Basics of Light-Cone Quantization and Light-Cone Perturbation Theory,* Part 2: From free fermions and photons to QCD

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1 Light-front quantization of free Dirac fermions

1.1 Constraint equation and quantization

Before addressing the QCD or QED case, let us consider the light-front quantization of free Dirac fermions. The Lagrangian density writes

$$\mathcal{L} = \overline{\Psi}(x)\left(i\partial\!\!\!/ - m\right)\Psi(x)\,, \tag{1}$$

which gives the classical equations of motion

$$\left(i\partial\!\!\!/ - m\right)\Psi(x) = 0 \tag{2}$$

$$\overline{\Psi}(x)\left(i\overleftarrow{\partial\!\!\!/} + m\right) = 0\,. \tag{3}$$

In view of the light-front quantization, one should introduce the following objects:

$$\mathcal{P}_G \equiv \frac{\gamma^-\gamma^+}{2} = \frac{\gamma^0\gamma^+}{\sqrt{2}}\,, \tag{4}$$

$$\mathcal{P}_B \equiv \frac{\gamma^+\gamma^-}{2} = \frac{\gamma^0\gamma^-}{\sqrt{2}}\,. \tag{5}$$

Using only the fundamental anti-commutator

$$\{\gamma^\mu\,,\,\gamma^\nu\} = 2\,g^{\mu\nu}\,, \tag{6}$$

it is straightforward to show that $\mathcal{P}_G$ and $\mathcal{P}_B$ are projections, meaning

$$\mathcal{P}_G\mathcal{P}_G = \mathcal{P}_G \qquad \text{and} \qquad \mathcal{P}_B\mathcal{P}_B = \mathcal{P}_B\,, \tag{7}$$

which are complimentary:

$$\mathcal{P}_G\mathcal{P}_B = \mathcal{P}_B\mathcal{P}_G = 0 \qquad \text{and} \qquad \mathcal{P}_G + \mathcal{P}_B = \mathbf{1}\,. \tag{8}$$

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Each of them projects out 2 of the 4 components of the Dirac spinors. Noting Ψ_G and Ψ_B the projected spinors, one has

$$\begin{aligned} \mathcal{P}_G\,\Psi &= \Psi_G\,, & \Psi^\dagger\,\mathcal{P}_G &= {\Psi_G}^\dagger\,, & \overline{\Psi}\,\gamma^+ &= \overline{\Psi_G}\,\gamma^+\,,\\ \mathcal{P}_B\,\Psi &= \Psi_B\,, & \Psi^\dagger\,\mathcal{P}_B &= {\Psi_B}^\dagger\,, & \overline{\Psi}\,\gamma^- &= \overline{\Psi_B}\,\gamma^-\,. \end{aligned} \tag{9}$$

Then, when multiplying the equation of motion (2) by γ^+ (Remember that Eq. (6) implies that $\gamma^+\gamma^+ = g^{++} = 0$ and $\gamma^-\gamma^- = g^{--} = 0$.):

$$\begin{aligned} 0 &= \gamma^+\left(i\gamma^-\partial_- + i\gamma^j\partial_j - m\right)\Psi(x)\\ &= 2i\partial_-\Psi_B(x) + \gamma^+\left(i\gamma^j\partial_j - m\right)\Psi_G(x)\,. \end{aligned} \tag{10}$$

That equation has no derivatives with respect to x^+. Hence, for the light-front formulation, it is not an equation of motion but a constraint equation. By contrast, when multiplying the equation of motion (2) by γ^-, we have

$$\begin{aligned} 0 &= \gamma^-\left(i\gamma^+\partial_+ + i\gamma^j\partial_j - m\right)\Psi(x)\\ &= 2i\partial_+\Psi_G(x) + \gamma^-\left(i\gamma^j\partial_j - m\right)\Psi_B(x)\,, \end{aligned} \tag{11}$$

which is the equation of motion for $\Psi_G(x)$ along x^+. None of the projections (10) and (11) of the Dirac equation contain derivatives of $\Psi_B(x)$ in x^+. Hence, in the front form of the dynamics, the projection $\Psi_B(x)$ is an auxiliary field, which can be determined in terms of $\Psi_G(x)$ at each x^+ thanks to the constraint equation (10), as

$$\Psi_B(x) = \left(\frac{i}{2\partial_-}\right)\gamma^+\left(i\gamma^j\partial_j - m\right)\Psi_G(x)\,. \tag{12}$$

Therefore, only the $\Psi_G(x)$ components (called good components) of Dirac spinors should be considered as independent degree of freedom for the light-front quantization, by contrast to the $\Psi_B(x)$ components (called bad components).

The momentum density conjugate to $\Psi_G(x)$ is

$$\pi^+_{\Psi_G}(x) = \frac{\partial\mathcal{L}}{\partial(\partial_+\Psi_G(x))} = i\,\overline{\Psi_G}(x)\,\gamma^+\,. \tag{13}$$

One can thus quantize $\Psi_G(x)$ with the anti-commutation relations

$$\Big\{\left(\Psi_G(x)\right)_a, \left(\Psi_G(y)\right)_b\Big\} = 0 \tag{14}$$

$$\Big\{\left(i\,\overline{\Psi_G}(x)\,\gamma^+\right)_a, \left(i\,\overline{\Psi_G}(y)\,\gamma^+\right)_b\Big\} = 0 \tag{15}$$

$$\Big\{\left(\Psi_G(x)\right)_a, \left(i\,\overline{\Psi_G}(y)\,\gamma^+\right)_b\Big\} = i\left(\mathcal{P}_G\right)_{ab}\delta(x^- - y^-)\delta^{(2)}(\mathbf{x}-\mathbf{y}) \tag{16}$$

for $x^+ = y^+$, where a, b correspond to Dirac spinor indices. The relation Eq. (16) can also be written as

$$\Big\{\left(\Psi_G(x)\right)_a, \left({\Psi_G}^\dagger(y)\right)_b\Big\} = \frac{1}{\sqrt{2}}\left(\mathcal{P}_G\right)_{ab}\delta(x^- - y^-)\delta^{(2)}(\mathbf{x}-\mathbf{y}) \tag{17}$$

or as

$$\Big\{\left(\Psi_G(x)\right)_a, \left(\overline{\Psi_G}(y)\right)_b\Big\} = \frac{1}{\sqrt{2}}\,(\gamma^0)_{ab}\delta(x^- - y^-)\delta^{(2)}(\mathbf{x}-\mathbf{y})\,. \tag{18}$$

1.2 Poincaré currents and charges

For the free Dirac spinor, the Noether currents for Poincaré symmetries are given by

$$T^{\mu\nu}(x) = \frac{\partial\mathcal{L}}{\partial(\partial_\mu\Psi(x))}\,\partial^\nu\Psi(x) - g^{\mu\nu}\,\mathcal{L} = i\overline{\Psi}(x)\,\gamma^\mu\partial^\nu\Psi(x) - g^{\mu\nu}\,\mathcal{L}\,, \tag{19}$$

and

$$\begin{aligned} J^{\rho\mu\nu}(x) &= x^\mu\,T^{\rho\nu}(x) - x^\nu\,T^{\rho\mu}(x) + \frac{\partial\mathcal{L}}{\partial(\partial_\rho\Psi(x))}\,\Sigma^{\mu\nu}_{\text{sp.}}\Psi(x) \\ &= x^\mu\,T^{\rho\nu}(x) - x^\nu\,T^{\rho\mu}(x) + i\overline{\Psi}(x)\,\gamma^\rho\,\frac{1}{4}\,[\gamma^\mu,\gamma^\nu]\,\Psi(x)\,. \end{aligned} \tag{20}$$

The kinematic components P^μ are thus

$$P^\nu = \int dx^- \!\int d^2\mathbf{x}\;T^{+\nu}(x) = i\int dx^- \!\int d^2\mathbf{x}\;\overline{\Psi_G}(x)\,\gamma^+\partial^\nu\Psi_G(x)\,, \tag{21}$$

for $\nu \neq -$. Moreover, one finds the generators of transverse Galilean boosts as

$$\begin{aligned} M^{+i} &= \int dx^- \!\int d^2\mathbf{x}\;J^{++i}(x) \\ &= i\int dx^- \!\int d^2\mathbf{x}\;\overline{\Psi_G}(x)\,\gamma^+\Big(-x^+\,\partial_i - x^i\,\partial_-\Big)\Psi_G(x)\,, \end{aligned} \tag{22}$$

and the one of the rotations in the transverse plane as

$$\begin{aligned} M^{ij} &= \int dx^- \!\int d^2\mathbf{x}\;J^{+ij}(x) \\ &= i\int dx^- \!\int d^2\mathbf{x}\;\overline{\Psi_G}(x)\,\gamma^+\Big(-x^i\,\partial_j + x^j\,\partial_i + \frac{1}{4}\,\big[\gamma^i,\gamma^j\big]\Big)\Psi_G(x)\,. \end{aligned} \tag{23}$$

In all of the kinematic Poincaré charges P^+, P^j, M^{+i} and M^{ij}, there is a γ^+ matrix (and no γ^- matrix) which is projecting out the bad components Ψ_B, so that the kinematic generators have simple expressions in terms of the good components Ψ_G. Thanks to the commutation relations (14), (15) and (16), one can check that the kinematic Poincaré generators at quantum level have the correct commutation relations between themselves, and with $\Psi_G(x)$.

As in the scalar case, the dynamic Poincaré charges P^-, M^{-i} and M^{-+} involve the component $T^{+-}(x)$ of the energy momentum tensor, which would be sensitive to the interactions in an interacting theory. For the free Dirac fermion, it writes

$$\begin{aligned} T^{+-}(x) = &-\overline{\Psi}(x)\,\big(i\gamma^j\partial_j + i\gamma^-\partial_- - m\big)\,\Psi(x) \\ = &-\overline{\Psi_B}(x)\,\big(i\gamma^j\partial_j - m\big)\,\Psi_G(x) - \overline{\Psi_G}(x)\,\big(i\gamma^j\partial_j - m\big)\,\Psi_B(x) \\ &-\overline{\Psi_B}(x)\,\big(i\gamma^-\partial_-\big)\,\Psi_B(x)\,. \end{aligned} \tag{24}$$

As it involves γ^j and γ^-, both the good and the bad components contribute to the dynamic generators. Using the solution (12) of the constraint equation (10), the Hamiltonian density $T^{+-}(x)$ can be rewritten in terms of the good components only, as

$$T^{+-}(x) = -\,\overline{\Psi_G}(x)\,\big(i\gamma^j\partial_j - m\big)\left(\frac{i}{2\partial_-}\right)\gamma^+\,\big(i\gamma^l\partial_l - m\big)\,\Psi_G(x)\,, \tag{25}$$

at the price of becoming non-local due to the $1/\partial_-$.

1.3 Fourier representation and Fock states

A natural choice of base of the 2 dimensional space spanned by the good components of spinors is provided thanks to the matrix $\Sigma^{ij}_{\text{sp.}}$. The 2×2 bloc associated with good components of the matrix

$$i\,\Sigma^{12}_{\text{sp.}} \equiv \frac{i}{4}\,\left[\gamma^1,\gamma^2\right] \tag{26}$$

can be diagonalized, and the two eigenvalues are $+1/2$ and $-1/2$. Since $\Sigma^{12}_{\text{sp.}}$ is the matrix generating the rotations around the longitudinal axis, these eigenvalues correspond to the component of the spin along the longitudinal axis, also called light-front helicity.

Hence, there are two eigenvectors $u_G(k^+,h)$ defined as

$$\frac{i}{4}\,\left[\gamma^1,\gamma^2\right] u_G(k^+,h) = h\, u_G(k^+,h)\,, \tag{27}$$

with $h=\pm 1/2$, forming a basis for the good components of spinors. For later convenience, the normalization of the eigenvectors is chosen such that

$$\overline{u_G}(k'^+,h')\gamma^+ u_G(k^+,h) = \sqrt{2k^+}\,\sqrt{2k'^+}\,\delta_{h,h'}\,, \tag{28}$$

for $k^+>0$ and $k'^+>0$. For later convenience again, let us introduce as well the two $v_G(k^+,h)$, defined as

$$v_G(k^+,h) \equiv u_G(k^+,-h)\,. \tag{29}$$

One has the completeness relations

$$\sum_{h=\pm\frac{1}{2}} u_G(k^+,h)\,\overline{u_G}(k^+,h)\,\gamma^+ = \sum_{h=\pm\frac{1}{2}} v_G(k^+,h)\,\overline{v_G}(k^+,h)\,\gamma^+ = 2k^+\,\mathcal{P}_G\,. \tag{30}$$

Thanks to these eigenvectors, one can define the operators

$$\hat{b}(\underline{k},h,x^+) = \int dx^- \int d^2\mathbf{x}\; e^{i\underline{k}\cdot x}\, \overline{u_G}(k^+,h)\gamma^+\Psi_G(x) \tag{31}$$

$$\hat{b}^\dagger(\underline{k},h,x^+) = \int dx^- \int d^2\mathbf{x}\; e^{-i\underline{k}\cdot x}\, \overline{\Psi_G}(x)\gamma^+ u_G(k^+,h) \tag{32}$$

$$\hat{d}(\underline{k},h,x^+) = \int dx^- \int d^2\mathbf{x}\; e^{i\underline{k}\cdot x}\, \overline{\Psi_G}(x)\gamma^+ v_G(k^+,h) \tag{33}$$

$$\hat{d}^\dagger(\underline{k},h,x^+) = \int dx^- \int d^2\mathbf{x}\; e^{-i\underline{k}\cdot x}\, \overline{v_G}(k^+,h)\gamma^+\Psi_G(x)\,, \tag{34}$$

or equivalently, the Fourier representation

$$\Psi_G(x) = \sum_{h=\pm\frac{1}{2}} \int \frac{d^3\underline{k}}{(2\pi)^3}\,\frac{\theta(k^+)}{2k^+}\,\Big\{ e^{-i\underline{k}\cdot x}\,\hat{b}(\underline{k},h,x^+)u_G(k^+,h) + e^{i\underline{k}\cdot x}\,\hat{d}^\dagger(\underline{k},h,x^+)v_G(k^+,h)\Big\}\,. \tag{35}$$

The non-trivial anti-commutation relations of the operators $\hat{b}$, $\hat{b}^\dagger$, $\hat{d}$ and $\hat{d}^\dagger$ are

$$\Big\{\hat{b}(\underline{k},h,x^+),\hat{b}^\dagger(\underline{k}',h',x^+)\Big\} = 2k^+\,(2\pi)^3\,\delta^{(3)}(\underline{k}-\underline{k}')\,\delta_{h,h'} \tag{36}$$

$$\Big\{\hat{d}(\underline{k},h,x^+),\hat{d}^\dagger(\underline{k}',h',x^+)\Big\} = 2k^+\,(2\pi)^3\,\delta^{(3)}(\underline{k}-\underline{k}')\,\delta_{h,h'}\,, \tag{37}$$

as expected for annihilation and creation operators of fermions. They also satisfy the correct commutation relations with the kinematic Poincaré generators, the same as the $\hat{a}$ and $\hat{a}^\dagger$ operators in the scalar case.

Studying $\hat{P}^+$ and $\hat{P}^j$ using the same method as in the scalar case,[1] one find the same situation as in the scalar theory: $\hat{P}^+$ and $\hat{P}^j$ are normal-ordered up to c-number contributions which vanish when in proper UV regularization scheme preserving Poincaré invariance. Then

$$\hat{P}^\mu = \sum_{h=\pm\frac{1}{2}} \int \frac{d^3\underline{k}}{(2\pi)^3}\,\frac{\theta(k^+)}{2k^+}\;k^\mu\;\Big(\hat{b}^\dagger(\underline{k},h,x^+)\,\hat{b}(\underline{k},h,x^+) + \hat{d}^\dagger(\underline{k},h,x^+)\,\hat{d}(\underline{k},h,x^+)\Big) \tag{38}$$

for $\mu \neq -$.

Similarly to the scalar case, the vacuum state can be defined by

$$\hat{b}(\underline{k},h,x^+)|0\rangle = \hat{d}(\underline{k},h,x^+)|0\rangle = 0\,, \tag{39}$$

and the construction of the Fock space proceeds in the same way as in the scalar case, except that now both fermions and antifermions can be included, thanks to the operators $\hat{b}^\dagger(\underline{k},h)$ and $\hat{d}^\dagger(\underline{k},h)$ respectively.

1.4 Full non-interacting spinor field

Essentially all the results so far are also valid for interacting Dirac spinors as well. In the free Dirac spinor case, one can in addition find a Fourier representation

$$\Psi(x) = \sum_{h=\pm\frac{1}{2}} \int \frac{d^3\underline{k}}{(2\pi)^3}\,\frac{\theta(k^+)}{2k^+}\;\Big\{e^{-ik\cdot x}\,\hat{b}(\underline{k},h)u(\underline{k},h) + e^{ik\cdot x}\,\hat{d}^\dagger(\underline{k},h)v(\underline{k},h)\Big\}\,, \tag{40}$$

of the full spinor field. Here, the creation and annihilation operators are taken at $x^+ = 0$, and the x^+ dependence of the field is accounted for by the phase factors, with $k^- \equiv (\mathbf{k}^2 + m^2)/(2k^+)$. The spinors $u(\underline{k},h)$ and $v(\underline{k},h)$ are extensions of the ones of the previous section:

$$\begin{aligned}\mathcal{P}_G u(\underline{k},h) &= u_G(k^+,h)\\ \mathcal{P}_G v(\underline{k},h) &= v_G(k^+,h)\,.\end{aligned} \tag{41}$$

And their bad components are defined in such a way that $\Psi(x)$ satisfies the Dirac equation (2). Hence, they have to obey the momentum space Dirac equations

$$\begin{aligned}(\not{k} - m)\,u(\underline{k},h) = (\not{k} + m)\,v(\underline{k},h) = 0\\ \overline{u}(\underline{k},h)\,(\not{k} - m) = \overline{v}(\underline{k},h)\,(\not{k} + m) = 0\,.\end{aligned} \tag{42}$$

[1] At quantum level, remember that one should anti-symmetrize the expressions of the Poincaré generators in terms of the $\Psi_G(x)$ field.

The solutions for the bad component of these spinors, fixed by the momentum-space version of the constraint equation (10) are

$$u_B(\underline{k},h) = \frac{\gamma^+}{2k^+}\,\left(\mathbf{k}^j\gamma^j+m\right)\,u_G(k^+,h), \tag{43}$$

$$v_B(\underline{k},h) = \frac{\gamma^+}{2k^+}\,\left(\mathbf{k}^j\gamma^j-m\right)\,v_G(k^+,h)\,, \tag{44}$$

which corresponds, for the conjugate spinors, to

$$\overline{u_B}(\underline{k},h) = \overline{u_G}(k^+,h)\,\left(\mathbf{k}^j\gamma^j+m\right)\,\frac{\gamma^+}{2k^+}\,, \tag{45}$$

$$\overline{v_B}(\underline{k},h) = \overline{v_G}(k^+,h)\,\left(\mathbf{k}^j\gamma^j-m\right)\,\frac{\gamma^+}{2k^+}\,.. \tag{46}$$

Then, the spinors $u(\underline{k},h)$ and $v(\underline{k},h)$ satisfy the completeness relations

$$\begin{aligned}\sum_{h=\pm 1/2} u(\underline{k},h)\;\overline{u}(\underline{k},h)\; &= \not{k}+m\\ \sum_{h=\pm 1/2} v(\underline{k},h)\;\overline{v}(\underline{k},h)\; &= \not{k}-m\,,\end{aligned} \tag{47}$$

with the again the notation $k^- \equiv (\mathbf{k}^2+m^2)/(2k^+))$.

2 Light-front quantization of free photons

For free photons, the Lagrangian density is

$$\mathcal{L} = -\frac{1}{4}F_{\mu\nu}F^{\mu\nu}\,, \tag{48}$$

with

$$F_{\mu\nu} \equiv \partial_\mu A_\nu(x) - \partial_\nu A_\mu(x)\,. \tag{49}$$

One has the equations of motion

$$\partial_\mu F^{\mu\nu} = 0\,, \tag{50}$$

as well as the Noether currents for Poincaré symmetries

$$T^{\mu\nu}(x) = \frac{\partial\mathcal{L}}{\partial(\partial_\mu A_\rho(x))}\,\partial^\nu A_\rho(x) - g^{\mu\nu}\,\mathcal{L} = -F^{\mu\rho}(x)(\partial^\nu A_\rho(x)) - g^{\mu\nu}\,\mathcal{L}\,, \tag{51}$$

and

$$\begin{aligned}J^{\rho\mu\nu}(x) &= x^\mu\,T^{\rho\nu}(x) - x^\nu\,T^{\rho\mu}(x) + \frac{\partial\mathcal{L}}{\partial(\partial_\rho A^\sigma(x))}\,\left(\Sigma^{\mu\nu}_{\text{vect.}}\right)^\sigma{}_\eta A^\eta(x)\\ &= x^\mu\,T^{\rho\nu}(x) - x^\nu\,T^{\rho\mu}(x) - F^{\rho\mu}(x)\,A^\nu(x) + F^{\rho\nu}(x)\,A^\mu(x)\,.\end{aligned} \tag{52}$$

The momentum density conjugate to $A_\nu(x)$ is

$$\pi^+_{A_\nu}(x) = \frac{\partial\mathcal{L}}{\partial(\partial_+ A_\nu(x))} = -F^{+\nu}(x)\,. \tag{53}$$

This vanishes for $\nu = +$, so that $A^-(x)$ is not an independent degree of freedom. Obviously, since the free photon theory is a gauge theory, it has to be a constrained dynamical system. As usual, one has to choose a gauge. In the context of light-front quantization, the most convenient gauge is by far the light-cone gauge

$$A^+(x) = 0\,. \tag{54}$$

In that gauge, one finds from Eqs. (51) and (52) the following expressions for the kinematic Poincaré generators

$$P^\nu = \int dx^- \int d^2\mathbf{x}\ \left(\partial_- A^j(x)\right) \left(\partial^\nu A^j(x)\right)\,, \tag{55}$$

for $\nu \neq -$,

$$M^{+i} = \int dx^- \int d^2\mathbf{x}\ \left(\partial_- A^j(x)\right) \Big(-x^+\,\partial_i - x^i\,\partial_-\Big) A^j(x)\,, \tag{56}$$

and

$$M^{ij} = \int dx^- \int d^2\mathbf{x}\ \left(\partial_- A^l(x)\right) \bigg(\left[-x^i\,\partial_j + x^j\,\partial_i\right] A^l(x) - g^i{}_l\, A^j(x) + g^j{}_l\, A^i(x) \bigg)\,. \tag{57}$$

Hence, in the light-cone gauge Eq. (54), the kinematic Poincaré generators dependent only on the components $A^j(x)$ of the gauge field, and not on the dependent component $A^-(x)$. Moreover, the expressions (55) and (56) are the same as in the scalar case, but with a summation over j. The expression (57) for M^{ij} has also the same form, but with extra terms, induced by $\Sigma^{\mu\nu}_{\text{vect.}}$.

In the light-cone gauge, the component $\nu = +$ of the equations of motion (50) becomes

$$-\partial_-{}^2 A^-(x) - \partial_-\partial_j A^j(x) = 0\,. \tag{58}$$

This is a constraint equation, allowing to constrain $A^-(x)$ with the independent components $A^j(x)$. In order to fully determine the component $A^-(x)$, the equation (58) is actually not enough. One also needs to specify some boundary conditions at $x^- \to \pm\infty$. This is related with the residual gauge symmetry left after imposing the condition (54), which does not entirely fix the gauge (see for example appendix D in Ref. [1]). Different choice lead to results for $A^-(x)$ which differ by zero modes, independent of x^-. For simplicity, we will not discuss further this technical (but important) issue, and simply solve the constraint equation (58) by writing

$$A^-(x) = -\left(\frac{1}{\partial_-}\right) \partial_j A^j(x)\,. \tag{59}$$

Concerning P^-, in light-cone gauge, one finds from (51) after some algebra and integration by parts

$$\begin{aligned} P^- &= \frac{1}{2}\int dx^- \int d^2\mathbf{x}\ \left[\left(\partial_i A^j(x)\right) \left(\partial_i A^j(x)\right) - \Big(\partial_- A^-(x) + \partial_j A^j(x)\Big)^2 \right] \\ &= \frac{1}{2}\int dx^- \int d^2\mathbf{x}\ \left(\partial_i A^j(x)\right) \left(\partial_i A^j(x)\right)\,, \end{aligned} \tag{60}$$

using the solution (59) of the constraint in the last step.

At this point, it is quite clear that the free photon field behaves as a collection of two free massless scalar fields. Then, it can be quantized on the light-front by applying the same procedure as in the scalar field case for each transverse component $A^j(x)$ of the photon field, and thus using the commutation relation

$$\begin{aligned}\Big[A^j(x), A^l(y)\Big] &= -\frac{i}{4}\,\delta^{jl}\epsilon(x^- - y^-)\,\delta^{(2)}(\mathbf{x}-\mathbf{y})\\ &= +\frac{i}{4}\,g^{jl}\epsilon(x^- - y^-)\,\delta^{(2)}(\mathbf{x}-\mathbf{y})\end{aligned} \tag{61}$$

for $x^+ = y^+$.

Introducing any basis of 2-dimensional polarization vectors ε^j_λ (for example linear or circular polarization vectors), one can then write the Fourier representation

$$A^j(x) = \sum_\lambda \int \frac{d^3\underline{k}}{(2\pi)^3}\,\frac{\theta(k^+)}{2k^+}\,\Big\{e^{-i\underline{k}\cdot x}\,\hat{a}(\underline{k},\lambda,x^+)\,\varepsilon^j_\lambda + e^{i\underline{k}\cdot x}\,\hat{a}^\dagger(\underline{k},\lambda,x^+)\,\varepsilon^{j\,*}_\lambda\Big\}\,, \tag{62}$$

or equivalently

$$\hat{a}(\underline{k},\lambda,x^+) = 2i\int dx^- \int d^2\mathbf{x}\; e^{i\underline{k}\cdot x}\partial_- A^j(x)\,\varepsilon^{j\,*}_\lambda \tag{63}$$

$$\hat{a}^\dagger(\underline{k},\lambda,x^+) = -2i\int dx^- \int d^2\mathbf{x}\; e^{-i\underline{k}\cdot x}\partial_- A^j(x)\,\varepsilon^{j}_\lambda\,. \tag{64}$$

The 2-dimensional polarization vectors have to obey

$$\sum_\lambda \varepsilon^i_\lambda\,\varepsilon^{j\,*}_\lambda = -g^{ij} = \delta^{ij} \tag{65}$$

$$-g_{ij}\,\varepsilon^i_{\lambda_1}\,\varepsilon^{j\,*}_{\lambda_2} = \varepsilon^j_{\lambda_1}\,\varepsilon^{j\,*}_{\lambda_2} = \delta_{\lambda_1,\lambda_2}\,. \tag{66}$$

The vacuum and the Fock space for photons is then constructed in the same way as for scalars. Again, that construction will also be valid in the presence of interactions, which would modify only the dynamical Poincaré generators and the constraint equation (58).

In the specific case of free photons, one can write the Fourier representation

$$A^\mu(x) = \sum_\lambda \int \frac{d^3\underline{k}}{(2\pi)^3}\,\frac{\theta(k^+)}{2k^+}\,\left\{e^{-ik\cdot x}\,\hat{a}(\underline{k},\lambda)\epsilon^\mu_\lambda(\underline{k}) + e^{ik\cdot x}\,\hat{a}^\dagger(\underline{k},\lambda)\epsilon^{\mu\,*}_\lambda(\underline{k})\right\}\,, \tag{67}$$

including all of the components of the photon field, with the creation and annihilation operators taken at $x^+ = 0$, and with the x^+ dependence of the field accounted for by the phase factors, where $k^- \equiv \mathbf{k}^2/(2k^+)$ in order to solve the equations of motion (50).

The 4-dimensional polarization vectors $\epsilon^\mu_\lambda(\underline{k})$ are given by

$$\epsilon^+_\lambda(\underline{k}) = 0, \tag{68}$$

$$\epsilon^j_\lambda(\underline{k}) = \varepsilon^j_\lambda\,, \tag{69}$$

$$\epsilon^-_\lambda(\underline{k}) = \frac{\mathbf{k}^j\,\varepsilon^j_\lambda}{k^+}\,. \tag{70}$$

The relation (68) is imposed by the gauge condition, whereas the relation (70) provides the solution to the constraint equation (58). The 4-dimensional polarization vectors obey the completeness relation

$$\sum_{\lambda} \epsilon_{\lambda}^{\mu}(\underline{k})\, \epsilon_{\lambda}^{\nu\,*}(\underline{k}) \;= -g^{\mu\nu} + \frac{k^{\mu}g^{\nu+} + g^{\mu+}k^{\nu}}{k^{+}} \tag{71}$$

where $k^{-} \equiv \mathbf{k}^2/(2k^+)$. As a remark, a specific choice of condition to fix the residual gauge invariance, and thus the boundary conditions to solve Eq. (58) would determine how the k^+ denominators in Eqs. (70) and (71) should be regularized at $k^+ = 0$.

3 QCD and QED on the light-front

The Lagrangian density for QCD is

$$\mathcal{L} = -\frac{1}{4} F_{\mu\nu}^{a} F_{a}^{\mu\nu} + \overline{\Psi}(x) \left(i\not{D} - m \right) \Psi(x)\,, \tag{72}$$

with[2]

$$D_{\mu}\Psi(x) \equiv \partial_{\mu}\Psi(x) + igA_{\mu}^{a}(x)t^{a}\,\Psi(x) \tag{73}$$

and

$$F_{\mu\nu}^{a} \equiv \partial_{\mu}A_{\nu}^{a}(x) - \partial_{\nu}A_{\mu}^{a}(x) - gf^{abc}A_{\mu}^{b}(x)A_{\nu}^{c}(x)\,. \tag{74}$$

By contrast, for QED

$$\mathcal{L} = -\frac{1}{4} F_{\mu\nu} F^{\mu\nu} + \overline{\Psi}(x) \left(i\not{D} - m \right) \Psi(x)\,, \tag{75}$$

with now

$$D_{\mu}\Psi(x) \equiv \left(\partial_{\mu} + iee_f A_{\mu}(x)\right) \Psi(x) \tag{76}$$

and

$$F_{\mu\nu} \equiv \partial_{\mu}A_{\nu}(x) - \partial_{\nu}A_{\mu}(x)\,. \tag{77}$$

These two theories have a very similar expressions. We can obtain the QED case by performing the replacements $gt^a \mapsto ee_f$ and $f^{abc} \mapsto 0$ in QCD expressions, as well as dropping the color indices $a, b, \ldots$.

3.1 Light-front quantization of QCD

For QCD, the Noether currents for Poincaré symmetries are

$$T^{\mu\nu}(x) = i\overline{\Psi}(x)\,\gamma^{\mu}\partial^{\nu}\Psi(x) - F_{a}^{\mu\rho}(x)(\partial^{\nu}A_{\rho}^{a}(x)) - g^{\mu\nu}\,\mathcal{L}\,, \tag{78}$$

[2] In the literature, another definition for the covariant derivative is also very common, with the opposite sign instead: $-igA_{\mu}^{a}(x)t^{a}$. They lead to the same final perturbative results expressed in terms of $\alpha_s = g^2/4\pi$, but different signs in many intermediate expressions.

and

$$
\begin{aligned}
J^{\rho\mu\nu}(x) =& x^\mu\, T^{\rho\nu}(x) - x^\nu\, T^{\rho\mu}(x) + \frac{\partial\mathcal{L}}{\partial(\partial_\rho \Psi(x))}\, \Sigma^{\mu\nu}_{\text{sp.}} \Psi(x) \\
&+ \frac{\partial\mathcal{L}}{\partial(\partial_\rho A_a^\sigma(x))}\, (\Sigma^{\mu\nu}_{\text{vect.}})^\sigma{}_\eta A_a^\eta(x) \\
=& x^\mu\, T^{\rho\nu}(x) - x^\nu\, T^{\rho\mu}(x) + i\overline{\Psi}(x)\,\gamma^\rho\, \frac{1}{4}\, [\gamma^\mu, \gamma^\nu]\, \Psi(x) \\
&- F_a^{\rho\mu}(x)\, A_a^\nu(x) + F_a^{\rho\nu}(x)\, A_a^\mu(x)\,.
\end{aligned}
\tag{79}
$$

From these expressions, and choosing the light-cone gauge

$$
A_a^+(x) = 0\,, \tag{80}
$$

it is easy to find

$$
\begin{aligned}
\hat{P}^\nu =& \int dx^- \!\int d^2\mathbf{x}\; T^{+\nu}(x) \\
=& \int dx^- \!\int d^2\mathbf{x}\, \Big\{ i\overline{\Psi_G}(x)\,\gamma^+ \partial^\nu \Psi_G(x) + \big(\partial_- A_a^j(x)\big)\, \big(\partial^\nu A_a^j(x)\big) \Big\}
\end{aligned}
\tag{81}
$$

for $\nu \neq -$,

$$
\begin{aligned}
\hat{M}^{+i} =& \int dx^- \!\int d^2\mathbf{x}\; J^{++i}(x) \\
=& \int dx^- \!\int d^2\mathbf{x}\, \Big\{ i\overline{\Psi_G}(x)\,\gamma^+ \Big(- x^+\, \partial_i - x^i\, \partial_- \Big) \Psi_G(x) \\
&+ \big(\partial_- A_a^j(x)\big)\, \Big(- x^+\, \partial_i - x^i\, \partial_- \Big) A_a^j(x) \Big\}\,,
\end{aligned}
\tag{82}
$$

and

$$
\begin{aligned}
\hat{M}^{ij} =& \int dx^- \!\int d^2\mathbf{x}\; J^{+ij}(x) \\
=& \int dx^- \!\int d^2\mathbf{x}\, \Big\{ i\overline{\Psi_G}(x)\,\gamma^+ \Big(- x^i\, \partial_j + x^j\, \partial_i + \frac{1}{4}\, \big[\gamma^i, \gamma^j\big] \Big) \Psi_G(x) \\
&+ \big(\partial_- A_a^l(x)\big) \Big(\big[-x^i\, \partial_j + x^j\, \partial_i\big]\, A_a^l(x) - g^i{}_l\, A_a^j(x) + g^j{}_l\, A_a^i(x) \Big) \Big\}\,.
\end{aligned}
\tag{83}
$$

These expressions are indeed simple sums of terms identical to the corresponding results in the free Dirac spinor or free photon case, as expected. Note that they depend only on the independent components Ψ_G and A_a^i of the fields, and not on the constrained components Ψ_B and A_a^-.

Hence, we can quantize Ψ_G and A_a^j using the same commutation relations as in the free Dirac spinor and free photon cases, and get a quantum theory invariant under the kinematical Poincaré subalgebra. Ignoring the issues of residual gauge fixing and zero modes, on can then define the vacuum and construct the Fock space at a given x^+ as in the scalar case, but now with 3 different species of particles: quarks, antiquarks and gluons.

3.2 Constraint equations in QCD on the light-front

In order to access the x^+ evolution of the theory, one should now study the constrained components Ψ_B and A_a^- of the fields, and then the dynamical Poincaré generator $\hat{P}^-$. The equations of motion of QCD (72) write

$$\begin{aligned}\left(i\not{D} - m\right)\Psi(x) &= \left(i\not{\partial} - m\right)\Psi(x) - gA_\mu^a(x)\gamma^\mu t^a \Psi(x) = 0 && (84)\\ \partial_\mu F_a^{\mu\nu} &= gJ_a^\nu(x)\,, && (85)\end{aligned}$$

with the definition

$$J_a^\nu(x) = \overline{\Psi}(x)\,\gamma^\nu t^a \Psi(x) + f^{abc}A_\mu^b(x)F_c^{\mu\nu} \tag{86}$$

for the total color current.

As in the free photon case, the $\nu = +$ component of Eq. (85) in the light-cone gauge (80) gives a constraint equation for $A_a^-(x)$, which now writes

$$-\partial_-{}^2 A_a^-(x) - \partial_-\partial_j A_a^j(x) = gJ_a^+(x)\,. \tag{87}$$

Noting that $J_a^+(x)$ has the expression

$$J_a^+(x) = \overline{\Psi_G}(x)\,\gamma^+ t^a \Psi_G(x) + f^{abc}A_b^j(x)\partial_- A_c^j(x) \tag{88}$$

involving only independent components Ψ_G and A_a^j, one can find $A_a^-(x)$ a function of independent fields as

$$A_a^-(x) = -\left(\frac{1}{\partial_-}\right)\partial_j A_a^j(x) - \left(\frac{1}{\partial_-}\right)^2 gJ_a^+(x)\,, \tag{89}$$

up to zero modes that we will ignore.

Multiplying the QCD Dirac equation (84) by γ^+, one finds the constraint equation

$$\begin{aligned}0 &= \gamma^+\left(i\gamma^-\partial_- + i\gamma^j\partial_j - m\right)\Psi(x) + g\gamma^+\left(A_a^j(x)\gamma^j t^a\right)\Psi(x)\\ &= 2i\partial_-\Psi_B(x) + \gamma^+\left(i\gamma^j\partial_j - m\right)\Psi_G(x) + g\gamma^+\left(A_a^j(x)\gamma^j t^a\right)\Psi_G(x)\,, \qquad (90)\end{aligned}$$

which can be solved as

$$\Psi_B(x) = \left(\frac{i}{2\partial_-}\right)\gamma^+\left(i\gamma^j\partial_j - m\right)\Psi_G(x) + g\left(\frac{i}{2\partial_-}\right)\gamma^+\left(A_a^j(x)\gamma^j t^a\right)\Psi_G(x)\,, \tag{91}$$

neglecting again zero modes issues.

The constrained components $A_a^-(x)$ and $\Psi_B(x)$ given by Eqs. (89) and (91) have some extra term of order g compared to their expressions in the free field case, whereas the independent components $\Psi_G(x)$ and $A_a^j(x)$ are the same as in the free field case, when restricted on a x^+ hyperplane. We also have convenient solutions for the free field versions of $\Psi(x)$ and $A_a^\mu(x)$, using 4-components spinors and 4-components polarization vectors. It will be useful in the following to introduce the notations $\widetilde{\Psi}(x)$ and $\widetilde{A}_a^\mu(x)$ for the free field analogs of $\Psi(x)$

and $A_a^\mu(x)$, which share the same independent components, but have their constrained components $\widetilde{A}_a^-(x)$ and $\widetilde{\Psi_B}(x)$ determined from the free field constraint equations instead of the QCD ones. Hence, we have

$$\Psi_G(x) = \widetilde{\Psi_G}(x) \tag{92}$$

$$A_a^j(x) = \widetilde{A}_a^j(x) \tag{93}$$

$$\Psi(x) = \widetilde{\Psi}(x) - g\left(\frac{\gamma^+}{2i\partial_-}\right)\left(A_a^j(x)\gamma^j t^a\right)\Psi_G(x) \tag{94}$$

$$A_a^\mu(x) = \widetilde{A}_a^\mu(x) + g^{\mu+}\,\frac{g}{(i\partial_-)^2}\,J_a^+(x)\,, \tag{95}$$

3.3 Interaction operator for LFPT in QCD and QED

Starting from the expression (78), and using Eqs. (94) and (95), one can evaluate the light-front QCD Hamiltonian P^- in terms of the independent components $\Psi_G(x)$ and $A_a^j(x)$, and of $\widetilde{\Psi}(x)$ and $\widetilde{A}_a^\mu(x)$. After a quite long and tricky calculation (see section 2G of Ref. [2] for details), and after splitting the result for P^- into the free part T and the interaction part V, one finds

$$\begin{aligned} V^{\mathrm{QCD}}(x^+) = \int d^2\mathbf{x}\int dx^-\Bigg\{& g\,\widetilde{A}_\mu^a(x)\,\overline{\widetilde{\Psi}}(x)\,\gamma^\mu t^a\,\widetilde{\Psi}(x) \\ & - g\,f^{abc}\,\widetilde{A}_a^\mu(x)\,\Big(\partial_\mu A_b^j(x)\Big)\,A_c^j(x) + \frac{g^2}{4}\,f^{abe}f^{cde}\,A_a^i(x)\,A_b^j(x)\,A_c^i(x)\,A_d^j(x) \\ & + \frac{g^2}{2}\,J_a^+(x)\,\frac{1}{(i\partial_-)^2}\,J_a^+(x) + \frac{g^2}{2}\,\left[\overline{\Psi_G}(x)\,\gamma^i t^a\,A_a^i(x)\right]\,\frac{\gamma^+}{i\partial_-}\,\left[\gamma^j t^b\,A_b^j(x)\Psi_G(x)\right]\Bigg\}\,, \end{aligned} \tag{96}$$

The first three terms correspond to the standard $q\bar{q}g$, ggg and $gggg$ interaction vertices, whereas the fourth in eq. (96) corresponds to instantaneous Coulomb interactions between quarks or gluons and the last term corresponds to an instantaneous quark exchange. These two non-local types of interactions arise as direct consequences of the solutions (94) and (95) of the constraint equations.

The interaction part of P^- in QED can be obtain in the same way, or directly from the QCD result (96) by taking f^{abc} to 0, replacing the fundamental color generator t^a by the fractional charge e_f of the considered flavor of quark (or lepton), the coupling g by e, and the gluon field by the photon field. One gets

$$\begin{aligned} V^{\mathrm{QED}}(x^+) = \int d^2\mathbf{x}\int dx^-\Bigg\{& e\,e_f\,\widetilde{A}_\mu(x)\,\overline{\widetilde{\Psi}}(x)\,\gamma^\mu\widetilde{\Psi}(x) \\ & + \frac{e^2\,e_f^2}{2}\,\left[\overline{\Psi_G}(x)\,\gamma^+\Psi_G(x)\right]\,\frac{1}{(i\partial_-)^2}\,\left[\overline{\Psi_G}(x)\,\gamma^+\Psi_G(x)\right] \\ & + \frac{e^2\,e_f^2}{2}\,\left[\overline{\Psi_G}(x)\,\gamma^i\,A^i(x)\right]\,\frac{\gamma^+}{i\partial_-}\,\left[\gamma^j\,A^j(x)\Psi_G(x)\right]\Bigg\}\,. \end{aligned} \tag{97}$$

Hence, the two types of non-local vertices also exist in QED.

In problems involving both QCD and QED effects, like DIS, Drell-Yan or photon production in hadronic collisions, one should use the interaction operator

$$V_I(x^+) = \hat{V}_I^{\mathrm{QCD}}(x^+) + V_I^{\mathrm{QED}}(x^+) + V_I^{\mathrm{mixed}}(x^+)\,. \tag{98}$$

Indeed, when both QCD and QED are included, each one give a separate contribution to $\Psi_B(x)$ see Eq. (94). Since such corrections appear quadratically in Eqs. (96) and (97), in the non-local fermion exchange term, there is also a mixed term, which appear as a non-local $q\bar{q}g\gamma$ interaction term. It writes

$$V^{\rm mixed}(x^+) = \int d^2\mathbf{x} \int dx^- \Bigg\{ \frac{g\, e\, e_f}{2} \left[\overline{\Psi_G}(x)\, \gamma^i t^a\, A_a^i(x)\right] \frac{\gamma^+}{i\partial_-} \left[\gamma^j\, A^j(x) \Psi_G(x)\right] + \frac{g\, e\, e_f}{2} \left[\overline{\Psi_G}(x)\, \gamma^j\, A^j(x)\right] \frac{\gamma^+}{i\partial_-} \left[\gamma^i t^a\, A_a^i(x) \Psi_G(x)\right] \Bigg\}. \tag{99}$$

In this section, we have completely ignored the issue of ordering of the field in the Hamiltonian at quantum level, for simplicity. Like in the scalar case, after renormalization of the vacuum energy, most of the terms present in the interaction part $\hat{V}_I(x^+)$ of the Hamiltonian $\hat{P}^-$ at quantum level can be obtained by simply normal-ordering its classical expression, apart from tadpole insertion vertices, which would contribute only to mass renormalization.

3.4 LFPT vertices

The last step needed before performing perturbative calculations in QCD/QED on the light front is to evaluate the matrix elements of the interaction operator $\hat{V}_I(0)$ in the Fock state basis. There are a large number of them to evaluate, so we will not list them all. As an example, all the $q\bar{q}g$ vertices generated by the first term in (96) write

$$\begin{aligned}
\langle 0|a_2 b_1\, \hat{V}_I(0)\, b_0^\dagger|0\rangle &= (2\pi)^3 \delta^{(3)}(\underline{k_1}+\underline{k_2}-\underline{k_0})\;\, g\, t^{a_2}_{\alpha_1\alpha_0}\, \overline{u}(1)\, \not\epsilon^*_{\lambda_2}(\underline{k_2})\, u(0)\\
\langle 0|b_1\, \hat{V}_I(0)\, b_0^\dagger a_2^\dagger|0\rangle &= (2\pi)^3 \delta^{(3)}(\underline{k_1}-\underline{k_2}-\underline{k_0})\;\, g\, t^{a_2}_{\alpha_1\alpha_0}\, \overline{u}(1)\, \not\epsilon_{\lambda_2}(\underline{k_2})\, u(0)\\
\langle 0|a_2 d_0\, \hat{V}_I(0)\, d_1^\dagger|0\rangle &= (2\pi)^3 \delta^{(3)}(\underline{k_0}+\underline{k_2}-\underline{k_1})\;\, g\, t^{a_2}_{\alpha_1\alpha_0}\, (-1)\, \overline{v}(1)\, \not\epsilon^*_{\lambda_2}(\underline{k_2})\, v(0)\\
\langle 0|d_0\, \hat{V}_I(0)\, d_1^\dagger a_2^\dagger|0\rangle &= (2\pi)^3 \delta^{(3)}(\underline{k_0}-\underline{k_2}-\underline{k_1})\;\, g\, t^{a_2}_{\alpha_1\alpha_0}\, (-1)\, \overline{v}(1)\, \not\epsilon_{\lambda_2}(\underline{k_2})\, v(0)\\
\langle 0|d_0 b_1\, \hat{V}_I(0)\, a_2^\dagger|0\rangle &= (2\pi)^3 \delta^{(3)}(\underline{k_0}+\underline{k_1}-\underline{k_2})\;\, g\, t^{a_2}_{\alpha_1\alpha_0}\, \overline{u}(1)\, \not\epsilon_{\lambda_2}(\underline{k_2})\, v(0)\\
\langle 0|a_2\, \hat{V}_I(0)\, b_0^\dagger d_1^\dagger|0\rangle &= (2\pi)^3 \delta^{(3)}(\underline{k_2}-\underline{k_0}-\underline{k_1})\;\, g\, t^{a_2}_{\alpha_1\alpha_0}\, \overline{v}(1)\, \not\epsilon^*_{\lambda_2}(\underline{k_2})\, u(0)\, ,
\end{aligned} \tag{100}$$

using compact notations $b_0 \equiv \hat{b}(\underline{k_0}, h_0, \alpha_0)$, where α_0 corresponds to the fundamental color index.

From the ggg term in the interaction operator (96), one has the gluon splitting vertex

$$\begin{aligned}
&\langle 0|a_2 a_1\, \hat{V}_I(0)\, a_0^\dagger|0\rangle = (2\pi)^3 \delta^{(3)}(\underline{k_1}+\underline{k_2}-\underline{k_0})\;\, (-ig)\, f^{a_0 a_1 a_2}\\
&\times \Big\{ \varepsilon^j_{\lambda_0} \varepsilon^{j\,*}_{\lambda_2}\, \epsilon^{\mu\,*}_{\lambda_1}(\underline{k_1})\, (k_{0\,\mu} + k_{2\,\mu}) - \varepsilon^j_{\lambda_0} \varepsilon^{j\,*}_{\lambda_1}\, \epsilon^{\mu\,*}_{\lambda_2}(\underline{k_2})\, (k_{0\,\mu} + k_{1\,\mu})\\
&+ \varepsilon^{j\,*}_{\lambda_1} \varepsilon^{j\,*}_{\lambda_2}\, \epsilon^{\mu\,*}_{\lambda_0}(\underline{k_0})\, (k_{1\,\mu} - k_{2\,\mu}) \Big\}\, ,
\end{aligned} \tag{101}$$

as well as the gluon merging vertex, which simply the conjugate of (101).

As an example for the instantaneous Coulomb interaction one has the $q\bar{q} \to q\bar{q}$ vertex

$$\langle 0|d_1 b_0\, \hat{V}_I(0)\, b_{0'}^\dagger d_{1'}^\dagger|0\rangle = (2\pi)^3 \delta^{(3)}(\underline{k_{0'}} + \underline{k_{1'}} - \underline{k_0} - \underline{k_1})\ g^2$$
$$\times \Bigg\{ - (t^{a_2})_{\alpha_0\, \alpha_{0'}}\ (t^{a_2})_{\alpha_{1'}\, \alpha_1}\ \frac{1}{(k_0^+ - k_{0'}^+)^2}\ \overline{u}(0)\, \gamma^+ u(0')\ \ \overline{v}(1')\, \gamma^+ v(1)$$
$$+ (t^{a_2})_{\alpha_0\, \alpha_1}\ (t^{a_2})_{\alpha_{1'}\, \alpha_{0'}}\ \frac{1}{(k_0^+ + k_1^+)^2}\ \overline{u}(0)\, \gamma^+ v(1)\ \ \overline{v}(1')\, \gamma^+ u(0') \Bigg\}\,. \quad (102)$$

In this expression, only the first term corresponds to an instantaneous Coulomb interaction between a quark and an antiquark in the t channel. By contrast, the second term stands for a $q\bar{q}$ annihilation followed (instantaneously) by a $q\bar{q}$ pair creation annihilation, via an s-channel intermediate gluon.

Finally, the mixed QCD/QED part with instantaneous quark exchange gives for example the $\gamma \to q\bar{q}g$ vertex

$$\langle 0|a_2 d_1 b_0\, V_I(0)\, a_\gamma^\dagger|0\rangle = (2\pi)^3 \delta^{(3)}(\underline{k_2} + \underline{k_1} + \underline{k_0} - \underline{q})\ \frac{e\, g}{2}\, e_f\ (t^{a_2})_{\alpha_0\, \alpha_1}$$
$$\times\ \ \overline{u}(0) \left[\frac{\not\epsilon_{\lambda_2}^*(\underline{k_2})\ \gamma^+\ \not\epsilon_\lambda(\underline{q})}{(k_0^+ + k_2^+)} - \frac{\not\epsilon_\lambda(\underline{q})\ \gamma^+\ \not\epsilon_{\lambda_2}^*(\underline{k_2})}{(k_1^+ + k_2^+)} \right] v(1)\,. \quad (103)$$

where the incoming photon has a momentum $\underline{q}$ and a polarization λ.

References

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