Basics of Light-Cone Quantization and Light-Cone Perturbation Theory,* Part 1: Generalities and scalar field case

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1 Introduction

In a relativistic quantum field theory, the dynamics of the system is usually described as an evolution along a timelike direction, either explicitly in the Hamiltonian formalism, or implicitly in the path integral approach. In these cases, the quantization of the theory is said to be performed on the instant form of relativistic dynamics. However, this is not the only possibility [1]. In particular, one can use instead the front form of relativistic dynamics, in which the dynamics of the system is described as an evolution along a lightlike direction. All of these approaches should be equivalent, even though detailed proofs of equivalence are often difficult to obtain. All of these approaches are also complementary, since they have different advantages and shortcomings.

For example, the quantization in the front form, (a.k.a. light-front quantization or light-cone quantization)[2, 3], typically leads to a simpler structure of the vacuum of the theory, and gives a radically different account of spontaneous symmetry breaking symmetry (see for example Ref. [4] for the scalar theory case, and Ref. [5] for chiral symmetry breaking in QCD). Moreover, it leads in a very intuitive way to the parton model picture of QCD. Light-cone perturbation theory for QCD has also been one of the main methods for practical calculations in the context of high-energy (or low x) QCD, from early studies of the dipole model [6, 7, 8] and of the Color Glass Condensate effective theory [9, 10, 11], to recent calculations of NLO corrections (see for example [12, 13, 14, 15, 16, 17, 18]) in that context.

In these lectures, we will present the light-front quantization and light-front perturbation theory first for scalar theories, and then for QCD and QED. Applications of this formalism to meson wavefunctions, high-energy scattering in QCD, and jet quenching in heavy ions collisions will be covered by the other lecturers.

Other useful lecture notes about the light-front quantization are Refs. [19], [20] and [4]. The original papers [2, 3] are also very interesting to read, but sometimes difficult to follow due to outdated notations.

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2 Poincaré invariance and covariance

2.1 Cartesian and Light-cone coordinates

In the usual cartesian coordinates (x^0, x^1, x^2, x^3), the metric of Minkowski space writes

$$g_{\mu\nu} = g^{\mu\nu} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix} . \tag{1}$$

When performing a general coordinates change

$$x^\mu \mapsto \tilde{x}^\mu(x) , \tag{2}$$

the metric transforms as

$$\tilde{g}_{\mu\nu} = \left(\frac{\partial x^\rho}{\partial \tilde{x}^\mu}\right) \left(\frac{\partial x^\sigma}{\partial \tilde{x}^\nu}\right) g_{\rho\sigma} \tag{3}$$

and, equivalently

$$\tilde{g}^{\mu\nu} = \left(\frac{\partial \tilde{x}^\mu}{\partial x^\rho}\right) \left(\frac{\partial \tilde{x}^\nu}{\partial x^\sigma}\right) g^{\rho\sigma} , \tag{4}$$

and such rules can be generalized to find the transformation of arbitrary tensor quantities.

In the following, we will mostly use the light-cone coordinates (x^+, x^1, x^2, x^-), defined as[1]

$$x^\pm = \frac{x^0 \pm x^3}{\sqrt{2}} \tag{5}$$

Then, the metric in light-cone coordinates (x^+, x^1, x^2, x^-) writes

$$g_{\mu\nu} = g^{\mu\nu} = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix} , \tag{6}$$

so that in particular $x^\pm = x_\mp$.

In the following, we will always use $i, j, \cdots = 1, 2$ for the transverse plane indices , $I, J, \cdots = 1, 2, 3$ for the space indices, and of course $\mu, \nu, \ldots$ for the Minkowski spacetime indices. The transverse vectors will be written in bold, as $\mathbf{x} \equiv (x^1, x^2)$.

[1] Note that two definitions are widespread in the literature, with or without the $\sqrt{2}$ in the denominator. We choose that definition so that the change of variables between cartesian and light-cone coordinates has a Jacobian equal to 1.

2.2 Poincaré transformations in classical field theory

2.2.1 Coordinate transformations

The Lorentz group corresponds to the space-time transformations of the type

$$x^\mu \mapsto \tilde{x}^\mu(x) = \Lambda^\mu{}_\nu\, x^\nu \tag{7}$$

which preserve the metric, meaning

$$\tilde{g}^{\mu\nu} = \Lambda^\mu{}_\rho\, \Lambda^\nu{}_\sigma\, g^{\rho\sigma} = g^{\mu\nu}\ , \tag{8}$$

or equivalently

$$(\Lambda^{-1})^\mu{}_\nu = g_{\nu\rho}\, \Lambda^\rho{}_\sigma\, g^{\sigma\mu}\ . \tag{9}$$

Such transformations include space rotations, for example the rotation of angle θ within the (x^1, x^2) plane (or around the x^3 axis), which writes

$$\Lambda^\mu{}_\nu = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & \cos(\theta) & -\sin(\theta) & 0 \\ 0 & \sin(\theta) & \cos(\theta) & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \tag{10}$$

both in cartesian coordinates and in light-cone coordinates.

There are also the Lorentz boosts, which mix the x^0 time direction with one of the space directions. For example the Lorentz boost of rapidity ω along the x^3 writes

$$\Lambda^\mu{}_\nu = \begin{pmatrix} \cosh(\omega) & 0 & 0 & \sinh(\omega) \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ \sinh(\omega) & 0 & 0 & \cosh(\omega) \end{pmatrix} \tag{11}$$

in cartesian coordinates and

$$\Lambda^\mu{}_\nu = \begin{pmatrix} e^\omega & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & e^{-\omega} \end{pmatrix} \tag{12}$$

in light-cone coordinates.

For a generic infinitesimal transformation $\Lambda^\mu{}_\nu \simeq g^\mu{}_\nu + \lambda^\mu{}_\nu$, so that $(\Lambda^{-1})^\mu{}_\nu \simeq g^\mu{}_\nu - \lambda^\mu{}_\nu$, the relation (9) gives

$$\lambda_{\mu\nu} = -\lambda_{\nu\mu}\ . \tag{13}$$

The space of anti-symmetric 4×4 matrices is of dimension 6, so that the Lorentz algebra for a $4-$dimensional space-time has 6 generators. There are 3 generators for space rotations and 3 for Lorentz boosts. Hence, the Lorentz group is the group formed by rotations and boosts only[2].

The Poincaré group is defined as the extension of the Lorentz group which includes space-time translations

$$x^\mu \mapsto \tilde{x}^\mu(x) = x^\mu - a^\mu \tag{14}$$

as well.

[2]In principle, two discrete symmetries, the time reversal and the space parity are also part of the Lorentz group. But we will not discuss them further in these lectures, and focus instead on the Lorentz algebra and on the component of the Lorentz group connected to the identity, that we will call for simplicity the Lorentz group.

2.2.2 Active transformations

So far, so the Poincaré transformations were considered as mere reparametrizations of coordinates. This corresponds to passive transformations. Instead, one can consider transformations acting on the physical system, but keeping the same coordinate system. If the system is described by a scalar field $\varphi(x)$ an active Lorentz transformation acts as follows:

$$\varphi(x) \mapsto \tilde{\varphi}(x) = \varphi\left(\Lambda^{-1}x\right) . \tag{15}$$

Indeed, the system is only displaced in spacetime by the Lorentz transformation, so that the value of the scalar field at x after the transformation is the same as the value of the field before the transformation at the point which gets transformed into x.

The case of a vector field $V^\mu(x)$ is similar, except that in addition the direction of the field is changed by the Lorentz transformation, so that

$$V^\mu(x) \mapsto \tilde{V}^\mu(x) = \Lambda^\mu{}_\nu\, V^\nu\left(\Lambda^{-1}x\right) . \tag{16}$$

For higher rank tensors, the Lorentz transformation acts on each index separately, for example for a rank 2 tensor one has

$$T^{\mu\nu}(x) \mapsto \tilde{T}^{\mu\nu}(x) = \Lambda^\mu{}_\rho\, \Lambda^\nu{}_\sigma\, T^{\rho\sigma}\left(\Lambda^{-1}x\right) . \tag{17}$$

The action of the active version of the translation $x^\mu \mapsto x^\mu - a^\mu$ on a generic local quantity $\mathcal{O}(x)$ writes

$$\mathcal{O}(x) \mapsto \tilde{\mathcal{O}}(x) = \mathcal{O}\left(x + a\right) , \tag{18}$$

for the same reason as in Eq. (15).

Hence, an active infinitesimal translation provokes a change

$$\delta\mathcal{O}(x) \equiv \tilde{\mathcal{O}}(x) - \mathcal{O}(x) = \mathcal{O}\left(x+a\right) - \mathcal{O}(x) \simeq a^\mu\, \partial_\mu \mathcal{O}(x)\, , \tag{19}$$

to linear accuracy in a^μ .

Similarly, an active infinitesimal Lorentz transformation $\Lambda^\mu{}_\nu \simeq g^\mu{}_\nu + \lambda^\mu{}_\nu$ gives

$$\begin{aligned} \delta\varphi(x) &\equiv \tilde{\varphi}(x) - \varphi(x) = \varphi\left(\Lambda^{-1}x\right) - \varphi(x) \\ &\simeq \left(\left(\Lambda^{-1}\right)^\mu{}_\nu\, x^\nu - x^\mu\right)\, \partial_\mu\varphi\left(x\right) \simeq -\lambda^\mu{}_\nu\, x^\nu\, \partial_\mu\varphi\left(x\right) \\ &\simeq \frac{\lambda_{\mu\nu}}{2}\, \left(x^\mu\, \partial^\nu - x^\nu\, \partial^\mu\right)\varphi\left(x\right) \end{aligned} \tag{20}$$

in the scalar field case, and

$$\begin{aligned} \delta V^\rho(x) &\equiv \tilde{V}^\rho(x) - V^\rho(x) = \Lambda^\rho{}_\sigma\, V^\sigma\left(\Lambda^{-1}x\right) - V^\rho(x) \\ &\simeq -\lambda^\mu{}_\nu\, x^\nu\, \partial_\mu V^\rho\left(x\right) + \lambda^\rho{}_\sigma\, V^\sigma(x) \\ &\simeq \frac{\lambda_{\mu\nu}}{2}\left[\left(x^\mu\, \partial^\nu - x^\nu\, \partial^\mu\right)\, V^\rho\left(x\right) + \left(\Sigma^{\mu\nu}_{\text{vect.}}\right)^\rho{}_\sigma\, V^\sigma(x)\right] \end{aligned} \tag{21}$$

in the vector field case, with the definition

$$\left(\Sigma^{\mu\nu}_{\text{vect.}}\right)^\rho{}_\sigma \equiv g^{\rho\mu}\, g^\nu{}_\sigma - g^{\rho\nu}\, g^\mu{}_\sigma\, . \tag{22}$$

For higher rank tensors, there is a $\Sigma^{\mu\nu}_{\text{vect.}}$ term for each index, for example

$$\delta T^{\rho\sigma}(x) \simeq \frac{\lambda_{\mu\nu}}{2} \left[(x^\mu\, \partial^\nu - x^\nu\, \partial^\mu)\; T^{\rho\sigma}(x) + (\Sigma^{\mu\nu}_{\text{vect.}})^\rho{}_\eta\, T^{\eta\sigma}(x) + (\Sigma^{\mu\nu}_{\text{vect.}})^\sigma{}_\eta\, T^{\rho\eta}(x) \right] \tag{23}$$

for rank 2 tensors.

Note that generic finite Lorentz transformations can then be parametrized as

$$\Lambda^\rho{}_\sigma \equiv \left(\exp\left(\frac{\lambda_{\mu\nu}}{2}\, \Sigma^{\mu\nu}_{\text{vect.}} \right) \right)^\rho{}_\sigma \,, \tag{24}$$

where $\lambda_{\mu\nu}$ is antisymmetric but otherwise arbitrary.

For a Dirac spinor field $\Psi(x)$, active Lorentz transformations should have an action of the form

$$\Psi(x) \mapsto \tilde{\Psi}(x) = \mathbf{M}(\Lambda)\, \Psi\left(\Lambda^{-1} x\right) \,, \tag{25}$$

with a matrix $\mathbf{M}(\Lambda)$ to be determined. Consider for simplicity a constant spinor Ψ. Then, a Lorentz transformation gives

$$\Psi \mapsto \mathbf{M}(\Lambda)\, \Psi \tag{26}$$

$$\overline{\Psi} = \Psi^\dagger\, \gamma^0 \mapsto \Psi^\dagger\, \mathbf{M}(\Lambda)^\dagger\, \gamma^0 = \overline{\Psi}\gamma^0 \mathbf{M}(\Lambda)^\dagger\, \gamma^0 \tag{27}$$

and thus

$$\overline{\Psi}\gamma^\mu\Psi \mapsto \overline{\Psi}\gamma^0 \mathbf{M}(\Lambda)^\dagger\, \gamma^0\gamma^\mu \mathbf{M}(\Lambda)\, \Psi \,. \tag{28}$$

On the other hand, $\overline{\Psi}\gamma^\rho\Psi$ should behave as a vector under Lorentz boosts, meaning

$$\overline{\Psi}\gamma^\rho\Psi \mapsto \Lambda^\rho{}_\sigma\, \overline{\Psi}\gamma^\sigma\Psi \,. \tag{29}$$

The expressions (28) and (29) should be valid for any Dirac spinor Ψ. Hence, one gets the constraint

$$\gamma^0 \mathbf{M}(\Lambda)^\dagger\, \gamma^0\gamma^\rho \mathbf{M}(\Lambda) = \Lambda^\rho{}_\sigma\, \gamma^\sigma \,. \tag{30}$$

Looking for a solution of the form

$$\mathbf{M}(\Lambda) \equiv \left(\exp\left(\frac{\lambda_{\mu\nu}}{2}\, \Sigma^{\mu\nu}_{\text{sp.}} \right) \right) \,, \tag{31}$$

one finds, at the linear level in $\lambda_{\mu\nu}$

$$\frac{\lambda_{\mu\nu}}{2} \left(\gamma^\rho\, \Sigma^{\mu\nu}_{\text{sp.}} + \gamma^0 \left(\Sigma^{\mu\nu}_{\text{sp.}} \right)^\dagger \gamma^0\gamma^\rho \right) = \lambda^\rho{}_\sigma\, \gamma^\sigma \,. \tag{32}$$

This has to be true for any small Lorentz transformation, and thus for any antisymmetric $\lambda_{\mu\nu}$, so that

$$\gamma^\rho\, \Sigma^{\mu\nu}_{\text{sp.}} + \gamma^0 \left(\Sigma^{\mu\nu}_{\text{sp.}} \right)^\dagger \gamma^0\gamma^\rho = (\Sigma^{\mu\nu}_{\text{vect.}})^\rho{}_\sigma\, \gamma^\sigma \,. \tag{33}$$

Remembering that $\gamma^{\mu\dagger} = \gamma^0\gamma^\mu\gamma^0$, one finds that the constraint (33) is solved by

$$\Sigma^{\mu\nu}_{\text{sp.}} = \frac{1}{4}\,[\gamma^\mu, \gamma^\nu]\,. \tag{34}$$

With this expression, it is possible to check that

$$\gamma^0 \mathbf{M}(\Lambda)^\dagger\,\gamma^0 = \mathbf{M}(\Lambda^{-1}) = \mathbf{M}(\Lambda)^{-1}\,, \tag{35}$$

and then that the constraint (30) is satisfied for all finite Lorentz transformations. Hence, a Dirac spinor field $\Psi(x)$ transform as (25) under a finite Lorentz transformation, and as

$$\begin{aligned}\delta\Psi(x) &\equiv \tilde{\Psi}(x) - \Psi(x) = \mathbf{M}(\Lambda)\,\Psi\left(\Lambda^{-1}x\right) - \Psi(x)\\ &\simeq \frac{\lambda_{\mu\nu}}{2}\left[\left(x^\mu\,\partial^\nu - x^\nu\,\partial^\mu\right)\,\Psi(x) + \left(\Sigma^{\mu\nu}_{\text{sp.}}\right)\,\Psi(x)\right]\end{aligned} \tag{36}$$

under an infinitesimal Lorentz transformation.

As a summary, for a field $\phi_{\mathcal{R}}(x)$ in a representation $\mathcal{R}$ of the Lorentz group, the action of a finite Lorentz transformation can be written as

$$\phi_{\mathcal{R}}(x) \mapsto \tilde{\phi}_{\mathcal{R}}(x) = \mathbf{M}_{\mathcal{R}}(\Lambda)\,\phi_{\mathcal{R}}\left(\Lambda^{-1}x\right)\,, \tag{37}$$

with

$$\mathbf{M}_{\mathcal{R}}(\Lambda) \equiv \exp\left(\frac{\lambda_{\mu\nu}}{2}\,\Sigma^{\mu\nu}_{\mathcal{R}}\right)\,, \tag{38}$$

and the action of an infinitesimal Lorentz transformation writes

$$\begin{aligned}\delta\phi_{\mathcal{R}}(x) &\equiv \tilde{\phi}_{\mathcal{R}}(x) - \phi_{\mathcal{R}}(x)\\ &\simeq \frac{\lambda_{\mu\nu}}{2}\left[\left(x^\mu\,\partial^\nu - x^\nu\,\partial^\mu\right)\,\phi_{\mathcal{R}}(x) + \left(\Sigma^{\mu\nu}_{\mathcal{R}}\right)\,\phi_{\mathcal{R}}(x)\right].\end{aligned} \tag{39}$$

The matrix $\Sigma^{\mu\nu}_{\mathcal{R}}$ is different depending on the representation, and is the only data needed to determine the action of Lorentz transformations. Moreover, the action of translations is given by Eqs. (18) and (19).

2.2.3 Noether's theorem and currents

According to Noether's theorem, for each continuous symmetry of a field theory, one can construct a current which is conserved. The current associated with translational invariance is the energy-momentum tensor

$$T^{\mu\nu}(x) = \frac{\partial\mathcal{L}}{\partial(\partial_\mu\phi(x))}\,\partial^\nu\phi(x) - g^{\mu\nu}\,\mathcal{L}\,, \tag{40}$$

and the one associated with Lorentz invariance is

$$J^{\rho\mu\nu}(x) = x^\mu\,T^{\rho\nu}(x) - x^\nu\,T^{\rho\mu}(x) + \frac{\partial\mathcal{L}}{\partial(\partial_\rho\phi_{\mathcal{R}}(x))}\,\left(\Sigma^{\mu\nu}_{\mathcal{R}}\right)\,\phi_{\mathcal{R}}(x)\,. \tag{41}$$

Hence, a Poincaré invariant theory has the conservation laws

$$\partial_\mu\,T^{\mu\nu}(x) = 0 \tag{42}$$

and

$$\partial_\rho\,J^{\rho\mu\nu}(x) = 0\,. \tag{43}$$

2.2.4 Conserved charges

For each current $J^\mu(x)$, one can define a charge either according to the cartesian coordinates, as

$$Q(x^0)_{\mathrm{cart.}} \equiv \int d^3\vec{x}\; J^0(x)\,, \tag{44}$$

or according to the light-cone coordinates, as

$$Q(x^+)_{\mathrm{LC}} \equiv \int dx^- \!\int d^2\mathbf{x}\; J^+(x)\,. \tag{45}$$

In both cases, if the current is conserved, meaning $\partial_\mu\, J^\mu(x) = 0$, the charged is conserved as well, meaning $\partial_0\, Q(x^0)_{\mathrm{cart.}} = 0$ or $\partial_+\, Q(x^+)_{\mathrm{LC}} = 0$. Indeed, one has for example

$$\begin{aligned}\partial_+\, Q(x^+)_{\mathrm{LC}} &= \int dx^- \!\int d^2\mathbf{x}\; \partial_+\, J^+(x)\\ &= \int dx^- \!\int d^2\mathbf{x} \left\{ -\, \partial_-\, J^-(x) - \partial_j\, J^j(x) \right\} = 0\,,\end{aligned} \tag{46}$$

assuming that the fields and thus the current decay fast enough in the space directions (and in the x^- direction) so that the boundary terms vanish. Note that $Q_{\mathrm{cart.}}$ and Q_{LC} are in principle different objects, even though they are built from the same current.

Hence, thanks to the conservation of the Noether currents $T^{\mu\nu}(x)$ and $J^{\rho\mu\nu}(x)$, one can define the constant charges associated with the translational and Lorentz invariances either as

$$P^\nu_{\mathrm{cart.}} \equiv \int d^3\vec{x}\; T^{0\nu}(x) \tag{47}$$

$$M^{\mu\nu}_{\mathrm{cart.}} \equiv \int d^3\vec{x}\; J^{0\mu\nu}(x) \tag{48}$$

in cartesian coordinates or as

$$P^\nu_{\mathrm{LC}} \equiv \int dx^- \!\int d^2\mathbf{x}\; T^{+\nu}(x) \tag{49}$$

$$M^{\mu\nu}_{\mathrm{LC}} \equiv \int dx^- \!\int d^2\mathbf{x}\; J^{+\mu\nu}(x) \tag{50}$$

in light-cone coordinates. The charges (47) and (48) are the ones used in the instant form of relativistic dynamics, whereas the charges (49) and (50) are the ones used in the front form [1], that we will mainly focus on, in the following. The subscripts cart. and LC will usually be dropped to avoid cluttered notations, and it should be clear according to the context if we are talking about the cartesian or the light-cone version, or if the discussion is generic and applies to both.

2.3 Poincaré algebra in quantum field theory

2.3.1 Generic case

When quantizing a field theory, both the fields and the conserved charges should be promoted to operators. Moreover, the action of an infinitesimal transformation on the fields and other quantities should now be encoded via their commutation relations with the corresponding Noether charge. For example, for a generic local operator $\hat{\mathcal{O}}(x)$, the action of infinitesimal translations writes

$$\delta\hat{\mathcal{O}}(x) = a_\mu \, i\Big[\hat{P}^\mu, \hat{\mathcal{O}}(x)\Big] \tag{51}$$

and the action of infinitesimal Lorentz transformations writes

$$\delta\hat{\mathcal{O}}(x) = \frac{\lambda_{\mu\nu}}{2} \, i\Big[\hat{M}^{\mu\nu}, \hat{\mathcal{O}}(x)\Big]\,. \tag{52}$$

These relations can be extended to finite Poincaré transformations. For finite translations by a^μ, one has

$$\hat{\mathcal{O}}(x) \mapsto \hat{\mathcal{O}}(x+a) = e^{i\,a_\mu\,\hat{P}^\mu}\,\hat{\mathcal{O}}(x)\,e^{-i\,a_\mu\,\hat{P}^\mu} \tag{53}$$

For finite Lorentz transformations acting on an operator in a representation $\mathcal{R}$, one has

$$\hat{\mathcal{O}}_{\mathcal{R}}(x) \mapsto \mathbf{M}_{\mathcal{R}}(\Lambda)\,\hat{\mathcal{O}}_{\mathcal{R}}\left(\Lambda^{-1}x\right) = e^{i\,\frac{\lambda_{\mu\nu}}{2}\,\hat{M}^{\mu\nu}}\,\hat{\mathcal{O}}_{\mathcal{R}}(x)\,e^{-i\,\frac{\lambda_{\mu\nu}}{2}\,\hat{M}^{\mu\nu}}\,. \tag{54}$$

By comparing the expressions (19) and (51), one finds that the commutation relation of any local operator $\hat{\mathcal{O}}(x)$ with $\hat{P}^\mu$ should be

$$\Big[\hat{P}^\mu, \hat{\mathcal{O}}(x)\Big] = -i\partial^\mu\hat{\mathcal{O}}(x)\,. \tag{55}$$

Similarly, by comparison of the Eqs. (39) and (52), the commutation relation of an operator $\hat{\mathcal{O}}_{\mathcal{R}}(x)$ in the representation $\mathcal{R}$ of the Lorentz group with $\hat{M}^{\mu\nu}$ should be

$$\Big[\hat{M}^{\mu\nu}, \hat{\mathcal{O}}_{\mathcal{R}}(x)\Big] = -i\Big[\,(x^\mu\,\partial^\nu - x^\nu\,\partial^\mu) + (\Sigma^{\mu\nu}_{\mathcal{R}})\,\Big]\,\hat{\mathcal{O}}_{\mathcal{R}}(x)\,. \tag{56}$$

Thanks to the Jacobi identity

$$\Big[\hat{A}, \Big[\hat{B}, \hat{C}\Big]\Big] + \Big[\hat{B}, \Big[\hat{C}, \hat{A}\Big]\Big] + \Big[\hat{C}, \Big[\hat{A}, \hat{B}\Big]\Big] = 0\,, \tag{57}$$

it is possible to show that, if $\hat{P}^\mu$ and $\hat{M}^{\mu\nu}$ obey the commutation relations (55) and (56) with any local operator $\hat{\mathcal{O}}_{\mathcal{R}}(x)$, then they obey the Poincaré algebra

$$\Big[\hat{P}^\mu, \hat{P}^\nu\Big] = 0 \tag{58}$$

$$\Big[\hat{M}^{\mu\nu}, \hat{P}^\rho\Big] = i\Big(g^{\nu\rho}\,\hat{P}^\mu - g^{\mu\rho}\,\hat{P}^\nu\Big) \tag{59}$$

$$\Big[\hat{M}^{\mu\nu}, \hat{M}^{\rho\sigma}\Big] = i\Big(g^{\mu\sigma}\,\hat{M}^{\nu\rho} + g^{\nu\rho}\,\hat{M}^{\mu\sigma} - g^{\mu\rho}\,\hat{M}^{\nu\sigma} - g^{\nu\sigma}\,\hat{M}^{\mu\rho}\Big)\,, \tag{60}$$

provided that $i\,\Sigma^{\mu\nu}_{\mathcal{R}}$ satisfies (as $\mathcal{R}-$matrices) the same commutation relations (60) as $\hat{M}^{\mu\nu}$ (as quantum operators).

2.3.2 Poincaré algebra in the instant-form dynamics

The most common ways to quantize a field theory are based on the *instant form* of relativistic dynamics [1]. In that case, the hamiltonian dynamics of the system is formulated as an evolution along the timelike x^0 axis, in some particular inertial frame. Then, for each x^0 = constant hypersurface one can associate a state of the system, characterized by variables on that hypersurface only.

Then, the Poincaré transformations leaving x^0 = constant hypersurfaces invariant or not play a different role. The space translations as well as the space rotations leave x^0 = constant hyperplanes invariant, and thus have nothing to do with the evolution of the system in the instant-form dynamics. These transformations, and the corresponding generators $\hat{P}^I$ and $\hat{M}^{IJ}$ are called kinematic. Obviously, they form form a subalgebra of the Poincaré algebra.

By contrast, x^0 translations transforms any x^0 = constant hyperplane into a different one. Lorentz boosts mix the x^0 direction with one of the space directions, so that they are not leaving x^0 = constant hyperplanes invariants. The generators $\hat{P}^0$ of x^0 translations and $\hat{M}^{0J}$ of Lorentz boosts are called dynamical, since they describe how the system change from a x^0 = constant hyperplane to another hypersurface. Hence, $\hat{P}^0$ and $\hat{M}^{0J}$ can be considered as generalized Hamiltonians.

As a summary, in the instant-form based on x^0 = constant hyperplanes, there are 6 kinematic Poincaré generators $\hat{P}^I$ and $\hat{M}^{IJ}$, which are by definition blind to the dynamics of the system, and 4 dynamic Poincaré generators $\hat{P}^0$ and $\hat{M}^{0J}$, which encode the dynamics of the system, as x^0 changes.

2.3.3 Poincaré algebra in the front-form dynamics

Alternatively, one can use the *front form* of relativistic dynamics [1, 21], with x^+ as evolution variable. In that case, the states of the system are associated to x^+ = constant hyperplanes.

The translations along x^- and along the transverse directions $\mathbf{x}^j$ leave these planes invariant, but obviously not the translations along x^+.

Writing $x^+ = n^\mu x_\mu$, with $n^\mu \equiv g^{\mu +}$, Lorentz transformations leave x^+ = constant hyperplanes invariant if they leave the vector n^μ invariant. Of course, rotations within the transverse plane leave x^+ = constant hyperplanes invariant, so that $\hat{M}^{ij} = \hat{M}_{ij} = \epsilon^{ij}\, \hat{M}^{12}$ is a kinematic generator. Moreover, one finds that $\hat{M}^{+i} = \hat{M}_{i-}$ also are kinematic generators. They are associated to some combinations of transverse Lorentz boosts with rotations.

So far, there are 6 kinematic generators in the front form: $\hat{P}^+$, $\hat{P}^j$, $\hat{M}^{ij}$ and $\hat{M}^{+i}$.

By contrast, the 3 generators $\hat{P}^-$ and $\hat{M}^{-i} = \hat{M}_{i+}$ are dynamical.

The last generator, $\hat{M}^{-+} = \hat{M}_{+-}$, which corresponds to longitudinal Lorentz boosts, has a peculiar status in the front-form dynamics. Indeed, under a longitudinal boost,

$$\begin{aligned} x^+ &\mapsto e^{\omega}\, x^+ \\ \mathbf{x} &\mapsto \mathbf{x} \\ x^- &\mapsto e^{-\omega}\, x^- \,, \end{aligned} \tag{61}$$

so that an hyperplane $x^+ =$ constant is transformed into an hyperplane $x^+ = e^{-\omega}$ constant. In general, this is a different hyperplane, meaning that $\hat{M}^{-+}$ is dynamic. However, the hyperplane $x^+ = 0$ is invariant under such longitudinal boost. Hence, in the case of the plane $x^+ = 0$, $\hat{M}^{-+}$ becomes kinematic, and one obtains an enhanced kinematical Poincaré subalgebra with 7 generators.

The non-trivial commutation relations between the kinematic generators (for generic x^+) $\hat{P}^+$, $\hat{P}^j$, $\hat{M}^{ij}$ and $\hat{M}^{+i}$ are

$$\left[\hat{M}^{ij}, \hat{P}^l\right] = i\Big(g^{jl}\,\hat{P}^i - g^{il}\,\hat{P}^j\Big) \tag{62}$$

$$\left[\hat{M}^{ij}, \hat{M}^{+l}\right] = i\Big(g^{jl}\,\hat{M}^{+i} - g^{il}\,\hat{M}^{+j}\Big) \tag{63}$$

$$\left[\hat{M}^{+i}, \hat{P}^j\right] = i\,g^{ij}\,\hat{P}^+\,. \tag{64}$$

Interestingly, one obtains another closed subalgebra by adding the dynamic generator $\hat{P}^-$, with the new non-trivial commutation relation

$$\left[\hat{M}^{+i}, \hat{P}^-\right] = -i\,\hat{P}^i\,. \tag{65}$$

That algebra formed by $\hat{P}^\mu$, $\hat{M}^{+i}$ and $\hat{M}^{ij}$ is isomorphic to the Galilean algebra in $2+1$ dimensions, with the following correspondences:

- $\hat{P}^j \mapsto$ 2D momentum
- $\hat{P}^- \mapsto$ energy
- $\hat{P}^+ \mapsto$ mass
- $\hat{M}^{+i} \mapsto$ 2D Galilean boosts
- $\hat{M}^{12} \mapsto$ angular momentum.

Finally, one finds that the commutation relations of the generator $\hat{M}^{-+}$ of longitudinal boosts with the others are such that longitudinal boosts simply rescale the generators according to the number of $+$ or $-$ indices they have, as

$$e^{i\,\omega\,\hat{M}^{-+}}\,\hat{P}^+\,e^{-i\,\omega\,\hat{M}^{-+}} = e^{\omega}\,\hat{P}^+ \tag{66}$$

$$e^{i\,\omega\,\hat{M}^{-+}}\,M^{+i}\,e^{-i\,\omega\,\hat{M}^{-+}} = e^{\omega}\,M^{+i} \tag{67}$$

$$e^{i\,\omega\,\hat{M}^{-+}}\,\hat{P}^j\,e^{-i\,\omega\,\hat{M}^{-+}} = \hat{P}^j \tag{68}$$

$$e^{i\,\omega\,\hat{M}^{-+}}\,\hat{M}^{ij}\,e^{-i\,\omega\,\hat{M}^{-+}} = \hat{M}^{ij} \tag{69}$$

$$e^{i\,\omega\,\hat{M}^{-+}}\,\hat{P}^-\,e^{-i\,\omega\,\hat{M}^{-+}} = e^{-\omega}\,\hat{P}^- \tag{70}$$

$$e^{i\,\omega\,\hat{M}^{-+}}\,\hat{M}^{-i}\,e^{-i\,\omega\,\hat{M}^{-+}} = e^{-\omega}\,\hat{M}^{-i}\,. \tag{71}$$

3 Light-front quantization of a scalar field theory

Let us discuss the light-front quantization, using x^+ as evolution variable, for a generic real scalar field theory.

3.1 Classical scalar field theory in the front form

Let us consider the Lagrangian density for a scalar theory

$$\begin{aligned}\mathcal{L} &= \frac{1}{2}\left(\partial_\mu\varphi(x)\right)\left(\partial^\mu\varphi(x)\right) - V(\varphi(x)) \\ &= \left(\partial_+\varphi(x)\right)\left(\partial_-\varphi(x)\right) - \frac{1}{2}\left(\partial_j\varphi(x)\right)^2 - V(\varphi(x))\,,\end{aligned} \tag{72}$$

with a generic potential $V(\varphi(x))$. The momentum density conjugate to the scalar field $\varphi(x)$ is given by

$$\pi^+(x) = \frac{\partial\mathcal{L}}{\partial(\partial_+\varphi(x))} = \partial_-\varphi(x)\,, \tag{73}$$

and the Euler-Lagrange equation of motion for this model is

$$\partial_\mu\partial^\mu\varphi(x) = -\,V'(\varphi(x))\,, \tag{74}$$

or equivalently

$$\partial_-\partial_+\varphi(x) = \frac{1}{2}\Big(\Delta_\perp\varphi(x) - V'(\varphi(x))\Big)\,. \tag{75}$$

At this stage, there are already two crucial differences between this front form of the dynamics of the scalar theory and its more usual instant form:

- The evolution equation (75) is of first order in x^+ (parabolic) in the front form, whereas in the instant form, one obtains a second order equation in x^0 (hyperbolic).
- In the front form, the conjugate momentum $\pi^+(x) = \partial_-\varphi(x)$ can be obtained from the knowledge of $\varphi(x)$ restricted to the hyperplane of fixed x^+. By contrast, in the instant form, the conjugate momentum $\pi^0(x) = \partial_0\varphi(x)$ cannot be determined from $\varphi(x)$ restricted to the hyperplane of fixed x^0 alone: more *initial data* is required.

These two differences are closely related to each other obviously.

Concerning the conserved currents, from the Lagrangian density (72), one finds

$$T^{\mu\nu}(x) = \frac{\partial\mathcal{L}}{\partial(\partial_\mu\varphi(x))}\,\partial^\nu\varphi(x) - g^{\mu\nu}\,\mathcal{L} = \left(\partial^\mu\varphi(x)\right)\left(\partial^\nu\varphi(x)\right) - g^{\mu\nu}\,\mathcal{L}\,, \tag{76}$$

and thus the light-front version of P^μ as

$$P^+ = \int dx^- \!\int d^2\mathbf{x}\; T^{++}(x) = \int dx^- \!\int d^2\mathbf{x}\;\left(\partial_-\varphi(x)\right)\left(\partial_-\varphi(x)\right)\,, \tag{77}$$

$$P^j = \int dx^- \!\int d^2\mathbf{x}\; T^{+j}(x) = -\int dx^- \!\int d^2\mathbf{x}\;\left(\partial_-\varphi(x)\right)\left(\partial_j\varphi(x)\right) \tag{78}$$

and

$$P^- = \int dx^- \!\int d^2\mathbf{x}\; T^{+-}(x)\,, \tag{79}$$

where

$$T^{+-}(x) = \frac{1}{2}\,(\partial_j\varphi(x))\,(\partial_j\varphi(x)) + V(\varphi(x))\,. \tag{80}$$

For a scalar theory,

$$J^{\rho\mu\nu}(x) = x^{\mu}\,T^{\rho\nu}(x) - x^{\nu}\,T^{\rho\mu}(x)\,, \tag{81}$$

and thus one finds

$$\begin{aligned} M^{+i} &= \int dx^- \!\int d^2\mathbf{x}\; J^{++i}(x) = \int dx^- \!\int d^2\mathbf{x}\,\Big(x^+\,T^{+i}(x) - x^i\,T^{++}(x)\Big) \\ &= \int dx^- \!\int d^2\mathbf{x}\;(\partial_-\varphi(x))\,\Big(-x^+\,\partial_i\varphi(x) - x^i\,\partial_-\varphi(x)\Big)\,, \end{aligned} \tag{82}$$

$$\begin{aligned} M^{ij} &= \int dx^- \!\int d^2\mathbf{x}\; J^{+ij}(x) = \int dx^- \!\int d^2\mathbf{x}\,\Big(x^i\,T^{+j}(x) - x^j\,T^{+i}(x)\Big) \\ &= \int dx^- \!\int d^2\mathbf{x}\;(\partial_-\varphi(x))\,\Big(-x^i\,\partial_j\varphi(x) + x^j\,\partial_i\varphi(x)\Big)\,, \end{aligned} \tag{83}$$

$$\begin{aligned} M^{-+} &= \int dx^- \!\int d^2\mathbf{x}\; J^{+-+}(x) = \int dx^- \!\int d^2\mathbf{x}\,\Big(x^-\,T^{++}(x) - x^+\,T^{+-}(x)\Big) \\ &= \int dx^- \!\int d^2\mathbf{x}\,\Big(x^-\,(\partial_-\varphi(x))^2 - x^+\,T^{+-}(x)\Big) \end{aligned} \tag{84}$$

and

$$\begin{aligned} M^{-i} &= \int dx^- \!\int d^2\mathbf{x}\; J^{+-i}(x) = \int dx^- \!\int d^2\mathbf{x}\,\Big(x^-\,T^{+i}(x) - x^i\,T^{+-}(x)\Big) \\ &= \int dx^- \!\int d^2\mathbf{x}\,\Big(-x^-\,(\partial_-\varphi(x))\,(\partial_i\varphi(x)) - x^i\,T^{+-}(x)\Big)\,. \end{aligned} \tag{85}$$

We can see that the expressions of the kinematical Poincaré charges P^+, P^j, M^{+i} and M^{ij} are independent of the potential, and are thus identical to the ones obtained in a free scalar theory. This justifies the name kinematical charges: they are fixed by the kinematics, independently of the dynamics of the theory.

By contrast, the dynamical charges P^- and M^{-i} depend on the potential, and thus on the precise dynamics of the theory, through $T^{+-}(x)$. Moreover, the charge M^{-+} associated with longitudinal boost depend in general of the potential, except on the surface $x^+ = 0$. This is consistent with the expectations based on the action of boosts on the surfaces of constant x^+.

3.2 Quantization of the scalar theory in the front form

At quantum level, both the scalar field and the Poincaré charges become quantum operators: $\hat{\varphi}(x)$, $\hat{P}^{\mu}$ and $\hat{M}^{\mu\nu}$. Quantizing the theory basically means specifying all the commutation relations between them in a consistent way. In particular, one has the following requirements :

- The commutation relations of the Poincaré charges $\hat{P}^\mu$ and $\hat{M}^{\mu\nu}$ should be the Poincaré algebra (58), (59), (60)
- The Poincaré transformations of the scalar field $\hat{\varphi}(x)$ should be encoded by its commutations relations with $\hat{P}^\mu$ and $\hat{M}^{\mu\nu}$, as

$$\partial_\mu \hat{\varphi}(x) = i\Big[\hat{P}_\mu, \hat{\varphi}(x)\Big] \tag{86}$$

$$(x^\mu\, \partial^\nu - x^\nu\, \partial^\mu)\; \hat{\varphi}(x) = i\Big[\hat{M}^{\mu\nu}, \hat{\varphi}(x)\Big] \tag{87}$$

- In particular, $\partial_+ \hat{\varphi}(x) = i\Big[\hat{P}^-, \hat{\varphi}(x)\Big]$ should encode the equation of motion (75) of the scalar field.
- The expression of the generators $\hat{P}^\mu$ and $\hat{M}^{\mu\nu}$ in terms of the scalar field should be obtained from the classical ones (77-85) by symmetrizing products of (bosonic) operators, so that for example

$$\hat{P}^j = -\frac{1}{2} \int dx^- \int d^2\mathbf{x}\, \Big((\partial_- \hat{\varphi}(x))\, (\partial_j \hat{\varphi}(x)) + (\partial_j \hat{\varphi}(x))\, (\partial_- \hat{\varphi}(x)) \Big) \tag{88}$$

At this stage, the only thing left to be specified is the commutator of the field with itself, $[\hat{\varphi}(x), \hat{\varphi}(y)]$ at $x^+ = y^+$. The traditional way to derive this commutator is to follow the Dirac-Bergmann method (see for example the appendix E of Ref. [20]). That method is powerful but rather cumbersome, in particular in the case of interest. Indeed, it is constructed specially for theories with second order equations of motion. When applied to a theory with first order equation of motion, like the scalar theory (72) in the front form, the theory is first reinterpreted as a constrained theory with second order equation of motion, in order to proceed further.

Alternatively, one can follow the much simpler method [22, 23] proposed by Floreanini, Faddeev and Jackiw (FFJ) for the quantization of systems with first order equations of motions, which avoids to introduce unnecessary complications. Instead of following precisely one one these two systematic methods, let us adopt a further simplified version of the FFJ method, which provides the same result in this case.

Assuming that the unkown commutator $[\hat{\varphi}(x), \hat{\varphi}(y)]$ at $x^+ = y^+$ is a c-number (instead of an operator), the quantum version of the expression (77) for $\hat{P}^+$ leads to

$$\begin{aligned}\Big[\hat{P}^+, \hat{\varphi}(x)\Big] &= \int d^4y\; \delta(y^+ \!-\! x^+) \Big[(\partial_- \hat{\varphi}(y))\, (\partial_- \hat{\varphi}(y))\, ,\, \hat{\varphi}(x)\Big] \\ &= 2 \int d^4y\; \delta(y^+ \!-\! x^+) \Big[(\partial_- \hat{\varphi}(y))\, ,\, \hat{\varphi}(x)\Big]\; (\partial_- \hat{\varphi}(y))\; .\end{aligned} \tag{89}$$

Similarly, one obtains from (88)

$$\begin{aligned}\Big[\hat{P}^j,\hat{\varphi}(x)\Big] &= -\frac{1}{2}\int d^4y\ \delta(y^+\!-x^+)\Big[\,(\partial_-\hat{\varphi}(y))\,(\partial_j\hat{\varphi}(y))+(\partial_j\hat{\varphi}(y))\,(\partial_-\hat{\varphi}(y))\,,\,\hat{\varphi}(x)\Big]\\ &= -\int d^4y\ \delta(y^+\!-x^+)\bigg(\Big[\,(\partial_-\hat{\varphi}(y))\,,\,\hat{\varphi}(x)\Big]\ (\partial_j\hat{\varphi}(y))\\ &\qquad\qquad + \Big[\,(\partial_j\hat{\varphi}(y))\,,\,\hat{\varphi}(x)\Big]\ (\partial_-\hat{\varphi}(y))\bigg)\\ &= -2\int d^4y\ \delta(y^+\!-x^+)\Big[\,(\partial_-\hat{\varphi}(y))\,,\,\hat{\varphi}(x)\Big]\ (\partial_j\hat{\varphi}(y))\ ,\end{aligned} \tag{90}$$

assuming that one can neglect boundary terms when integrating by part in $\mathbf{y}^j$ or in y^-. Hence, in order to recover the correct commutation relations $\partial_-\hat{\varphi}(x)=i\Big[\hat{P}^+,\hat{\varphi}(x)\Big]$ and $\partial_j\hat{\varphi}(x)=-i\Big[\hat{P}^j,\hat{\varphi}(x)\Big]$, it is sufficient to postulate the commutation relation

$$\Big[\hat{\varphi}(x),\partial_-\hat{\varphi}(y)\Big]=\frac{i}{2}\,\delta(x^-\!-y^-)\delta^{(2)}(\mathbf{x}\!-\!\mathbf{y}) \tag{91}$$

for $x^+=y^+$. Integrating that relation with respect to y^-, one gets

$$\Big[\hat{\varphi}(x),\hat{\varphi}(y)\Big]=\frac{i}{2}\,\Big(\theta(y^-\!-x^-)+\text{constant}\Big)\delta^{(2)}(\mathbf{x}\!-\!\mathbf{y})\,. \tag{92}$$

The integration constant is determined by requiring the antisymmetry of the commutator. One then finds

$$\Big[\hat{\varphi}(x),\hat{\varphi}(y)\Big]=-\frac{i}{4}\,\epsilon(x^-\!-y^-)\,\delta^{(2)}(\mathbf{x}\!-\!\mathbf{y})\,, \tag{93}$$

with the notation

$$\epsilon(x^-\!-y^-)\equiv\theta(x^-\!-y^-)-\theta(y^-\!-x^-)\,, \tag{94}$$

for the sign function.

Using the commutation relations (91) and (93), it is a simple exercise to check that the other kinematical Poincaré generators $\hat{M}^{+i}$ and $\hat{M}^{ij}$ satisfy the correct commutation relations with the scalar field (neglecting again boundary terms produced by the integrations by parts in $\mathbf{y}^j$ or y^-), meaning

$$\Big[\hat{M}^{+i},\hat{\varphi}(x)\Big]=-i(x^+\,\partial^i-\mathbf{x}^i\,\partial^+)\hat{\varphi}(x)=i(x^+\,\partial_i+\mathbf{x}^i\,\partial_-)\hat{\varphi}(x) \tag{95}$$

$$\Big[\hat{M}^{ij},\hat{\varphi}(x)\Big]=-i(\mathbf{x}^i\,\partial^j-\mathbf{x}^j\,\partial^i)\hat{\varphi}(x)=i(\mathbf{x}^i\,\partial_j-\mathbf{x}^j\,\partial_i)\hat{\varphi}(x)\,, \tag{96}$$

and also that $\hat{P}^+$, $\hat{P}^j$, $\hat{M}^{+i}$ and $\hat{M}^{ij}$ have the correct commutation relations between themselves. Hence, when quantizing the scalar theory (72) on a surface of fixed x^+ using the commutation relations (91) and (93), the symmetries associated with the kinematical subalgebra are automatically preserved at the quantum level, independently of the dynamics of the theory.

Concerning the generator $\hat{M}^{-+}$ of longitudinal boosts, one finds in the same way

$$\Big[\hat{M}^{-+},\hat{\varphi}(x)\Big]=-ix^-\,\partial_-\hat{\varphi}(x)-x^+\,\Big[\hat{P}^-,\hat{\varphi}(x)\Big]\,. \tag{97}$$

On the one hand $\hat{M}^{-+}$ has the proper commutation relation with $\hat{\varphi}(x)$ on the $x^+=0$ surface. On the other hand, if $\hat{P}^-$ has the proper commutation relation with $\hat{\varphi}(x)$ for any x^+, so does $\hat{M}^{-+}$.

It remains now to study the dynamics of the theory at the quantum level, given the commutator (93). From the classical expression of the Light-front Hamiltonian P^- (see eqs. (79) and (80)), one finds at the quantum level

$$\begin{aligned}\Big[\hat{P}^-,\hat{\varphi}(x)\Big] &= \int d^4y\; \delta(y^+\!-x^+)\Big[\frac{1}{2}\,(\partial_j\hat{\varphi}(y))\,(\partial_j\hat{\varphi}(y)) + V(\hat{\varphi}(y)),\, \hat{\varphi}(x)\Big]\\ &= \int d^4y\; \delta(y^+\!-x^+)\Big[\hat{\varphi}(y),\, \hat{\varphi}(x)\Big]\,\Big(-\Delta_\perp\hat{\varphi}(y) + V'(\hat{\varphi}(y))\Big)\,, \end{aligned} \tag{98}$$

so that

$$\begin{aligned}\partial_{x^-} i\Big[\hat{P}^-,\hat{\varphi}(x)\Big] &= \int d^4y\; \delta(y^+\!-x^+) i\Big[\hat{\varphi}(y),\, (\partial_-\hat{\varphi}(x))\Big]\,\Big(-\Delta_\perp\hat{\varphi}(y) + V'(\hat{\varphi}(y))\Big)\\ &= \frac{1}{2}\,\Big(\Delta_\perp\hat{\varphi}(x) - V'(\hat{\varphi}(x)\Big)\,. \end{aligned} \tag{99}$$

Hence, postulating that the x^+ evolution of the field is defined by its commutation with $\hat{P}^-$ as $\partial_+\hat{\varphi}(x) = i\Big[\hat{P}^-,\hat{\varphi}(x)\Big]$, one recovers from (99)

$$\partial_-\partial_+\hat{\varphi}(x) = \frac{1}{2}\Big(\Delta_\perp\hat{\varphi}(x) - V'(\hat{\varphi}(x))\Big)\,, \tag{100}$$

which is indeed the quantum version of the classical equation of motion (75).

More precisely, the evolution of the field is then given by

$$\begin{aligned}\partial_+\hat{\varphi}(x) &= i\Big[\hat{P}^-,\hat{\varphi}(x)\Big]\\ &= \frac{1}{4}\int d^4y\; \delta(y^+\!-x^+)\,\delta^{(2)}(\mathbf{x}\!-\!\mathbf{y})\,\epsilon(x^-\!-y^-)\,\Big(\Delta_\perp\hat{\varphi}(y) - V'(\hat{\varphi}(y))\Big)\\ &= \frac{1}{2\partial_-}\,\Big(\Delta_\perp\hat{\varphi}(x) - V'(\hat{\varphi}(x))\Big)\,, \end{aligned} \tag{101}$$

with $1/\partial_-$ defined as

$$\frac{1}{\partial_-}\,f(x) \equiv \frac{1}{2}\int dy^-\;\; \epsilon(x^-\!-y^-)\,f(x^+,\mathbf{x},y^-) \tag{102}$$

so that $\partial_-(1/\partial_-) = 1$. In the same way, one can find that

$$\begin{aligned}\Big[\hat{M}^{-i},\hat{\varphi}(x)\Big] &= ix^-\,\partial_i\hat{\varphi}(x) + i\,\mathbf{x}^i\,\frac{1}{2\partial_-}\,\Big(\Delta_\perp\hat{\varphi}(x) - V'(\hat{\varphi}(x))\Big)\\ &= ix^-\,\partial_i\hat{\varphi}(x) + i\,\mathbf{x}^i\,\partial_+\hat{\varphi}(x)\,. \end{aligned} \tag{103}$$

Hence, with the expression (93) found for the commutator $[\hat{\varphi}(x),\hat{\varphi}(y)]$ at $x^+=y^+$, one obtains the correct commutation relations of $\hat{\varphi}(x)$ with the Poincaré generators $\hat{P}^\mu$ and $\hat{M}^{\mu\nu}$, with in particular the commutation relations with the dynamical generators encoding the correct equations of motion for $\hat{\varphi}(x)$. Moreover, it is then possible [24] (but lengthy) to check explicitly that the commutation relations of the Poincaré generators $\hat{P}^\mu$ and $\hat{M}^{\mu\nu}$ form the Poincaré algebra, see Eqs. (58), (59) and (60).

3.3 Construction of the Fock space

From now on, for simplicity, let us assume that the scalar potential $V(\varphi(x))$ has a unique and stable minimum at $\varphi(x)=0$ classical level, so that in particular

$$V'(0)=0\,,\quad V''(0)>0\,, \tag{104}$$

and there is no spontaneous symmetry breaking. Note that the conditions (104) imply that the considered theory has a classical mass gap $m_{\rm gap}=\sqrt{V''(0)}$.

3.3.1 Annihilation and creation operators

Within an hyperplane of fixed x^+, an interacting scalar field $\hat{\varphi}(x)$ cannot be distinguished from a free scalar field, since its dependence on x^- and $\mathbf{x}$ is driven by the kinematic generators $\hat{P}^+$ and $\hat{P}^j$, independent of interactions. It is then useful to Fourier transform the field with respect these variables, but keep the non-trivial x^+ dependence explicit. Hence, one can define

$$\hat{a}(\underline{k},x^+) = 2i\int dx^-\!\!\int d^2\mathbf{x}\; e^{i\underline{k}\cdot x}\partial_-\hat{\varphi}(x) \tag{105}$$

$$\hat{a}^\dagger(\underline{k},x^+) = -2i\int dx^-\!\!\int d^2\mathbf{x}\; e^{-i\underline{k}\cdot x}\partial_-\hat{\varphi}(x) \tag{106}$$

for $k^+>0$, with the notations $\underline{k}\equiv(k^+,\mathbf{k})$ and $\underline{k}\cdot x\equiv k^+\mathbf{x}^- - \mathbf{k}\cdot\mathbf{x}$. Equivalently, the field $\hat{\varphi}(x)$ can be written as

$$\hat{\varphi}(x) = \int \frac{d^3\underline{k}}{(2\pi)^3}\,\frac{\theta(k^+)}{2k^+}\,\left\{e^{-i\underline{k}\cdot x}\,\hat{a}(\underline{k},x^+) + e^{i\underline{k}\cdot x}\,\hat{a}^\dagger(\underline{k},x^+)\right\}\,. \tag{107}$$

From the commutation relations (91) or (93) for $\hat{\varphi}(x)$, it is easy to find that

$$\Big[\hat{a}(\underline{k},x^+),\hat{a}^\dagger(\underline{k}',x^+)\Big] = 2k^+\,(2\pi)^3\,\delta^{(3)}(\underline{k}-\underline{k}') \tag{108}$$

$$\Big[\hat{a}(\underline{k},x^+),\hat{a}(\underline{k}',x^+)\Big] = 0 \tag{109}$$

$$\Big[\hat{a}^\dagger(\underline{k},x^+),\hat{a}^\dagger(\underline{k}',x^+)\Big] = 0\,, \tag{110}$$

as well as

$$\Big[\hat{P}^\mu,\hat{a}^\dagger(\underline{k},x^+)\Big] = k^\mu\,\hat{a}^\dagger(\underline{k},x^+) \tag{111}$$

$$\Big[\hat{P}^\mu,\hat{a}(\underline{k},x^+)\Big] = -k^\mu\,\hat{a}(\underline{k},x^+) \tag{112}$$

for $\mu\neq -$. From the commutation relations (108), (109), (110), (111) and (112) it is clear that the operators $\hat{a}^\dagger(\underline{k},x^+)$ and $\hat{a}(\underline{k},x^+)$ correspond respectively to creation and annihilation operators of a quantum of momentum $\underline{k}$, at the time x^+.

For completeness, one can calculate as well their commutation relations with the remaining kinematical generators, and find

$$\left[\hat{M}^{ij}, \hat{a}^\dagger(\underline{k}, x^+)\right] = i\Big(\mathbf{k}^i\,\partial_{\mathbf{k}^j} - \mathbf{k}^j\,\partial_{\mathbf{k}^i}\Big)\,\hat{a}^\dagger(\underline{k}, x^+) \tag{113}$$

$$\left[\hat{M}^{ij}, \hat{a}(\underline{k}, x^+)\right] = i\Big(\mathbf{k}^i\,\partial_{\mathbf{k}^j} - \mathbf{k}^j\,\partial_{\mathbf{k}^i}\Big)\,\hat{a}(\underline{k}, x^+) \tag{114}$$

$$\left[\hat{M}^{+i}, \hat{a}^\dagger(\underline{k}, x^+)\right] = \Big(ik^+\,\partial_{\mathbf{k}^i} + x^+\,\mathbf{k}^j\Big)\,\hat{a}^\dagger(\underline{k}, x^+) \tag{115}$$

$$\left[\hat{M}^{+i}, \hat{a}(\underline{k}, x^+)\right] = \Big(ik^+\,\partial_{\mathbf{k}^i} - x^+\,\mathbf{k}^j\Big)\,\hat{a}(\underline{k}, x^+)\,. \tag{116}$$

3.3.2 Defining the vacuum

A consistent QFT should have a spectrum bounded from below, with one (or several) ground state(s) (also called vacuum) of minimal energy P^0_{vac}, with a vanishing momentum $\vec{P}_{\mathrm{vac}} = 0$. In the case of a potential with a unique and stable minimum, that we are considering, the ground state of the quantum theory should be unique as well.

A physical quantum excitation above the vacuum, with a cartesian momentum $\vec{k}$, corresponds to an increase of energy by k^0 above the vacuum, and should satisfy that $k^\mu k_\mu \geqslant 0$ and $k^0 \geqslant 0$. In light-cone coordinates, these conditions imply $k^+ \geqslant 0$ and $k^- \geqslant 0$ for an excitation above the ground state.

In particular, in a theory with a mass gap, excitations above the ground state obey $2k^+k^- \geqslant m^2_{\mathrm{gap}} + \mathbf{k}^2 > 0$. Hence, there is no excitation of finite k^- which has $k^+ = 0$, and vice-versa. Therefore, the vacuum can be defined uniquely[3] as the state of minimal P^+, or equivalently of minimum P^- . Since the annihilation operators $\hat{a}(\underline{k}, x^+)$ lower the P^+ of the state they act on, they annihilate states of minimum P^+, and thus the vacuum state $|0\rangle$, as

$$\hat{a}(\underline{k}, x^+)|0\rangle = 0\,. \tag{117}$$

Excited states are then obtained by acting with creation operators $\hat{a}^\dagger(\underline{k}, x^+)$ on the vacuum $|0\rangle$. Of course the vacuum is chosen to be normalized as

$$\langle 0|0\rangle = 1\,. \tag{118}$$

It is thus possible to construct, on each hypersurface of constant x^+, a Fock space for such an interacting scalar theory quantized in the front form. The most crucial ingredient here is the existence of a lower bound on the k^+ of the physical excitations, which is entirely independent on interactions, since $\hat{P}^+$ is a kinematic generator. The situation is quite remarkable, by comparison to case of quantization in the instant form. In that case, none of the three kinematic space components of $\hat{P}^\mu$ would have a bounded spectrum, but only the dynamical component $\hat{P}^0$. For that reason, in the instant form, none of the candidate annihilation operator that we could build from the interacting field at a given x^0 could annihilate the vacuum in a similar way as in eq. (117), and thus the Fock space construction would fail. In the instant form, such a Fock space can be constructed only for a free theory, and after Fourier transform to full momentum space, including from x^0 to k^0, and moreover free and interacting theory have different vacua.

[3] In the case of a theory without mass gap, like QED or perturbative QCD, the vacuum state (or states) still have minimum P^+ and $\mathbf{P} = 0$. However, this does not characterize the vacuum uniquely anymore.

3.3.3 Normal ordering, vacuum energy and Poincaré invariance of the vacuum

At this stage, it is useful to rewrite the Poincaré generators in terms of $\hat{a}(\underline{k},x^+)$ and $\hat{a}^\dagger(\underline{k},x^+)$ by substituting $\hat{\varphi}(x)$ by its Fourier decomposition (107). As an example, let us consider the case of $\hat{P}^+$. Its density becomes

$$\hat{T}^{++}(x) = \left(\partial_-\hat{\varphi}(x)\right)\left(\partial_-\hat{\varphi}(x)\right) = \int \frac{d^3\underline{k}}{(2\pi)^3}\,\frac{\theta(k^+)}{2k^+}\,\int \frac{d^3\underline{k}'}{(2\pi)^3}\,\frac{\theta(k'^+)}{2k'^+}\; k^+\,k'^+$$
$$\times\bigg\{ - e^{-i(\underline{k}+\underline{k}')\cdot x}\,\hat{a}(\underline{k}',x^+)\,\hat{a}(\underline{k},x^+) + e^{-i(\underline{k}-\underline{k}')\cdot x}\,\hat{a}^\dagger(\underline{k}',x^+)\,\hat{a}(\underline{k},x^+)$$
$$+\, e^{i(\underline{k}-\underline{k}')\cdot x}\,\hat{a}(\underline{k}',x^+)\,\hat{a}^\dagger(\underline{k},x^+) - e^{i(\underline{k}+\underline{k}')\cdot x}\,\hat{a}^\dagger(\underline{k}',x^+)\,\hat{a}^\dagger(\underline{k},x^+)\bigg\}\,. \qquad (119)$$

Due to the relation $\hat{a}(\underline{k},x^+)|0\rangle = 0$, it is more convenient to work with normal-ordered operators, meaning that all the $\hat{a}$ are on the right of all the $\hat{a}^\dagger$. In the expression (119), only the third term violates the normal ordering. Hence, $\hat{T}^{++}(x)$ differs from its normal-ordered version, noted $:\hat{T}^{++}(x):$, by a c-number term as

$$\hat{T}^{++}(x) = \,:\hat{T}^{++}(x):+\int \frac{d^3\underline{k}}{(2\pi)^3}\,\frac{\theta(k^+)}{2k^+}\,\int \frac{d^3\underline{k}'}{(2\pi)^3}\,\frac{\theta(k'^+)}{2k'^+}\; k^+\,k'^+$$
$$\times\, e^{i(\underline{k}-\underline{k}')\cdot x}\,\left[\hat{a}(\underline{k}',x^+)\,,\,\hat{a}^\dagger(\underline{k},x^+)\right]$$
$$= \,:\hat{T}^{++}(x):+\int \frac{d^2\mathbf{k}}{(2\pi)^2}\int \frac{dk^+}{2\pi}\,\frac{\theta(k^+)}{2k^+}\,\left(k^+\right)^2\,. \qquad (120)$$

At first sight, the extra term looks awkward: if one evaluates the transverse and k^+ integrals separately, the first one is a pure quadratic UV divergence, whereas the second one has a linear UV divergence for $k^+\rightarrow+\infty$. One might invoke transverse dimensional regularization to argue that the transverse integral vanishes, but even then, one is lead to the product of a vanishing quantity by a divergent one, which might be dangerous. The main issue here, is that by evaluating the transverse and k^+ integrals separately, one is lead to introduce different UV regularization for each, which partially breaks Poincaré invariance.

The most elegant way to address this issue is to recognize in eq. (120) the one-particle Lorentz invariant phase-space measure, after integrating over k^-. Restoring this integration over k^-, with a mass m (which might be the physical mass of the field or not), one can fully apply dimensional regularization, as

$$\hat{T}^{++}(x) = \,:\hat{T}^{++}(x):+\int \frac{d^Dk}{(2\pi)^D}\;2\pi\delta(k^2-m^2)\;\theta(k^0)\;\left(k^+\right)^2\,. \qquad (121)$$

However, one has

$$\int \frac{d^Dk}{(2\pi)^D}\;2\pi\delta(k^2-m^2)\;\theta(k^0)\;k^\mu\,k^\nu = \int \frac{d^Dk}{(2\pi)^D}\;2\pi\delta(k^2-m^2)\;\theta(k^0)\;\frac{g^{\mu\nu}}{D}\,k^2$$
$$= \frac{g^{\mu\nu}}{D}\,m^2 \int \frac{d^Dk}{(2\pi)^D}\;2\pi\delta(k^2-m^2)\;\theta(k^0)$$
$$= \frac{g^{\mu\nu}}{D}\,\frac{m^4}{(4\pi)^2}\,\Gamma\left(1-\frac{D}{2}\right)\,\left(\frac{m^2}{4\pi}\right)^{\frac{D}{2}-2}\,. \qquad (122)$$

The extra term in eq. (121) is then obtained by taking $\mu = \nu = +$ in eq. (122), which gives zero since $g^{++} = 0$. Hence, $\hat{T}^{++}(x)$ is automatically normal-ordered, provided the chosen UV regularization does not break Poincaré invariance.

Finally, $\hat{P}^+$ is obtained as

$$\begin{aligned}
\hat{P}^+ &= \int dx^- \int d^2\mathbf{x}\, \hat{T}^{++}(x) \\
&= \int \frac{d^3\underline{k}}{(2\pi)^3} \frac{\theta(k^+)}{2k^+} \int \frac{d^3\underline{k}'}{(2\pi)^3} \frac{\theta(k'^+)}{2k'^+}\, k^+\, k'^+ \\
&\quad \times \Big\{ -(2\pi)^3\delta^{(3)}(\underline{k}+\underline{k}')\, \hat{a}(\underline{k}',x^+)\, \hat{a}(\underline{k},x^+) + (2\pi)^3\delta^{(3)}(\underline{k}-\underline{k}')\ \hat{a}^\dagger(\underline{k}',x^+)\, \hat{a}(\underline{k},x^+) \\
&\qquad + (2\pi)^3\delta^{(3)}(\underline{k}-\underline{k}')\ \hat{a}^\dagger(\underline{k},x^+)\, \hat{a}(\underline{k}',x^+) - (2\pi)^3\delta^{(3)}(\underline{k}+\underline{k}')\ \hat{a}^\dagger(\underline{k}',x^+)\, \hat{a}^\dagger(\underline{k},x^+) \Big\}\,.
\end{aligned} \tag{123}$$

In the $\hat{a}\hat{a}$ and in the $\hat{a}^\dagger\hat{a}^\dagger$ terms, k^+ and k'^+ are constrained to be opposite, whereas they are both positive. These terms thus vanish, and one is left with

$$\hat{P}^+ = \int \frac{d^3\underline{k}}{(2\pi)^3} \frac{\theta(k^+)}{2k^+}\, k^+\ \hat{a}^\dagger(\underline{k},x^+)\, \hat{a}(\underline{k},x^+)\,. \tag{124}$$

Since $\hat{a}(\underline{k},x^+)|0\rangle = 0$, it is now clear that $\hat{P}^+|0\rangle = 0$.

The other kinematic Poincaré generators can be studied along the same lines. One finds that all of them are automatically normal-ordered as well, and that they annihilate the vacuum as well. Hence, the vacuum $|0\rangle$, that we have constructed for the Fock space at x^+ is invariant under the kinematic Poincaré transformations at x^+.

In order to discuss to case of the dynamical generators, let us focus on the case of the φ^4 theory as an example, which has the potential

$$V(\hat{\varphi}(x)) = \frac{\lambda}{4!}\,\hat{\varphi}(x)^4 + \frac{m^2}{2}\,\hat{\varphi}(x)^2 + \Lambda_0\,. \tag{125}$$

Here, Λ_0 is an arbitrary constant, which can be interpreted as a bare cosmological constant. Indeed, it is always possible to add such constant in the Lagrangian density, since it will have no effect on the equations of motions (at least in theories without gravity). However, a Λ_0 term will appear in the Hamiltonian density $\hat{T}^{+-}(x)$, via the potential, and thus in the expression of the dynamical generators. It turns out that, by contrast to the kinematic generators, the dynamical Poincaré generators are not automatically normal-ordered, and do not automatically annihilate the vacuum.

Quadratic terms in $\hat{\varphi}(x)$ can always be written as their normal-ordered version plus a c-number term, as we have seen in the case of $T^{++}(x)$. In general, interaction terms of degree higher than 2 in $\hat{\varphi}(x)$ are present in $\hat{T}^{+-}(x)$. In the example (125), this is the case of the quartic term. For interaction terms, normal-ordering does not simply amount to extract a c-number term. For example, one has

$$\hat{\varphi}(x)^4 =: \hat{\varphi}(x)^4 : +6\,\langle 0|\hat{\varphi}(x)^2|0\rangle : \hat{\varphi}(x)^2 : +\langle 0|\hat{\varphi}(x)^4|0\rangle\,, \tag{126}$$

with an extra normal-ordered quadratic contribution. At this stage, $\hat{T}^{+-}(x)$ can be written as a sum of normal-ordered terms and of c-number terms. By tuning Λ_0, the total c-number contribution to $\hat{T}^{+-}(x)$ can be removed, leading to

$$\begin{aligned}\hat{T}^{+-}(x) = &\frac{1}{2} : (\partial_j \hat{\varphi}(x)) \, (\partial_j \hat{\varphi}(x)) : + \frac{m^2}{2} : \hat{\varphi}(x)^2 : \\ &+ \frac{\lambda}{4!} : \hat{\varphi}(x)^4 : + \frac{\lambda}{4} \langle 0|\hat{\varphi}(x)^2|0\rangle : \hat{\varphi}(x)^2 : \, . \end{aligned} \tag{127}$$

Such tuning of Λ_0 can be interpreted as renormalizing the vacuum energy (k^-). When integrating over $\hat{T}^{+-}(x)$ over x^- and $\mathbf{x}$ in order to obtain $\hat{P}^-$, terms with only annihilation operators or only creation operators will drop, like in eq. (123). For that reason, when written in terms of $\hat{a}^\dagger(\underline{k}, x^+)$ and $\hat{a}(\underline{k}, x^+)$, each term in $\hat{P}^-$ will contain a $\hat{a}(\underline{k}, x^+)$ as the rightmost operator, and thus $\hat{P}^-|0\rangle = 0$. The situation is the same when calculating $\hat{M}^{-i}$ from (127).

All in all, it is sufficient to fix a single counterterm, Λ_0, in order to obtain the ten relations

$$\hat{P}^\mu|0\rangle = 0 \tag{128}$$

$$\hat{M}^{\mu\nu}|0\rangle = 0 \, , \tag{129}$$

so that the vacuum $|0\rangle$ is fully Poincaré invariant. In particular, this means that the same state $|0\rangle$ is the vacuum for all Fock spaces, contructed at each value of x^+.

3.3.4 Fock states

Now that the vacuum state $|0\rangle$ is understood, it is possible to construct Fock states by acting with creation operators $\hat{a}^\dagger(\underline{k}, x^+)$ on $|0\rangle$. Hence, one has

- the vacuum $|0\rangle$,
- 1-particle states $\hat{a}^\dagger(\underline{k}_1, x^+)|0\rangle$,
- 2-particles states $\hat{a}^\dagger(\underline{k}_1, x^+)\hat{a}^\dagger(\underline{k}_2, x^+)|0\rangle$,
- and so on.

The set of all of these Fock states form a basis of the Hilbert space of the theory. That construction can be done for any value of x^+ independently. However, the basis obtained for each value of x^+ are isomorphic to each other, with the mapping provided by the action of the dynamical generator $\hat{P}^-$. In particular, the vacuum is the same in any of these bases, but the other states can mix, in an interacting theory.

For convenience, the base we will choose as reference is the one corresponding to the Fock space at $x^+ = 0$, and the notations will be simplified as follows

$$\begin{aligned}\hat{a}^\dagger(\underline{k}) \equiv &\hat{a}^\dagger(\underline{k}, x^+ = 0) \\ \hat{a}(\underline{k}) \equiv &\hat{a}(\underline{k}, x^+ = 0) \, . \end{aligned} \tag{130}$$

A $x^+ = 0$ Fock state will be generically noted $|\mathcal{F}\rangle$, so that

$$|\mathcal{F}\rangle \equiv \prod_{n=1}^{N} \Big[\hat{a}^\dagger(\underline{k}_n)\Big]\,|0\rangle \tag{131}$$

for some $N \geqslant 0$.

Using that $x^+ = 0$ Fock state basis, the identity operator can be written as

$$\begin{aligned}\mathbf{1} =&|0\rangle\langle 0| + \sum_{N=1}^{+\infty} \frac{1}{N!} \int \prod_{n=1}^{N} \Big[\frac{d^3\underline{k}_n}{(2\pi)^3}\,\frac{\theta(k_n^+)}{2k_n^+}\Big] \prod_{n=1}^{N} \Big[\hat{a}^\dagger(\underline{k}_n)\Big]\,|0\rangle\langle 0| \prod_{n=1}^{N} \Big[\hat{a}(\underline{k}_n)\Big]\\ \equiv& \sum_{\mathcal{F}} |\mathcal{F}\rangle\langle\mathcal{F}|\,.\end{aligned} \tag{132}$$

It is easy to check that

$$\sum_{\mathcal{F}} |\mathcal{F}\rangle\langle\mathcal{F}|\mathcal{F}'\rangle = |\mathcal{F}'\rangle \tag{133}$$

as it should, thanks to the commutation relations (108), (109) and (110) of the creation and annihilation operators. Note that the $1/N!$ factor in Eq. (132) is there to compensate the number of ways to associate an $\hat{a}^\dagger$ from $|\mathcal{F}'\rangle$ with an $\hat{a}$ from $\langle\mathcal{F}|$ in Eq. (133).

3.4 Light-cone perturbation theory: LFWFs and S-matrix elements

3.4.1 Heisenberg picture

So far, we have been using the light-front version of the Heisenberg picture, in which the operators evolve in x^+ according to

$$\hat{\mathcal{O}}_H(x^+) = e^{i\,x^+\hat{P}^-}\;\hat{\mathcal{O}}_H(0)\,e^{-i\,x^+\hat{P}^-}\,. \tag{134}$$

Since the evolution is already taken into account at the operator level, the states upon which the operators act should have no explicit x^+ dependence in the Heisenberg picture.

In an interacting theory, the physical states $\big|\,i_H\big\rangle$ one considers are typically not the Fock states constructed previously, but complicated linear combinations of them, which can be written as

$$\big|\,i_H\big\rangle = \sum_{\mathcal{F}} \big|\mathcal{F}\big\rangle\;\Phi_{i\to\mathcal{F}}\,, \tag{135}$$

defining the light-front wave-function (LFWF) $\Phi_{i\to\mathcal{F}}$ as

$$\Phi_{i\to\mathcal{F}} = \big\langle\mathcal{F}\big|\,i_H\big\rangle\,. \tag{136}$$

Here, $\big|\,i_H\big\rangle$ can typically be an eigenstate of $\hat{P}^-$ corresponding to a *dressed* 1−particle state, for example a hadron in the QCD case. The LFWFs $\Phi_{i\to\mathcal{F}}$ describe the content of the *dressed* particle $\big|\,i_H\big\rangle$ in terms of elementary *partons*.

Moreover, at this stage, the S-matrix element for a scattering process with initial state i and final state f is given by the overlap between these two states in the Heisenberg picture, as

$$\mathcal{S}_{fi} = \big\langle f_H \big| i_H \big\rangle \,. \tag{137}$$

Here, by contrast, the chosen $\big| i_H \big\rangle$ and $\big\langle f_H \big|$ are typically not eigenstates of $\hat{P}^-$, but states containing two (or more) dressed particles.

3.4.2 Interaction picture

Usually, in an interacting theory, the $\hat{P}^-$ operator can be decomposed as $\hat{P}^- = \hat{T} + \hat{V}$, where $\hat{T}$ is the part present in the corresponding free theory, whereas $\hat{V}$ collects the interaction terms. In order to formulate perturbation theory, it is convenient to switch from the Heisenberg picture to the interaction picture, defined as follows. In the interaction picture, the x^+ evolution of the operators is generated by the free term $\hat{T}$ only, as

$$\hat{\mathcal{O}}_I(x^+) = e^{i\, x^+ \hat{T}}\, \hat{\mathcal{O}}_I(0)\, e^{-i\, x^+ \hat{T}} \,. \tag{138}$$

As a particular case, the interaction term $\hat{V}$ evolves in the interaction picture as

$$\hat{V}_I(x^+) = e^{i\, x^+ \hat{T}}\, \hat{V}_I(0)\, e^{-i\, x^+ \hat{T}} \,. \tag{139}$$

In the interaction picture, the states are now x^+ dependent, as

$$\big| i_I(x_2^+) \big\rangle = \mathcal{P}_+ \, \exp\left(-i \int_{x_1^+}^{x_2^+} \mathrm{d}x^+ \, \hat{V}_I(x^+) \right) \, \big| i_I(x_1^+) \big\rangle \,. \tag{140}$$

where $\mathcal{P}_+$ indicates the ordering of the operators $\hat{V}_I(x^+)$ according to x^+: from smaller x^+ on the right to larger x^+ on the left.

By convention, the Heisenberg picture and the Interaction picture are matched at $x^+ = 0$, meaning

$$\begin{aligned} \hat{\mathcal{O}}_I(0) &= \hat{\mathcal{O}}_H(0) \\ \big| i_I(0) \big\rangle &= \big| i_H \big\rangle \,, \end{aligned} \tag{141}$$

so that

$$\big| i_H \big\rangle = \mathcal{P}_+ \, \exp\left(-i \int_{-\infty}^{0} \mathrm{d}x^+ \, \hat{V}_I(x^+) \right) \, \big| i_I(-\infty) \big\rangle \,. \tag{142}$$

In the example of scalar φ^4 theory (72), the light-front Hamiltonian density is written in eq.(127), after vacuum energy renormalization. The Hamiltonian $\hat{P}^-$ can then be split into a free part $\hat{T}$,

$$\hat{T} = \int dx^- \int d^2\mathbf{x} \left[\frac{1}{2} :(\partial_j \hat{\varphi}(x))\,(\partial_j \hat{\varphi}(x)): + \frac{m^2}{2} \; :\hat{\varphi}(x)^2: \right] \tag{143}$$

$$= \int \frac{d^3\underline{k}}{(2\pi)^3} \, \frac{\theta(k^+)}{2k^+} \, \frac{(\mathbf{k}^2 + m^2)}{2k^+} \, \hat{a}^\dagger(\underline{k}, x^+)\, \hat{a}(\underline{k}, x^+) \,, \tag{144}$$

and an interaction part $\hat{V}_I(x^+)$. Note that the Fock states are eigenstates of $\hat{T}$, with

$$\hat{T}\prod_{n=1}^{N}\left[\hat{a}^\dagger(\underline{k}_n)\right]|0\rangle = \left[\sum_{n=1}^{N}\frac{\mathbf{k}_n^2+m^2}{2k_n^+}\right]\prod_{n=1}^{N}\left[\hat{a}^\dagger(\underline{k}_n)\right]|0\rangle\,. \tag{145}$$

3.4.3 Light-cone perturbation theory for LFWFs

Using Eqs. (139) and (142) as well as the integral

$$\int_0^{+\infty} dt\; e^{it\,\Delta} = \frac{i}{(\Delta+i\epsilon)}\,, \tag{146}$$

one finds the perturbative expansion for the LFWFs

$$\begin{aligned}\Phi_{i\to\mathcal{F}} =&\langle\mathcal{F}\big|\mathcal{P}_+\,\exp\left(-i\int_{-\infty}^{0}\mathrm{d}x^+\,\hat{V}_I(x^+)\right)\,\big|\,i_I(-\infty)\big\rangle\\ =&\langle\mathcal{F}|i_I(-\infty)\rangle+\sum_{N=1}^{\infty}\sum_{\mathcal{F}_{N-1}}\cdots\sum_{\mathcal{F}_0}\frac{\langle\mathcal{F}|\hat{V}_I(0)|\mathcal{F}_{N-1}\rangle}{(T_{\mathcal{F}_0}-T_{\mathcal{F}}+i\epsilon)}\;\frac{\langle\mathcal{F}_{N-1}|\hat{V}_I(0)|\mathcal{F}_{N-2}\rangle}{\big(T_{\mathcal{F}_0}-T_{\mathcal{F}_{N-1}}+i\epsilon\big)}\cdots\\ &\cdots\frac{\langle\mathcal{F}_1|\hat{V}_I(0)|\mathcal{F}_0\rangle}{(T_{\mathcal{F}_0}-T_{\mathcal{F}_1}+i\epsilon)}\;\langle\mathcal{F}_0|i_I(-\infty)\rangle\,,\end{aligned} \tag{147}$$

with the notation $T_{\mathcal{F}_n}$ for the eigenvalue of $\hat{T}$ associated with the Fock state $|\,\mathcal{F}_n\rangle$, see for example Eq. (145). Intuitively, $T_{\mathcal{F}_n}$ is the estimate of the total k^- in $|\,\mathcal{F}_n\rangle$, assuming the theory is free.

Since the interactions are taken into account perturbatively, down to $x^+\to-\infty$, the initial state $|i_I(-\infty)\rangle$ is usually assumed to be a free Fock state, if the theory is not confining. It is then not necessary to introduce $\mathcal{F}_0$. For example, if one applies this formalism to the case of an electron LFWF in QED, $\big|\,i_H\big\rangle$ would correspond to a dressed electron state, and $|i_I(-\infty)\rangle$ to a free asymptotic electron state. In the QCD case, the expansion (147) can be used at the parton level, and instead is usually not very helpful for the case of an incoming hadron.

In Eq. (147), each interaction is ordered along the x^+ direction, even though their x^+ have been integrated over. Hence, Eq. (147) cannot be represented by standard Feynmann diagrams, but instead by diagrams with specific ordering for the vertices, like in old-fashioned perturbation theory.

Instead of propagators, each term in the expansion Eq. (147) contain one *energy denominator* for each intermediate Fock state, plus another one for the final Fock state $\mathcal{F}$. Each energy denominator is the difference of kinetic energy T between the incoming state and the current (intermediate or final) Fock state.

In the particular case where $\big|\,i_H\big\rangle$ is a (perturbative) one-particle state, and thus $|i_I(-\infty)\rangle$ is a one-particle Fock state, it is convenient to define the LFWFs in a slightly different way [3], extracting the wave-function renormalization constant Z_i, as

$$\big|\,i_H\big\rangle = \sqrt{Z_i}\,\left\{|i_I(-\infty)\rangle+\sum_{\mathcal{F}\neq i}|\mathcal{F}\rangle\;\Psi_{i\to\mathcal{F}}\right\}, \tag{148}$$

where the sum over Fock states is now excluding the $|i_I(-\infty)\rangle$ Fock state. The new LFWFs (for $\mathcal{F} \neq i$) admit the perturbative expansion

$$\begin{aligned}
\Psi_{i\to\mathcal{F}} =& \frac{\langle\mathcal{F}|\hat{V}_I(0)|i_I(-\infty)\rangle}{(T_i - T_\mathcal{F} + i\epsilon)} + \sum_{N=2}^{\infty} \sum_{\mathcal{F}_{N-1}\neq i} \cdots \sum_{\mathcal{F}_1\neq i} \frac{\langle\mathcal{F}|\hat{V}_I(0)|\mathcal{F}_{N-1}\rangle}{(T_i - T_\mathcal{F} + i\epsilon)} \\
&\times \frac{\langle\mathcal{F}_{N-1}|\hat{V}_I(0)|\mathcal{F}_{N-2}\rangle}{\big(T_i - T_{\mathcal{F}_{N-1}} + i\epsilon\big)} \cdots \frac{\langle\mathcal{F}_1|\hat{V}_I(0)|i_I(-\infty)\rangle}{(T_i - T_{\mathcal{F}_1} + i\epsilon)}\,, \qquad (149)
\end{aligned}$$

where $T_i = k_i^- \equiv (\mathbf{k}_i{}^2 + m_i^2)/(2k_i^+)$ is the eigenvalue of $\hat{T}$ corresponding to the Fock state $|i_I(-\infty)\rangle$.

The wave-function renormalization constant Z_i can be determined be enforcing the proper normalization for both the dressed state $\big|\, i_H\big\rangle$ and the Fock states simultaneously.

3.4.4 Light-cone perturbation theory for S-matrix elements

Similarly, S-matrix elements (137) can be calculated in pertubation theory in the interaction picture as

$$\mathcal{S}_{fi} = \big\langle f_I(+\infty)\big|\mathcal{P}_+ \exp\left(-i\int_{-\infty}^{+\infty} \mathrm{d}x^+\, \hat{V}_I(x^+)\right) \big|\, i_I(-\infty)\big\rangle \qquad (150)$$

$$\begin{aligned}
=&\big\langle f_I(+\infty)\big|i_I(-\infty)\big\rangle + \sum_{N=1}^{\infty} \sum_{\mathcal{F}_N} \cdots \sum_{\mathcal{F}_0} (-i) 2\pi\delta(T_{\mathcal{F}_N} - T_{\mathcal{F}_0})\, \langle f_I(+\infty)|\mathcal{F}_N\rangle \\
&\times \langle\mathcal{F}_N|\hat{V}_I(0)|\mathcal{F}_{N-1}\rangle\, \frac{\langle\mathcal{F}_{N-1}|\hat{V}_I(0)|\mathcal{F}_{N-2}\rangle}{\big(T_{\mathcal{F}_0} - T_{\mathcal{F}_{N-1}} + i\epsilon\big)} \cdots \frac{\langle\mathcal{F}_1|\hat{V}_I(0)|\mathcal{F}_0\rangle}{(T_{\mathcal{F}_0} - T_{\mathcal{F}_1} + i\epsilon)}\, \langle\mathcal{F}_0|i_I(-\infty)\rangle\,. \qquad (151)
\end{aligned}$$

The main difference in the derivation of Eq. (151) compared to the LFWF case Eq. (147), is that the integration range in x^+ is not bounded in Eq. (150). Nevertheless, the interactions are still ordered along x^+. Hence, when integrating over the x^+s of the interaction insertions, a single integration is not bounded, and thus not done thanks to Eq. (146), giving a delta function instead of an energy denominator.

3.4.5 Interaction vertices: φ^4 theory

The last ingredient to be specified in order to use the light-cone perturbative expansions (147), (149) or (151) are the interaction vertices $\langle\mathcal{F}'|\hat{V}_I(0)|\mathcal{F}\rangle$, which obviously depend on the theory. As an illustration, let us come back to the scalar theory case, and consider the φ^4 theory as an example. After renormalizing the vacuum energy as in eq. (127) and isolating the free part $\hat{T}$ given in eq. (143), one is left with the interaction part of the light-front Hamiltonian

$$\hat{V}_I(x^+) = \int dx^- \int d^2\mathbf{x} \left[\frac{\lambda}{4!} :\hat{\varphi}(x)^4: + \frac{\lambda}{4}\, \langle 0|\hat{\varphi}(x)^2|0\rangle :\hat{\varphi}(x)^2: \right]. \qquad (152)$$

From the Fourier representation (107) of the scalar field, one finds the commutation relations

$$\left[\hat{a}(\underline{k}), \hat{\varphi}(x)\right] = e^{i\underline{k}\cdot x} \tag{153}$$

$$\left[\hat{\varphi}(x), \hat{a}^\dagger(\underline{k})\right] = e^{-i\underline{k}\cdot x} \tag{154}$$

at $x^+ = 0$. With these, it is easy to get the 1-to-3 and the 3-to-1 vertices

$$\begin{aligned}\langle 0|\hat{a}(\underline{k}_3)\hat{a}(\underline{k}_2)\hat{a}(\underline{k}_1)\hat{V}_I(0)\hat{a}^\dagger(\underline{p})|0\rangle &= \frac{\lambda}{4!}\int dx^- \int d^2\mathbf{x}\langle 0|\hat{a}(\underline{k}_3)\hat{a}(\underline{k}_2)\hat{a}(\underline{k}_1) : \hat{\varphi}(x)^4 : \hat{a}^\dagger(\underline{p})|0\rangle\Big|_{x^+=0} \\ &= \lambda\,(2\pi)^3\delta^{(3)}(\underline{k}_1 + \underline{k}_2 + \underline{k}_3 - \underline{p}) \end{aligned} \tag{155}$$

and

$$\langle 0|\hat{a}(\underline{k})\,\hat{V}_I(0)\,\hat{a}^\dagger(\underline{p}_1)\hat{a}^\dagger(\underline{p}_2)\hat{a}^\dagger(\underline{p}_3)|0\rangle = \lambda\,(2\pi)^3\delta^{(3)}(\underline{k} - \underline{p}_1 - \underline{p}_2 - \underline{p}_3)\,. \tag{156}$$

The integration over x^- and $\mathbf{x}$ implies the k^+ and $\mathbf{k}$ are conserved at each vertex. Since all k^+s are positive, all vertices between the vacuum and a non-trivial Fock state vanish.

The connected 2-to-2 vertex is

$$\frac{\lambda}{4!}\int dx^- \int d^2\mathbf{x}\langle 0|\hat{a}(\underline{k}_2)\hat{a}(\underline{k}_1)\ : \hat{\varphi}(x)^4 :\ \hat{a}^\dagger(\underline{p}_1)\hat{a}^\dagger(\underline{p}_2)|0\rangle \tag{157}$$

$$=\lambda\,(2\pi)^3\delta^{(3)}(\underline{k}_1 + \underline{k}_2 - \underline{p}_1 - \underline{p}_2)\,. \tag{158}$$

Note that here, only the first term in (152) is included. Indeed, the second term in (152) would have produced disconnected contributions, with at least one a spectator particle.

Finally, the second term in (152) leads to a 1-to-1 vertex

$$\begin{aligned}\langle 0|\hat{a}(\underline{k})\,\hat{V}_I(0)\,\hat{a}^\dagger(\underline{p})|0\rangle &= \frac{\lambda}{2}\int dx^- \int d^2\mathbf{x}\, e^{i(\underline{k}-\underline{p})\cdot x}\,\langle 0|\hat{\varphi}(x)^2|0\rangle \\ &= (2\pi)^3\delta^{(3)}(\underline{k} - \underline{p})\,\frac{\lambda}{2}\left[\int \frac{d^3\underline{q}}{(2\pi)^3}\,\frac{\theta(q^+)}{2q^+}\right]. \end{aligned} \tag{159}$$

This corresponds to a divergent 1-loop tadpole insertion, which typically contributes only to mass renormalization.

References

[1] P. A. M. Dirac, *Forms of Relativistic Dynamics*, *Rev. Mod. Phys.* **21** (1949) 392–399.

[2] J. B. Kogut and D. E. Soper, *Quantum Electrodynamics in the Infinite Momentum Frame*, *Phys.Rev.* **D1** (1970) 2901–2913.

[3] J. Bjorken, J. B. Kogut, and D. E. Soper, *Quantum Electrodynamics at Infinite Momentum: Scattering from an External Field*, *Phys.Rev.* **D3** (1971) 1382.

[4] K. Yamawaki, *Zero mode problem on the light front*, in *QCD, light cone physics and hadron phenomenology. Proceedings, 10th Nuclear Summer School and Symposium, NuSS'97, Seoul, Korea, June 23-28, 1997*, pp. 116–199, 1998. `hep-th/9802037`.

[5] S. R. Beane, *Broken Chiral Symmetry on a Null Plane*, *Annals Phys.* **337** (2013) 111–142, [`arXiv:1302.1600`].

[6] A. H. Mueller, *Soft gluons in the infinite momentum wave function and the BFKL pomeron*, *Nucl. Phys.* **B415** (1994) 373–385.

[7] A. H. Mueller and B. Patel, *Single and double BFKL pomeron exchange and a dipole picture of high-energy hard processes*, *Nucl. Phys.* **B425** (1994) 471–488, [`hep-ph/9403256`].

[8] Y. V. Kovchegov, *Small-x F2 structure function of a nucleus including multiple pomeron exchanges*, *Phys. Rev.* **D60** (1999) 034008, [`hep-ph/9901281`].

[9] L. D. McLerran and R. Venugopalan, *Computing quark and gluon distribution functions for very large nuclei*, *Phys. Rev.* **D49** (1994) 2233–2241, [`hep-ph/9309289`].

[10] L. D. McLerran and R. Venugopalan, *Gluon distribution functions for very large nuclei at small transverse momentum*, *Phys. Rev.* **D49** (1994) 3352–3355, [`hep-ph/9311205`].

[11] L. D. McLerran and R. Venugopalan, *Green's functions in the color field of a large nucleus*, *Phys. Rev.* **D50** (1994) 2225–2233, [`hep-ph/9402335`].

[12] G. A. Chirilli, B.-W. Xiao, and F. Yuan, *Inclusive Hadron Productions in pA Collisions*, *Phys.Rev.* **D86** (2012) 054005, [`arXiv:1203.6139`].

[13] M. Lublinsky and Y. Mulian, *High Energy QCD at NLO: from light-cone wave function to JIMWLK evolution*, *JHEP* **05** (2017) 097, [`arXiv:1610.0345`].

[14] G. Beuf, *Dipole factorization for DIS at NLO: Loop correction to the $\gamma^*_{T,L} \to q\bar{q}$ light-front wave functions*, *Phys. Rev.* **D94** (2016), no. 5 054016, [`arXiv:1606.0077`].

[15] G. Beuf, *Dipole factorization for DIS at NLO: Combining the $q\bar{q}$ and $q\bar{q}g$ contributions*, *Phys. Rev.* **D96** (2017), no. 7 074033, [`arXiv:1708.0655`].

[16] H. Hanninen, T. Lappi, and R. Paatelainen, *One-loop corrections to light cone wave functions: the dipole picture DIS cross section*, *Annals Phys.* **393** (2018) 358–412, [`arXiv:1711.0820`].

[17] G. Beuf, T. Lappi, and R. Paatelainen, *Massive quarks in NLO dipole factorization for DIS: Longitudinal photon*, *Phys. Rev. D* **104** (2021), no. 5 056032, [`arXiv:2103.1454`].

[18] G. Beuf, T. Lappi, and R. Paatelainen, *Massive quarks in NLO dipole factorization for DIS: Transverse photon*, *Phys. Rev. D* **106** (2022), no. 3 034013, [`arXiv:2204.0248`].

[19] W.-M. Zhang, *Light front dynamics and light front QCD*, *Chin. J. Phys.* **32** (1994) 717–808, [hep-ph/9412244].

[20] S. J. Brodsky, H.-C. Pauli, and S. S. Pinsky, *Quantum Chromodynamics and Other Field Theories on the Light Cone*, *Phys. Rept.* **301** (1998) 299–486, [hep-ph/9705477].

[21] H. Leutwyler and J. Stern, *Relativistic Dynamics on a Null Plane*, *Annals Phys.* **112** (1978) 94.

[22] R. Floreanini and R. Jackiw, *Selfdual Fields as Charge Density Solitons*, *Phys. Rev. Lett.* **59** (1987) 1873.

[23] L. D. Faddeev and R. Jackiw, *Hamiltonian Reduction of Unconstrained and Constrained Systems*, *Phys. Rev. Lett.* **60** (1988) 1692–1694.

[24] S.-J. Chang, R. G. Root, and T.-M. Yan, *Quantum field theories in the infinite momentum frame. 1. Quantization of scalar and Dirac fields*, *Phys. Rev.* **D7** (1973) 1133–1148.