i \cdot k} \right) G_j^{\mu\nu} = \mathcal{O}(k)$$

The shifts modify the mass of the particles by $\mathrm{NNLP}$ terms.

$$p_j \cdot \delta p_j = 0 \Longrightarrow (p_j + \delta p_j)^2 = m_j^2 + \mathcal{O}\big(k^2\big)$$

This is consistent with the approximation, but not ideal for numerical implementations.

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We found an alternative way to do the shifts that:

- is consistent with LBK theorem at $\mathrm{NLP}$,
- satisfies four-momentum conservation,
- keeps the particles on-shell to all orders in the expansion of $k$.

# On-shell shifted kinematics

Consider a general shift

$$\delta p_i^\mu = \sum_j A_{ij} p_j^\mu + B_i k^\mu.$$

LBK theorem fixes the coefficients $A_{ij}$ and $B_i$ to

$$A_{ij} = \eta_i Q_i C^{-1} \frac{\eta_j Q_j}{k \cdot p_j} + \mathcal{O}\big(k^2\big), \qquad B_i = -\sum_j A_{ij} \frac{p_i \cdot p_j}{p_i \cdot k} + \mathcal{O}(k)$$

A suitable ansatz is $A_{ij} = A \eta_i Q_i \frac{\eta_j Q_j}{k \cdot p_j}$ with $A = C^{-1} + \mathcal{O}\big(k^3\big)$.

To keep $p^2$ invariant the following condition is imposed

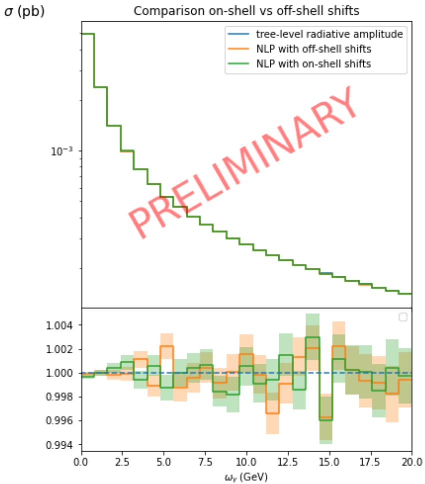
$$B_i = -A \sum_j \frac{(\eta_i Q_i p_i) \cdot (\eta_j Q_j p_j)}{(p_i \cdot k)(p_j \cdot k)} - \frac{A^2 Q_i^2 C}{2 p_i \cdot k}$$

Conservation of 4-momentum can thus be written as

$$AC + \frac{A^2 C}{2} \sum_i \frac{Q_i^2}{p_i \cdot k} = 1$$

which determines the constant $A$, giving the desired shifts.

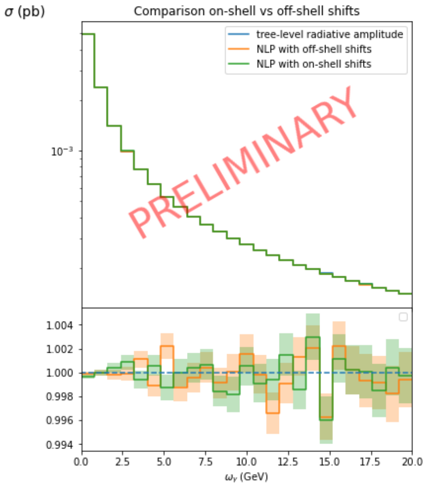
# Results




Radiative amplitude calculated with MadGraph5, non-radiative amplitude obtained analytically.

$e^- e^+ \to \mu^- \mu^+ \gamma$, $\sqrt{S} = 1\text{TeV}$

$p_{T,\gamma} > 0.1\text{GeV}$, $p_{T,\mu} > 10\text{GeV}$, $|\eta| < 2.5$, $\Delta R > 0.4$

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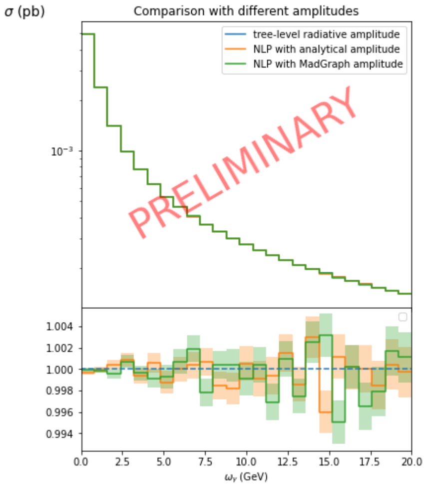
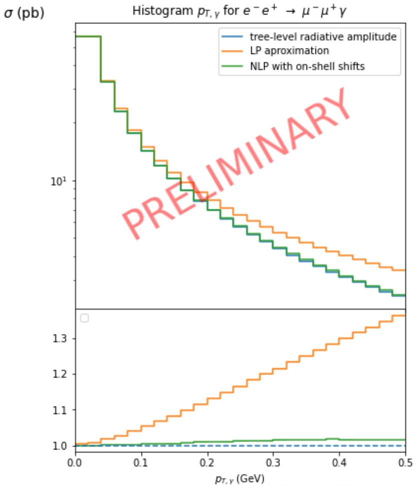



Radiative amplitude calculated with MadGraph5_aMC@NLO.
NLP amplitudes are obtained using the new on-shell shifts.

$e^- e^+ \to \mu^- \mu^+ \gamma, \quad \sqrt{S} = 1\text{TeV}$

$p_{T,\gamma} > 0.1\text{GeV}, \quad p_{T,\mu} > 10\text{GeV}, \quad |\eta| < 2.5, \quad \Delta R > 0.4$

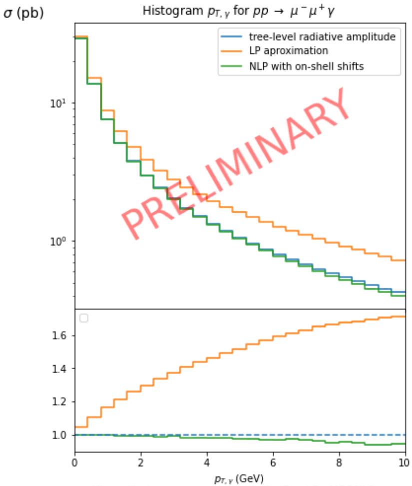
# Results




All amplitudes are generated with MadGraph5_aMC@NLO.

$e^-e^+ \to \mu^-\mu^+\gamma, \quad \sqrt{S} = 91.1\text{GeV}$

$10MeV < p_{T,\gamma} < 5\text{GeV}, \quad p_{T,\mu} > 10\text{GeV}, \quad |\eta| < 2.5, \quad \Delta R > 0.4$

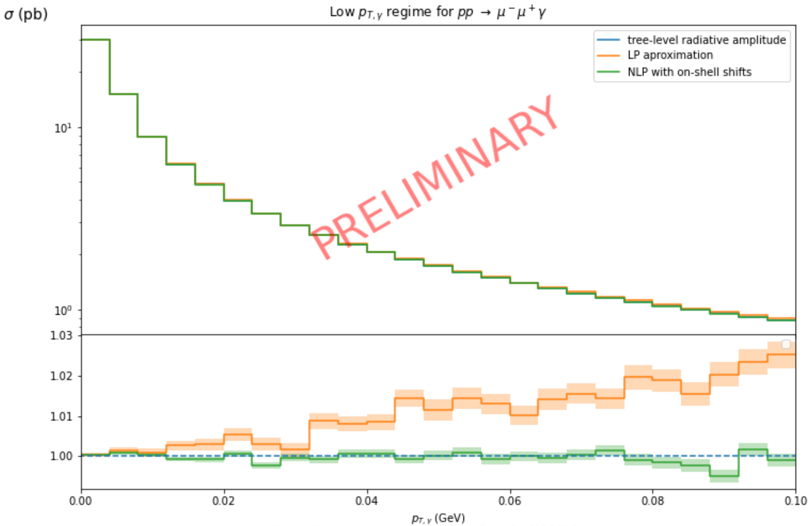



All amplitudes are generated with MadGraph5_aMC@NLO.

$pp \to \mu^-\mu^+\gamma, \quad \sqrt{S} = 13\text{TeV}$

$100MeV < p_{T,\gamma} < 50\text{GeV}, \quad p_{T,\mu} > 10\text{GeV}, \quad |\eta| < 2.5, \quad \Delta R > 0.4$

# Results




All amplitudes are generated with MadGraph5_aMC@NLO.

$pp \to \mu^-\mu^+\gamma$, $\sqrt{S} = 13\text{TeV}$

$1MeV < p_{T,\gamma} < 1\text{GeV}$, $p_{T,\mu} > 10\text{GeV}$, $|\eta| < 2.5$, $\Delta R > 0.4$

# Conclusions

- Precision predictions call for understanding the $\mathrm{NLP}$ terms.
- LBK theorem is free of inconsistencies and can be used safely for calculating soft photon spectra.
- Reformulation of LBK theorem using on-shell shifted kinematics opens the door to an efficient implementation for the $\mathrm{NLP}$ approximation for the emission of (ultra-)soft photons (e.g. as measured in the future by ALICE3 detector).
- More work has to be done in order to understand the origin of the soft photon anomaly observed at LEP.