Mesons on the light front
Part 1: Light-front Hamiltonian approach and the meson light-front wavefunctions

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This lecture note is written for "Courses on Light-Cone Techniques applied to QCD", Nov 21-25, IGFAE. It is intended to provide the basic knowledge and selective perspectives on the application of light-front Hamiltonian approach to mesons in two 1.5-hour lectures. This is part 1(of 2).

Studying mesons, the bound states of quantum chromodynamics (QCD), is crucial to increase our comprehension of the strong interaction and the constitution of matter. The meson system has attracted extensive experimental investigations, including the mass spectrum, transitions between excited and low-lying states, and photoproduction of the vector mesons in heavy-ion collisions. Theoretical efforts contribute from an array of complementary perspectives, both Euclidean and Minkowskian formalisms. Euclidean formulations of quantum field theories such as Dyson-Schwinger equations and lattice gauge theory offer methods of performing a first-principles computation. Here, we take the path of a Minkowskian formalism, the light-front(LF) Hamiltonian method, in which the light-front wavefunctions(LFWFs) play a central role in describing the bound states and computing physical observables.

In this Part I of the lecture, we will first derive the QCD Hamiltonian on the light front, then we can treat mesons as its eigenstates, lastly, we review two phenomenological approaches in addressing the meson LFWFs.

I. CANONICAL QUANTIZATION OF THE QCD HAMILTONIAN ON THE LIGHT FRONT (0.75 H')

[Comment] The content of this section is mainly based on Chapter 2 of Ref. [1] (the review paper on light front field theories by Brodsky, Pauli, and Pinsky). The lecture notes by Harindranath could also be helpful [2].

A. Light-front dynamics

From the viewpoint that the quantum field theory is formulated to reconcile quantum mechanics with special relativity, let us first study how symmetries like Lorentz invariance appear in quantum setting. In

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particular, we would like to combine the principle of relativity with the Hamiltonian formulation of dynamics.

Einstein's principle of relativity requires that physical laws shall be invariant under transformations from one space-time coordinate system to another, or in other words, invariant in all inertial frames of reference. The whole group of the transformations is the inhomogeneous Lorentz group, also known as the Poincaré group. Quantum theory postulates that physical states are represented by rays [1] in Hilbert space. Therefore we need to implement a representation of the Poincaré group. The Poincaré algebra is the Lie algebra of the Poincaré group, and it is given by the commutation relations:

$$\begin{aligned} &[P^\mu, P^\nu] = 0\,, \\ &[P^\mu, M^{\alpha\beta}] = i(g^{\mu\alpha}P^\beta - g^{\mu\beta}P^\alpha)\,, \\ &[M^{\mu\nu}, M^{\rho\sigma}] = i(g^{\mu\sigma}M^{\nu\rho} - g^{\nu\sigma}M^{\mu\rho} + g^{\nu\rho}M^{\mu\sigma} - g^{\mu\rho}M^{\nu\sigma})\,. \end{aligned} \tag{1}$$

It has ten generators, four generators of translations $P^\mu = (P^0, P^1, P^2, P^3)$ and six generators of Lorentz transformations $M^{\mu\nu}$. The latter can be further split into the three generators of rotations $J^i = 1/2\epsilon^{ijk}M^{jk}$ and 3 generators of boosts $K^i = M^{0i}$. [2]

In quantum mechanics, and also in the quantum field theory, the dynamical evolution of a quantum state satisfies the Schrödinger equation,

$$i\frac{\partial}{\partial t}\,|\psi(t)\rangle = H\,|\psi(t)\rangle\ . \tag{2}$$

For stationary states,

$$|\psi(t)\rangle = e^{-iEt}\,|\psi(0)\rangle\ , \tag{3}$$

and it leads to the bound-state equation

$$H\,|\psi(0)\rangle = E\,|\psi(0)\rangle\ , \tag{4}$$

where E is the bound state energy. Though in its original form the time t is the regular time, there are actually multiple choices of the time variable as a foliation of spacetime. [3] P. A. M. Dirac brought up three forms of relativistic dynamics, namely the instant form, the point form, and the front form [4].

In the instant form, one works with dynamical variables referring to physical conditions at some instant of time, x^0. The Hamiltonian is P^0. The transformations of coordinates associated with the momenta P^1,

[1] A ray is a set of normalized vectors differed by multiplying an arbitrary scalar of unit magnitude [3].

[2] The cyclic symbol ϵ^{ijk} is 1 if the indices ijk are in cyclic order, and 0 otherwise.

[3] By foliation it means that the manifold of spacetime is decomposed into hypersurfaces and there exists a smooth scalar field (the "time") which is regular in the sense that its gradient never vanishes, such that each hypersurface is a level surface of this scalar field.

P^2, P^3 and the rotations J^1, J^2, J^3, leave the instant invariant, and are thus kinematic. The energy P^0, and the boosts K^1, K^2, K^3 are dynamical. The instant form seems most intuitive since its time variable is the regular time. Although it is the conventional choice for quantizing field theories, it has many disadvantages. The experiment determining the wavefunction $\psi(t, \vec{x})$ solved from the evolution equation of Eq. (2) requires the simultaneous measurement of all positions of the state. A more practical experimental measurement scatters one plane-wave laser beam, and the signal reaches each part of the object at the same light-front time $x^+ = t + z/c$ (this is the same with the definition $x^+ = x^0 + x^3$ with the unit $c = 1$).

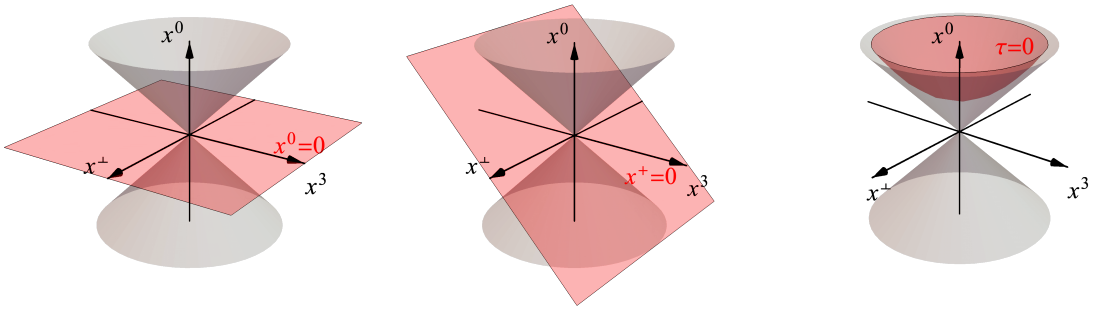
The point form of dynamics describes physical conditions on the three-dimensional surface, $\tau = \sqrt{x^\mu x_\mu - a^2} = \sqrt{(x^0)^2 - (x^1)^2 - (x^2)^2 - (x^3)^2 - a^2}$ with $x^0 > 0$. The energy P^0, and the momenta P^1, P^2, P^3 are all dynamical. The kinematic group consists of the boosts K^1, K^2, K^3 and the rotations J^1, J^2, J^3, which leave the origin point invariant. The point form of relativistic quantum mechanics has been advocated as an appropriate framework for calculating the electroweak structure of mesons and baryons within the scope of constituent-quark models [5–7].

The front form considers the three-dimensional surface in space-time formed by a plane wave front advancing with the velocity of light. The theory describes physical conditions at some constant light-front time $x^+ = x^0 + x^3$. The front form has the largest number(seven) of kinematic generators that leaves the light front invariant. They are, the transverse momentum P^1, P^2, the longitudinal momentum $P^+ = P^0 + P^3$, the transverse boosts $E^1 = K^1 + J^2$, $E^2 = K^2 - J^1$, the rotation in the x-y plane J^3, and the boost in the longitudinal direction K^3.[4] The remaining generators $\{P^- = P^0 - P^3, F^1 = J^1 + K^2, F^2 = J^2 - K^1\}$ are dynamical. P^- is the light-front Hamiltonian. It is usually convenient to use the light-front coordinates when implementing the light-front dynamics. We include the conventions of the light-front coordinates in Appendix A 1.

A visualization of the "time" in these three forms is presented in Fig. 1. Be aware that there also exists two other forms of dynamics, with the time defined as $\tau_z = \sqrt{(x^0)^2 - (x^3)^2 - a^2}$ with $x^0 > 0$ and $\tau_\perp = \sqrt{(x^0)^2 - (x^1)^2 - (x^2)^2 - a^2}$ with $x^0 > 0$ respectively, though they have a rather small kinematical group and are not commonly used [8].

The quantum field theory quantized on the light-front surface $x^+ = 0$ is the light front quantum field theory. In the next section, we will carry out the canonical quantization of QCD on the light front.

[4] The longitudinal boost is actually a scale transformation, seeing that $x^\pm \to \tilde{x}^\pm = e^{\pm\phi} x^\pm$ with the Lorentz factor $\gamma = \cosh\phi$. It therefore leaves the $x^+ = 0$ plane invariant.



(a)instant form (b)front form (c)point form

FIG. 1. "Time" in the three forms of dynamics. The gray cones are the reference surfaces of the light cones, $t = \sqrt{(x^0)^2 + (x^3)^2}$. The equal-"time" surfaces are in red. In (a), the instant form, time is defined as x^0 and the shown equal-time surface is $x^0 = 0$. In (b), the front form, time is defined as $x^+ = x^0 + x^3$ and the shown equal-light-front-time surface is $x^+ = 0$. In (c), the point form, time is defined as $\tau = \sqrt{x^\mu x_\mu - a^2}$ with $x^0 > 0$ and the shown equal-point-time surface is $\tau = 0$.

B. The light-front QCD Hamiltonian

The strong interaction between quarks and gluons is described by the non-Abelian gauge theory with symmetry group SU(3), known as quantum chromodynamics (QCD), and the Lagrangian reads

$$\mathcal{L}_{QCD} = -\frac{1}{4} F^{\mu\nu}{}_a F^a_{\mu\nu} + \bar{\Psi}(i\gamma^\mu D_\mu - m)\Psi \,. \tag{5}$$

A^ν_a is color vector potential, with the gluon index $a = 1, 2, \ldots, 8$. The quark field $\Psi_{\alpha,c}$, carries the Dirac index $\alpha = 1, 2, \ldots, 4$ and the color index $c = 1, 2, 3$, which are usually suppressed in expressions like $\bar{\Psi}\gamma^\mu D_\mu \Psi = \bar{\Psi}_c \gamma^\mu (D_\mu)_{cc'} \Psi_{c'}$. $m = mI_3 = m\delta_{cc'}$ is diagonal in color space. The vector potential can be parameterized as $(A_\mu)_{cc'} = T^a_{cc'} A^\mu_a$ by the color matrices $T^a_{cc'}$, and its matrix form can be found in Appendix A 3. $F^{\mu\nu}_a \equiv \partial^\mu A^\nu_a - \partial^\nu A^\mu_a - g f^{abc} A^\mu_b A^\nu_c$ is the field tensor, and $D^\mu \equiv \partial_\mu I_3 + igA^\mu$ is the covariant derivative. We follow the convention of the covariant derivative from Ref. [1], such that g is the chromo-electric charge of the anti-fermion. Note that there exists another widely used convention that assigns g to the chromo-electric charge of the fermion instead [9]. The structure constants f^{abc} are complete anti-symmetric, $f^{abc} = f^{cab} = -f^{acb}$. In the following derivations, we will drop the identity operator in the color space, I_3, for simplicity. Let us now derive the canonical QCD Hamiltonian according to the procedure in Ref. [1].

The QCD Lagrangian is a functional of the twelve components A^μ, Ψ_α, $\bar{\Psi}_\alpha$ and their space-time derivatives. We can denote them collectively as $\mathcal{L} = \mathcal{L}[\phi_r, \partial_\mu \phi_r]$. The equations of motion are

$$\partial_\kappa \Pi^\kappa_r - \delta\mathcal{L}/\delta\phi_r = 0 \,, \tag{6}$$

where the generalized momentum fields are $\Pi_r^\kappa \equiv \delta\mathcal{L}/\delta(\partial_\kappa\phi_r)$. Let us write out the equation of motions for each field.

1. A_a^μ and the color-Maxwell equations

   The variational derivatives and the generalized momentum fields of the vector field are

$$\frac{\delta\mathcal{L}}{\delta A_\kappa^s} = -\frac{1}{4}F_a^{\kappa\mu}(-gf^{asc}A_\mu^c)\times 4 + \bar{\Psi}(i\gamma^\kappa(igT^s))\Psi = -gf^{sac}F_a^{\kappa\mu}A_\mu^c - g\bar{\Psi}\gamma^\kappa T^s\Psi, \qquad \Pi^\lambda_{A_\kappa^s} = -F_s^{\lambda\kappa}\ . \tag{7}$$

   The first four equations of motion give the color-Maxwell equations,

$$\partial_\lambda F_s^{\lambda\kappa} = gJ_s^\kappa\ , \tag{8}$$

   with the current density $J_s^\kappa \equiv f^{sac}F_a^{\kappa\mu}A_\mu^c + \bar{\Psi}\gamma^\kappa T^s\Psi$. In the light-cone gauge of $A_a^+ = 0$, the $\kappa = +$ component of Eq. (8) does not contain time derivatives, and can be written as

$$gJ_a^+ = \partial_\lambda F_a^{\lambda+} = -\partial^+\partial_- A_a^- - \partial^+\partial_i A_a^i\ . \tag{9}$$

   By inverting the above equation, we get

$$\frac{1}{2}A_a^- = -g\frac{1}{(\partial^+)^2}J_a^+ - \frac{1}{\partial^+}\partial_i A_a^i\ . \tag{10}$$

   We define the free solution $\tilde{A}_a^\mu$ such that $\lim_{g\to 0}A_a^\mu = \tilde{A}_a^\mu$. According to Eq. (10), the free field reads,

$$\tilde{A}_a^\mu = (0, \tilde{A}_a^-, A_a^i), \quad \text{with } \frac{1}{2}\tilde{A}_a^- \equiv \frac{1}{2}A_a^- + g\frac{1}{(\partial^+)^2}J_a^+ = -\frac{1}{\partial^+}\partial_i A_a^i\ . \tag{11}$$

   $\tilde{A}_a^\mu$ is thereby purely transverse.

2. Ψ_α and the (adjoint) color-Dirac equations

   The variational derivatives and the generalized momentum fields of the fermion field are

$$\frac{\delta\mathcal{L}}{\delta\Psi} = -g\bar{\Psi}\gamma^\mu A_\mu - m\bar{\Psi} - \frac{i}{2}\bar{\Psi}\gamma^\mu\overleftarrow{\partial}_\mu, \quad \Pi_\Psi^\lambda = \frac{i}{2}\bar{\Psi}\gamma^\lambda\ . \tag{12}$$

   Note that the second term of the Lagrangian in Eq. (5) written more explicitly is

$$\frac{1}{2}\left[\bar{\Psi}(i\gamma^\mu D_\mu - m)\Psi + h.c.\right] = \frac{1}{2}\left[\bar{\Psi}(i\gamma^\mu D_\mu - m)\Psi + \bar{\Psi}(-i\gamma^\mu\overleftarrow{D}_\mu - m)\Psi\right]\ , \tag{13}$$

   in which $\overleftarrow{D}^\mu \equiv \overleftarrow{\partial}_\mu I_3 - igA^\mu$.

   The equations of motion for Ψ give the adjoint color-Dirac equation,

$$\bar{\Psi}[i\gamma^\mu(\overleftarrow{\partial}_\mu - igA_\mu) + m] = 0\ . \tag{14}$$

Take Hermitian conjugate on the equation and use the relation $\bar{\Psi} = \Psi^\dagger\gamma^0$, we have

$$[-i\gamma^{\mu\dagger}(\partial_\mu + igA_\mu) + m]\gamma^0\Psi = 0 \,. \tag{15}$$

By moving γ^0 to the left, we arrive at the color-Dirac equation,

$$[i\gamma^\mu(\partial_\mu + igA_\mu) - m]\Psi = 0 \,. \tag{16}$$

Similar to the gluon field, we also want to separate the dynamical components of the fermion field. Define the projected spinors $\Psi_\pm = \Lambda^\pm\Psi$, with $\Lambda^\pm = \frac{1}{2}\gamma^0\gamma^\pm$, see more definitions of $\Lambda^\pm$ in Appendix A 2. First multiply Eq. (16) by γ^0 on the left,

$$\begin{aligned} &[i(\gamma^0\gamma^+D_+ + \gamma^0\gamma^-D_- + \alpha^iD_i) - m\beta]\Psi = 0 \,, \\ \text{which is, } &[i(2\Lambda^+D_+ + 2\Lambda^-D_- + \alpha^iD_i) - m\beta]\Psi = 0 \,. \end{aligned} \tag{17}$$

Then multiply the equation by $\Lambda^+(\Lambda^-)$ on the left, and bring it to the right,

$$[i(2D_\pm\Lambda^\pm + \alpha^iD_i\Lambda^\mp) - m\beta\Lambda^\mp]\Psi = 0 \,. \tag{18}$$

One thereby obtains a coupled set of spinor equations,

$$2i\partial_+\Psi_+ = (-i\alpha^iD_i + m\beta)\Psi_- + 2gA_+\Psi_+ \,, \tag{19}$$

$$2i\partial_-\Psi_- = (-i\alpha^iD_i + m\beta)\Psi_+ + 2gA_-\Psi_- \,. \tag{20}$$

Then, in the light-cone gauge, $2A_- = A^+ = 0$. Equation (20) does not contain time derivatives, and can be written as a constraint relation,

$$\Psi_- = \frac{1}{2i\partial_-}(m\beta - i\alpha^iD_i)\Psi_+ \,. \tag{21}$$

By substituting Eq. (21) into Eq. (19), we get

$$2iD_+\Psi_+ = (m\beta - i\alpha^iD_i)\frac{1}{2i\partial_-}(m\beta - i\alpha^iD_i)\Psi_+ \,. \tag{22}$$

In analogy to the free solution $\tilde{A}$, we define the free spinor $\tilde{\Psi} = \tilde{\Psi}_+ + \tilde{\Psi}_-$ with

$$\tilde{\Psi}_+ = \Psi_+, \quad \tilde{\Psi}_- = \frac{1}{2i\partial_-}(m\beta - i\alpha^i\partial_i)\Psi_+ \,. \tag{23}$$

The projection still holds, $\tilde{\Psi}_\pm = \Lambda^\pm\tilde{\Psi}$.

3. $\bar{\Psi}_\alpha$ and the color-Dirac equations

The variational derivatives and the generalized momentum fields of the anti-fermion field are

$$\frac{\delta\mathcal{L}}{\delta\bar{\Psi}} = -g\gamma^\mu A_\mu\Psi - m\Psi + \frac{i}{2}\gamma^\mu\partial_\mu\Psi, \quad \Pi^\lambda_{\bar{\Psi}} = -\frac{i}{2}\gamma^\lambda\Psi \,. \tag{24}$$

The equations of motion for $\bar{\Psi}$ give the color-Dirac equation, Eq. (16), which we have already arrived from Ψ, not surprisingly.

We now turn to the construction of the canonical Hamiltonian density through a Legendre transformation, [5]

$$\begin{aligned}\mathcal{P}_+ =&(\partial_+ A^s_\kappa)\Pi^+_{A^s_\kappa} + (\partial_+\Psi)\Pi^+_\Psi + (\partial_+\bar\Psi)\Pi^+_{\bar\Psi} - \mathcal{L}\\ =&- F^{+\kappa}_s\partial_+ A^s_\kappa + \frac{1}{2}[i\bar\Psi\gamma^+\partial_+\Psi + h.c.] + \frac{1}{4}F^{\mu\nu}{}_a F^a_{\mu\nu} - \frac{1}{2}[\bar\Psi(i\gamma^\mu D_\mu - m)\Psi + \bar\Psi(-i\gamma^\mu\overleftarrow{D}_\mu - m)\Psi]\\ =&- F^{+\kappa}_s\partial_+ A^s_\kappa + \frac{1}{2}[i\bar\Psi\gamma^+\partial_+\Psi + h.c.] + \frac{1}{4}F^{\mu\nu}{}_a F^a_{\mu\nu}\,,\end{aligned} \tag{25}$$

where we have used the color-Dirac equations as in Eqs. (14) and (16) in the last line. It is convenient to add a total derivative $-\partial_\kappa(F^{\kappa+}_s A^s_+)$ to the Hamiltonian $P^- = 2P_+$,

$$\begin{aligned}P^- =&2\int \mathrm{d}x_+\,\mathrm{d}^2x_\perp\,\mathcal{P}_+\\ =&\int \mathrm{d}x^-\,\mathrm{d}^2x_\perp\;- F^{+\kappa}_s\partial_+ A^s_\kappa + \frac{1}{2}[i\bar\Psi\gamma^+\partial_+\Psi + h.c.] + \frac{1}{4}F^{\mu\nu}{}_a F^a_{\mu\nu} - \partial_\kappa(F^{\kappa+}_s A^s_+)\,.\end{aligned} \tag{26}$$

We can rewrite the first and the last terms into

$$\begin{aligned}F^{\kappa+}_s\partial_+ A^s_\kappa - \partial_\kappa(F^{\kappa+}_s A^s_+) =&F^{\kappa+}_s\partial_+ A^s_\kappa - (\partial_\kappa F^{\kappa+}_s)A^s_+ - F^{\kappa+}_s\partial_\kappa A^s_+\\ =&F^{\kappa+}_s(\partial_+ A^s_\kappa - \partial_\kappa A^s_+) - (\partial_\kappa F^{\kappa+}_s)A^s_+\\ =&- F^{\kappa+}_s(F^s_{\kappa+} + gf^{sbc}A^b_\kappa A^c_+) - gJ^+_s A^s_+\\ =&- F^{\kappa+}_s F^s_{\kappa+} - g\bar\Psi\gamma^+ T^s A^s_+\Psi\,.\end{aligned} \tag{27}$$

The Hamiltonian becomes

$$\begin{aligned}P^- =&\int \mathrm{d}x^-\,\mathrm{d}^2x_\perp\,\frac{1}{4}F^{\mu\nu}{}_a F^a_{\mu\nu} - F^{\kappa+}_s F^s_{\kappa+} - g\bar\Psi\gamma^+ T^s A^s_+\Psi + \frac{1}{2}[i\bar\Psi\gamma^+\partial_+\Psi + h.c.]\\ =&\int \mathrm{d}x^-\,\mathrm{d}^2x_\perp\,\frac{1}{4}F^{\mu\nu}{}_a F^a_{\mu\nu} - F^{\kappa+}_s F^s_{\kappa+} + \frac{1}{2}[i\bar\Psi\gamma^+ D_+\Psi + h.c.]\,.\end{aligned} \tag{28}$$

1. 1st part of Eq. (28)

   Let us also rewrite the color-electro-magnetic energy density and separate the longitudinal and the transversal contributions,

$$\begin{aligned}\frac{1}{4}F^{\mu\nu}_a F^a_{\mu\nu} - F^{\mu+}_a F^a_{\mu+} =&\frac{1}{4}(F^{ij}_a F^a_{ij} + F^{\mu+}_a F^a_{\mu+} + F^{+\nu}_a F^a_{+\nu} + F^{\mu-}_a F^a_{\mu-} + F^{-\nu}_a F^a_{-\nu}\\ &- F^{+-}_a F_{+-} - F^{-+}_a F_{-+}) - F^{\mu+}_a F_{\mu+}\\ =&\frac{1}{4}F^{ij}_a F^a_{ij} + \frac{1}{2}(F^{\mu+}_a F^a_{\mu+} + F^{\mu-}_a F^a_{\mu-} - F^{+-}_a F^a_{+-}) - F^{\mu+}_a F^a_{\mu+}\\ =&\frac{1}{4}F^{ij}_a F^a_{ij} - \frac{1}{2}F^{+-}_a F^a_{+-}\,.\end{aligned} \tag{29}$$

[5] Note that we are taking the derivative ∂^+, in terms of x^+, and the conjugate quantity is P_+, or equivalently $P^-/2$.

Note that $F_a^{\mu+}F_{\mu+}^a = F_a^{\mu-}F_{\mu-}^a$ by $F_a^{\mu-} = g^{\mu\nu}F_{\nu+}^a g^{+-}$. Substituting A_a^- by Eq. (10), the color-electric part becomes,

$$\begin{aligned} F_a^{+-}F_{+-}^a =& -\partial^+ A_a^- \partial_- A_+^a \\ =& -\frac{1}{4}\partial^+ A_a^- \partial^+ A_a^- \\ =& -(-g\frac{1}{\partial^+}J_a^+ - \partial_i A_a^i)^2 \\ =& g^2 J_a^+ \frac{1}{(\partial^+)^2}J_a^+ - (\partial_i A_a^i)^2 - gJ_a^+\tilde{A}_a^- \ . \end{aligned} \tag{30}$$

In deriving the last line, we introduced an extra term $g^2\frac{1}{\partial^+}(J_a^+\frac{1}{\partial^+}J_a^+)$, taking that it should vanish under the integral of $\int \mathrm{d}x^-$. The color-magnetic part can be written as

$$\begin{aligned} F_a^{ij}F_{ij}^a =& 2\partial^i A_a^j \partial_i A_j^a - 2\partial^i A_a^j \partial_j A_i^a - 4gf^{abc}\partial^i A_a^j A_i^b A_j^c + g^2 f^{abc}A_i^b A_c^j f^{aef}A_i^e A_j^f \\ =& -2A_a^j \partial^i \partial_i A_j^a + 2A_a^j \partial^i \partial_j A_i^a - 4gf^{abc}\partial^i A_a^j A_i^b A_j^c + g^2 f^{abc}A_i^b A_c^j f^{aef}A_i^e A_j^f \\ =& 2A_a^j \nabla_\perp^2 A_j^a - 2(\partial_j A_a^j \partial^i A_i^a) - 4gf^{abc}\partial^i A_a^j A_i^b A_j^c + g^2 f^{abc}A_i^b A_c^j f^{aef}A_i^e A_j^f \ . \end{aligned} \tag{31}$$

2. 2nd part of Eq. (28) For the spinor terms,

$$i\bar{\Psi}\gamma^+ D_+ \Psi = i\Psi^\dagger \gamma^0 \gamma^+ D_+ \Psi = 2i\Psi^\dagger \Lambda^+ D_+ \Psi = 2i\Psi^\dagger \Lambda^+ D_+ \Lambda^+ \Psi = 2i\Psi_+^\dagger D_+ \Psi_+ \ . \tag{32}$$

Substitution of the time derivative in Eq. (22) and the free spinors defined in Eq. (23) leads to

$$\begin{aligned} & 2i\Psi_+^\dagger D_+ \Psi_+ \\ =& \Psi_+^\dagger (m\beta - i\alpha^i D_i)\frac{1}{2i\partial_-}(m\beta - i\alpha^i D_i)\Psi_+ \\ =& \Psi_+^\dagger (m\beta - i\alpha^i \partial_i)\frac{1}{2i\partial_-}(m\beta - i\alpha^i \partial_i)\Psi_+ + g^2\Psi_+^\dagger \alpha^i A_i \frac{1}{2i\partial_-}\alpha^i A_i \Psi_+ \\ & + g\Psi_+^\dagger \alpha^i A_i \frac{1}{2i\partial_-}(m\beta - i\alpha^i \partial_i)\Psi_+ + g\Psi_+^\dagger (m\beta - i\alpha^i \partial_i)\frac{1}{2i\partial_-}\alpha^i A_i \Psi_+ \\ =& \Psi_+^\dagger (m\beta - i\alpha^i \partial_i)\frac{1}{2i\partial_-}(m\beta - i\alpha^i \partial_i)\Psi_+ + g^2\Psi_+^\dagger \alpha^i A_i \frac{1}{2i\partial_-}\alpha^i A_i \Psi_+ + g\Psi_+^\dagger \alpha^i A_i \tilde{\Psi}_- + g\tilde{\Psi}_-^\dagger \alpha^i A_i \Psi_+ \ . \end{aligned} \tag{33}$$

The first term reads, with recalling that $\tilde{\Psi}_+ = \Psi_+$,

$$\begin{aligned} \tilde{\Psi}_+^\dagger (m\beta - i\alpha^i \partial_i)\frac{1}{2i\partial_-}(m\beta - i\alpha^j \partial_j)\tilde{\Psi}_+ =& \tilde{\Psi}^\dagger \Lambda_+ (m\beta - i\alpha^i \partial_i)\frac{1}{2i\partial_-}(m\beta - i\alpha^j \partial_j)\Lambda_+ \tilde{\Psi} \\ =& \frac{1}{2}\bar{\tilde{\Psi}}\gamma^+ (m\beta - i\alpha^i \partial_i)\frac{1}{2i\partial_-}(m\beta - i\alpha^j \partial_j)\tilde{\Psi} \\ =& \frac{1}{2}\bar{\tilde{\Psi}}\gamma^+ (m + i\gamma^i \partial_i)\frac{(\gamma^0)^2}{2i\partial_-}(m - i\gamma^j \partial_j)\tilde{\Psi} \\ =& \frac{1}{2}\bar{\tilde{\Psi}}\gamma^+ \frac{m^2 - \nabla_\perp^2}{2i\partial_-}\tilde{\Psi} \ . \end{aligned} \tag{34}$$

From the first to the second equation, the Λ_+ on the right is brought to the left, using the relations in Eq.(A22). The second term reads,

$$g^2\tilde{\Psi}_+^\dagger\alpha^i A_i\frac{1}{2i\partial_-}\alpha^j A_j\tilde{\Psi}_+ = \frac{g^2}{2}\bar{\tilde{\Psi}}\gamma^+\gamma^0\gamma^i A_i\frac{1}{2i\partial_-}\gamma^0\gamma^j A_j\tilde{\Psi} = \frac{g^2}{2}\bar{\tilde{\Psi}}\gamma^i A_i\frac{\gamma^+}{2i\partial_-}\gamma^j A_j\tilde{\Psi}\,. \tag{35}$$

The last two terms combine into

$$g\tilde{\Psi}_+^\dagger\alpha^i A_i\tilde{\Psi}_- + g\tilde{\Psi}_-^\dagger\alpha^i A_i\tilde{\Psi}_+ = g(\tilde{\Psi}_+^\dagger+\tilde{\Psi}_-)\alpha^i A_i(\tilde{\Psi}_+^\dagger+\tilde{\Psi}_-) = g\tilde{\Psi}^\dagger\alpha^i A_i\tilde{\Psi} = g\bar{\tilde{\Psi}}\gamma^i A_i\tilde{\Psi}\,. \tag{36}$$

We can also define the current density of free fields solution $\tilde{J}_a^\mu$ in analogy to J_a^μ, and notice that their "+" components are the same,

$$\begin{aligned} J_s^+ =& f^{sac}F_a^{+\mu}A_\mu^c + \bar{\Psi}\gamma^+T^s\Psi = f^{sac}\partial^+A_a^\mu A_\mu^c + \bar{\Psi}\gamma^+T^s\Psi \\ =& f^{sac}\partial^+A_a^i A_i^c + \bar{\Psi}\gamma^+T^s\Psi = f^{sac}\partial^+\tilde{A}_a^i\tilde{A}_i^c + \bar{\tilde{\Psi}}\gamma^+T^s\tilde{\Psi} = \tilde{J}_s^+\,. \end{aligned} \tag{37}$$

Let us also introduce the fermion current $\tilde{f}_a^\mu \equiv \bar{\tilde{\Psi}}\gamma^\mu T^a\tilde{\Psi}$ as part of the total current $\tilde{J}_a^\mu$. By substituting Eqs. (29) to (37) into Eq. (28), and with $\tilde{A}^i = A^i$, we finally get the front form Hamiltonian,

$$\begin{aligned} P_{QCD}^- = \int \mathrm{d}x^-\,\mathrm{d}^2x_\perp\; & -\frac{1}{2}\tilde{A}_a^j(i\nabla_\perp)^2\tilde{A}_j^a + \frac{1}{2}\bar{\tilde{\Psi}}\gamma^+\frac{m^2-\nabla_\perp^2}{i\partial^+}\tilde{\Psi} \\ & - gf^{abc}\partial^i\tilde{A}_a^j\tilde{A}_i^b\tilde{A}_j^c + g\tilde{J}_a^+\tilde{A}_+^a + g\bar{\tilde{\Psi}}\gamma^i\tilde{A}_i\tilde{\Psi} \\ & - \frac{1}{2}g^2\tilde{J}_a^+\frac{1}{(\partial^+)^2}\tilde{J}_a^+ + \frac{g^2}{4}f^{abc}\tilde{A}_b^i\tilde{A}_c^j f^{aef}\tilde{A}_i^e\tilde{A}_j^f \\ & + \frac{g^2}{2}\bar{\tilde{\Psi}}\gamma^i\tilde{A}_i\frac{\gamma^+}{i\partial^+}\gamma^j\tilde{A}_j\tilde{\Psi}\,. \end{aligned} \tag{38}$$

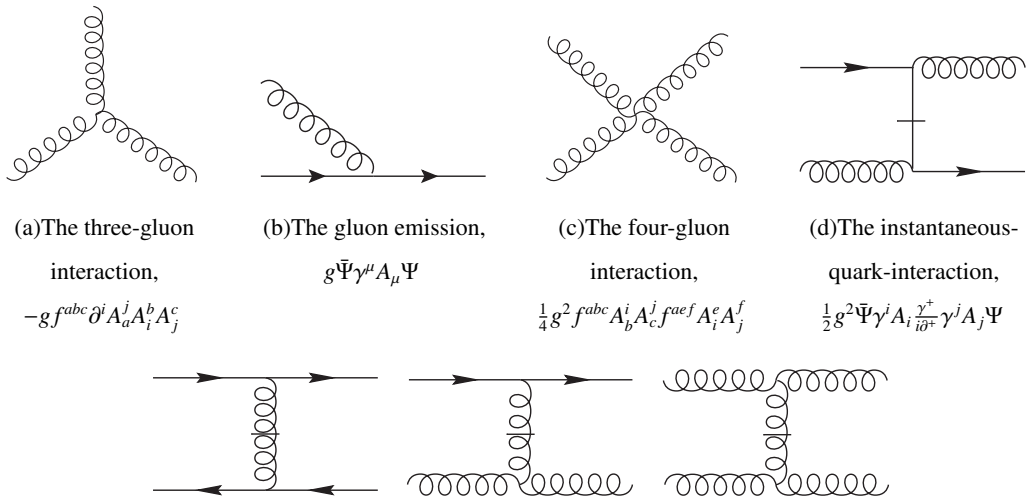
The two terms in the first line are the kinetic energy for the gauge field and the fermion respectively. The three terms in the second line can be written collectively as $gJ_a^\mu A_\mu^a$, which include the three-gluon-interaction, the gluon emission and quark-antiquark-pair-production processes. The two terms in the third line are the instantaneous-gluon-interaction and the four-gluon-interaction respectively. The last line contains the instantaneous-fermion-interaction. The vertex diagrams for these interactions are shown in Fig. 2

The fields for QCD admit free-field expansions at $x^+ = 0$ [1],

$$\Psi_{\alpha cf}(x) = \sum_{\lambda=\pm\frac{1}{2}}\int\frac{\mathrm{d}^2p_\perp\,\mathrm{d}p^+}{(2\pi)^3 2p^+}\theta(p^+)\left[b_q(p)u_\alpha(p,\lambda)e^{-ip\cdot x} + d_q^\dagger(p)v_\alpha(p,\lambda)e^{ip\cdot x}\right]\,, \tag{39}$$

$$A_{\mu a}(x) = \sum_{\lambda=\pm1}\int\frac{\mathrm{d}^2p_\perp\,\mathrm{d}p^+}{(2\pi)^3 2p^+}\theta(p^+)\left[a_q(p)\epsilon_\mu(p,\lambda)e^{-ip\cdot x} + a_q^\dagger(p)\epsilon_\mu^*(p,\lambda)e^{ip\cdot x}\right]\,, \tag{40}$$

where $\theta(p^+)$ is the Heaviside unit step function. α denotes the spinor components of Ψ, and μ denotes the vector components of A. λ is the light-front helicity of the corresponding field ($\lambda = \pm1/2$ for quarks and



(a)The three-gluon interaction, $-gf^{abc}\partial^i A^j_a A^b_i A^c_j$

(b)The gluon emission, $g\bar{\Psi}\gamma^\mu A_\mu \Psi$

(c)The four-gluon interaction, $\frac{1}{4}g^2 f^{abc}A^i_b A^j_c f^{aef}A^e_i A^f_j$

(d)The instantaneous-quark-interaction, $\frac{1}{2}g^2\bar{\Psi}\gamma^i A_i \frac{\gamma^+}{i\partial^+}\gamma^j A_j \Psi$

(e)The instantaneous-gluon-interaction, $-\frac{1}{2}g^2 J^+_a \frac{1}{(\partial^+)^2}J^+_a$

FIG. 2. Vertex diagram representation of the light-front QCD Hamiltonian in Eq. (38). The solid lines represent the quark operators, and the curly lines represent the gluon operators. The instantaneous quark (gluon) propagator $1/(i\partial^+)$ $[1/(\partial^+)^2]$ is represented by a quark (gluon) line with a bar across it.

$\lambda = \pm 1$ for gluons). $c = 1, 2, 3$ and $a = 1, 2, \ldots, 8$ are the color indices of quarks (antiquarks) and gluons respectively. q contains the quantum numbers of single particle state, for fermion $q = \{\lambda, c, f(\text{flavor})\}$ and for gluons $q = \{\lambda, a\}$. The creation and annihilation operators obey the commutation and anti-commutation relations. For gluons,

$$[a_{\lambda a}(p), a^\dagger_{\lambda' a'}(p')] = 2p^+\theta(p^+)(2\pi)^3\delta^3(p-p')\delta_{\lambda\lambda'}\delta_{aa'} \ , \tag{41}$$

where $\delta^3(p-p') = \delta(p^+ - p'^+)\delta^2(\vec{p}_\perp - \vec{p}'_\perp)$. For quarks and antiquarks,

$$\begin{aligned} \{b_{\lambda cf}(p), b^\dagger_{\lambda' c' f'}(p')\} &= 2p^+\theta(p^+)(2\pi)^3\delta^3(p-p')\delta_{\lambda\lambda'}\delta_{cc'}\delta_{ff'} \\ \{d_{\lambda cf}(p), d^\dagger_{\lambda' c' f'}(p')\} &= 2p^+\theta(p^+)(2\pi)^3\delta^3(p-p')\delta_{\lambda\lambda'}\delta_{cc'}\delta_{ff'} \ . \end{aligned} \tag{42}$$

All the other commutation and anti-commutation relations vanish,

$$[a_{\lambda a}(p), a_{\lambda' a'}(p')] = \{b_{\lambda cf}(p), b_{\lambda' c' f'}(p')\} = \{d_{\lambda cf}(p), d_{\lambda' c' f'}(p')\} = \{b_{\lambda cf}(p), d^\dagger_{\lambda' c' f'}(p')\} = \cdots = 0 \ . \tag{43}$$

The fields obey the standard equal-light-front-time commutation relations, and here we write it out for the dynamical components (suppressing the flavor indices):

$$\{\Psi_{+,c}(x), \Psi^\dagger_{+,c'}(y)\}_{x^+=y^+} = \Lambda_+\delta(x^- - y^-)\delta^2(\vec{x}_\perp - \vec{y}_\perp)\delta_{c,c'} \ , \tag{44}$$

in which recall that $\Lambda_+ = \gamma^0\gamma^+/2$ is the light-front projector , and

$$[A_{i,a}(x), A^\dagger_{j,a'}(y)]_{x^+=y^+} = -\frac{i}{4}\epsilon(x^- - y^-)\delta^2(\vec{x}_\perp - \vec{y}_\perp)\delta_{i,j}\delta_{a,a'} \ , \tag{45}$$

with $i, j = 1, 2$, and $\epsilon(x)$ is the sign function.

II. MESONS AS THE QCD BOUND STATES (0.5 H')

1. *Fock space representation*

The Hilbert space for the single-particle creation and destruction operators is the Fock space. The Fock space can be decomposed into sectors with n Fock particles, in which the number of quarks, antiquarks and gluons, N, $\bar{N}$ and $\tilde{N}$, respectively and $n = N+\bar{N}+\tilde{N}$. Fock states can be defined in terms of the eigenstates of the free-field Hamiltonian, i.e., the light-front kinetic operator, and can be obtained by applying the creation operators on the Fock vacuum $|0\rangle$. The hadron state vector $|\psi_h(P, j, m_j)\rangle$ can be expanded in the Fock space. We use j as the total spin of meson and m_j as its magnetic projection. In the single particle coordinates, it reads

$$\begin{aligned}|\psi_h(P, j, m_j)\rangle = \sum_{n=0}^{\infty} \int \prod_{i=1}^{n} \frac{\mathrm{d}\kappa_i^+ \, \mathrm{d}^2\kappa_{i\perp}}{(2\pi)^3 2\kappa_i^+} \theta(\kappa_i^+) 2P^+ \theta(P^+) (2\pi)^3 \delta^3(\kappa_1 + \kappa_2 + \cdots + \kappa_n - P) \\ \times \sum_{\{l_i, s_i\}} \psi_{n/h}^{(m_j)}(\{\kappa_i, s_i, l_i\}) c^\dagger_{s_1 l_1}(\kappa_1) \ldots c^\dagger_{s_n l_n}(\kappa_n) |0\rangle \ ,\end{aligned} \tag{46}$$

where i is the index of the Fock particle, and it takes values of $i = 1, \ldots, n$ for the n-particle sector. $c^\dagger_{s_i l_i}(\kappa_i)$ is the creation operator for the corresponding constituent (quark, antiquark or gluon). κ_i is the momentum, and each particle is on its mass-shell $\kappa_i^2 = m_i^2$. l is the color index, and s is the spin projection of the particle. $\sum_{\{l_i, s_i\}}$ means the sum of all color and spin arrangements in the string of the creation operators resulting in a sum over a unique set of creation operators with the restriction of producing color-singlet projected states. The construction of the global color singlets for multi-particle states can be found in Ref. [10]. We suppress flavor indices but they can be included in a straightforward manner. $\psi_{n/h}^{(m_j)}(\{\kappa_i, s_i, l_i\})$ are the projection of the physical states to the Fock states, called the light-front wavefunctions (LFWFs).

In the relative particle coordinates, we define

$$x_i \equiv \frac{\kappa_i^+}{P^+}, \qquad \vec{k}_{i\perp} \equiv \vec{\kappa}_{i\perp} - x_i \vec{P}_\perp \ . \tag{47}$$

x_i are known as the longitudinal momentum fractions; $\vec{k}_{i\perp}$ are the relative transverse momenta. They are independent of the total momentum of the bound state, and satisfy $0 < x_i < 1$, $\sum^n x_i = 1$ and $\sum \vec{k}_{i\perp} = \vec{0}$. **[Exercise]** Show that the quantities defined in Eq. (47) are invariant under Lorentz boosts

$$v^+ \to v^+, \quad \vec{v}_\perp \to \vec{v}_\perp + v^+ \vec{\beta}_\perp \ , \tag{48}$$

$$v^+ \to c_\omega v^+, \quad \vec{v}_\perp \to \vec{v}_\perp \ , \tag{49}$$

with c-numbers $\vec{\beta}_\perp$ and c_ω.

The hadron state vector now reads,

$$|\psi_h(P,j,m_j)\rangle = \sum_{n=0}^{\infty}\int\prod_{i=1}^{n}\frac{\mathrm{d}x_i\,\mathrm{d}^2k_{i\perp}}{(2\pi)^3 2x_i}2(2\pi)^3\delta(x_1+x_2+\cdots+x_n-1)\delta^2(\vec{k}_{1\perp}+\vec{k}_{2\perp}+\cdots+\vec{k}_{n\perp}) \\ \times\sum_{\{l_i,s_i\}}\psi^{(m_j)}_{n/h}(\{x_i,\vec{k}_{i\perp},s_i,l_i\})c^\dagger_{s_1l_1}(x_1P^+,\vec{k}_{1\perp}+x_1\vec{P}_\perp)\cdots c^\dagger_{s_nl_n}(x_nP^+,\vec{k}_{n\perp}+x_n\vec{P}_\perp)\,|0\rangle\ , \tag{50}$$

with the LFWFs $\psi^{(m_j)}_{n/h}(\{x_i,\vec{k}_{i\perp},s_i,l_i\})$ in the relative coordinates.

The hadron state vector is normalized as,

$$\langle\psi_h(P,j,m_j)|\psi_{h'}(P',j',m'_j)\rangle = 2P^+\theta(P^+)(2\pi)^3\delta^3(P-P')\delta_{m_j,m'_j}\delta_{j,j'}\delta_{h,h'}\ . \tag{51}$$

Then the normalization of the LFWFs reads,

$$\sum_{n=0}^{\infty}\int\prod_{i=1}^{n}\frac{\mathrm{d}x_i\,\mathrm{d}^2k_{i\perp}}{(2\pi)^3 2x_i}2(2\pi)^3\delta(x_1+\cdots+x_n-1)\delta^2(\vec{k}_{1\perp}+\cdots+\vec{k}_{n\perp})\sum_{\{l_i,s_i\}}\left|\psi^{(m_j)}_{n/h}(\{x_i,\vec{k}_{i\perp},s_i,l_i\})\right|^2 = 1\ . \tag{52}$$

[Exercise] For practical calculations, the infinite Fock space needs to be truncated. Consider a meson state in the $|q\bar{q}\rangle$ Fock sector, write out its light-front wavefunction representation in terms of the relative momenta,

$$x\equiv\frac{k^+_q}{P^+},\qquad \vec{k}_\perp\equiv\vec{k}_{q\perp}-x\vec{P}_\perp\ . \tag{53}$$

The state reads,

$$|h_{q\bar{q}}(P,j,m_j)\rangle = \sum_{s,\bar{s}}\int_0^1\frac{\mathrm{d}x}{2x(1-x)}\int\frac{\mathrm{d}^2k_\perp}{(2\pi)^3}\psi^{(m_j)}_{s\bar{s}/h}(\vec{k}_\perp,x) \\ \times\frac{1}{\sqrt{N_c}}\sum_{i=1}^{N_c}b^\dagger_{si}(xP^+,\vec{k}_\perp+x\vec{P}_\perp)d^\dagger_{\bar{s}i}((1-x)P^+,-\vec{k}_\perp+(1-x)\vec{P}_\perp)\,|0\rangle\ . \tag{54}$$

Here we write the color singlet configuration of the $q\bar{q}$ state, $1/\sqrt{3}(r\bar{r}+g\bar{g}+b\bar{b})$, explicitly with color index i and $N_c=3$ in the above equation. The normalization relation of the valence LFWF $\psi^{(m_j)}_{s\bar{s}/h}(\vec{k}_\perp,x)$ is

$$\sum_{s,\bar{s}}\int_0^1\frac{\mathrm{d}x}{2x(1-x)}\int\frac{\mathrm{d}^2k_\perp}{(2\pi)^3}\psi^{(m'_j)*}_{s\bar{s}/h'}(\vec{k}_\perp,x)\psi^{(m_j)}_{s\bar{s}/h}(\vec{k}_\perp,x) = \delta_{hh'}\delta_{m_j,m'_j}\delta_{h,h'}\ . \tag{55}$$

2. *The eigenvalue equation*

The quarkonium state $|\psi_h\rangle$ is an eigenstate of the light-front Hamiltonian, and satisfies

$$H_{LF}\,|\psi_h\rangle = M_h^2\,|\psi_h\rangle\ , \tag{56}$$

where $H_{LF} = P^+P^- + \vec{P}_\perp^2$ is the light-front Hamiltonian and M_h is the mass of the bound state. Each eigenstate $|\psi_h\rangle$ can be labeled with six eigenvalues, M_h, P^+, $\vec{P}_\perp$, the total spin j and its longitudinal projection m_j.

Projecting the Hamiltonian eigenvalue equation of Eq. (56) onto the Fock space results in an infinite number of coupled integral eigenvalue equations. The solutions of these equations consist of the spectrum and the corresponding wavefunctions, which could fully describe the bound state system. Fock states can be defined in terms of the eigenstates of the free-field Hamiltonian, i.e., the light-front kinetic operator, and can be obtained by applying the creation operators on the Fock vacuum $|0\rangle$:

$$\begin{aligned} Q_0 &\equiv |q\bar{q}:\ k_i^+, \vec{k}_{i\perp}, \lambda_i\rangle = b^\dagger_{\lambda_1}(k_1) d^\dagger_{\lambda_2}(k_2)\,|0\rangle \\ Q_1 &\equiv |q\bar{q}g:\ k_i^+, \vec{k}_{i\perp}, \lambda_i\rangle = b^\dagger_{\lambda_1}(k_1) d^\dagger_{\lambda_2}(k_2) a^\dagger_{\lambda_3}(k_3)\,|0\rangle \\ Q_2 &\equiv |q\bar{q}q\bar{q}:\ k_i^+, \vec{k}_{i\perp}, \lambda_i\rangle = b^\dagger_{\lambda_1}(k_1) d^\dagger_{\lambda_2}(k_2) b^\dagger_{\lambda_3}(k_3) d^\dagger_{\lambda_4}(k_4)\,|0\rangle \\ &\dots \end{aligned} \tag{57}$$

For convenience, we have labeled the various Fock states with index $n = 1, 2, \ldots$. Each Fock state Q_n is an eigenstate of P^+ and $\vec{P}_\perp$, satisfying $P^+ = \sum_i k_i^+$ and $\vec{P}_\perp = \sum_i \vec{k}_\perp$.

In practical calculations, only a finite number of the leading Fock sectors are considered. The eigenvalue equation, Eq. (56), can be written explicitly on the finite Fock basis truncated as,

$$\sum_{j=1}^{N} H_{ij}\,|\psi_j\rangle = M_h^2\,|\psi_i\rangle \qquad \text{for all } i = 1, 2, \ldots, N\ . \tag{58}$$

We define the block matrices $H_{ij} \equiv Q_i H_{LF} Q_j$, and the projected eigenstates $|\psi_i\rangle \equiv Q_i\,|\psi_h\rangle$. One could then proceed to solve the coupled matrix equations in Eq. (58). The resulting eigenstate can be written as $|\psi_h\rangle = \sum_{n=1}^{N} \int \mathrm{d}[k_i] Q_n\,|\psi_n\rangle$.

Even with a finite truncation scheme, solving the Hamiltonian matrix becomes a major challenge in numerical calculations with increasing number of Fock sectors. Could we include the physics from higher Fock sectors while carrying out the calculation at a smaller feasible Fock space? A well known and widely used method is the effective interactions. In field theories, it was first introduced by I.Tamm [11] and rediscovered by S.M.Dancoff [12] to describe the two nucleon forces. It reduces and solves the field equations according to the number of Fock particles.

Although the Tamm-Dancoff approach was applied originally in the instant form, we can derive it analogously in the front form. The Fock space could be arbitrarily divided into two parts, namely the P-space and the Q-space. By choosing a specific partition, we wish to formulate an effective potential acting only in the P-space but including the effects generated by the Q-space. The Hamiltonian matrix equation, Eq. (58),

can then be rewritten as a coupled matrix equation involving the block matrices $H_{\alpha\beta} \equiv \langle\alpha|H_{LF}|\beta\rangle$ and the projected eigenfunctions $|\psi_h\rangle_\alpha = \langle\alpha|\psi_h\rangle$ with $(\alpha, \beta = P, Q)$:

$$H_{PP}\,|\psi\rangle_P + H_{PQ}\,|\psi\rangle_Q = \omega\,|\psi\rangle_P \ , \tag{59a}$$

$$H_{QP}\,|\psi\rangle_P + H_{QQ}\,|\psi\rangle_Q = \omega\,|\psi\rangle_Q \ . \tag{59b}$$

The mass eigenvalue is unknown at this point, and it is written as $\omega = M_h^2$ in the above equations. One can express the Q-space wavefunction $|\psi_h\rangle_Q$ in terms of the P-space wavefunction $|\psi_h\rangle_P$ from Eq. (59b) as,

$$|\psi\rangle_Q = \frac{1}{\omega - H_{QQ}} H_{QP}\,|\psi\rangle_P \ . \tag{60}$$

Plugging it into Eq. (59a), we arrive at an eigenvalue equation with an "effective Hamiltonian" acting only in the P-space:

$$H_{\rm eff}\,|\psi\rangle_P = \omega\,|\psi\rangle_P \ , \tag{61}$$

with

$$H_{\rm eff} = H_{PP} + H_{PQ}\frac{1}{\omega - H_{QQ}} H_{QP} \ . \tag{62}$$

We can see that the effective interaction contains two parts: the original block matrix H_{PP}, and a contribution where the system is scattered virtually into the Q-space and then scattered back to the P-space.

One key problem now is to compute the energy denominator $(\omega - H_{QQ})^{-1}$, since the value of ω is unknown before solving the equations. One could start with some fixed value of ω as the "starting point energy" and calculate $M_h^2(\omega)$ from the eigenvalue equation. The true eigenvalues are determined by varying ω until $\omega = M_h^2(\omega)$ [13, 14]. This procedure, involving inverting a Q-space matrix, however, does not seem to reduce the numerical work of diagonalizing the (P+Q)-space matrix directly. An alternative way is to substitute the eigenvalue ω by T^*, the average kinetic energy of the initial and final P-space states [15]. The idea is to reduce the matrix $\omega - H_{QQ}$ to its dominant term as a c-number. The Q-space matrix H_{QQ} splits into a diagonal kinetic term T_{QQ} and an off-diagonal interaction term U_{QQ}. The inverse matrix could then be written as

$$\frac{1}{\omega - H_{QQ}} = \frac{1}{T^* - T_{QQ} - \delta U(\omega)}, \qquad \delta U(\omega) = \omega - T^* - U_{QQ} \ . \tag{63}$$

In the case of a sufficiently small $\delta U(\omega)$, the energy denominator can be approximated by the kinetic energy $T^* - T_{QQ}$, which no longer depends on the energy eigenvalue.

3. *Mesons in the valence Fock sector*

In solving bound state systems with effective Hamiltonian approaches, the simplest P-space one can choose is the valence Fock sector. For heavy quarkonium, constituent quark models have shown reasonable first approximations in non-relativistic potential models [16, 17]. In the following, we illustrate the formulation of the effective Hamiltonian in the valence Fock sector by choosing $Q_0 = |q\bar{q}\rangle$ as the P-space and $Q_1 = |q\bar{q}g\rangle$ as the Q-space. The eigenvalue equation now reads (signifying the Q_i by its index "i" in the following),

$$\left(H_{00} + H_{01}\frac{1}{\omega - H_{11}}H_{10}\right)\psi_0 = \omega\psi_0 \ . \tag{64}$$

We can write the Hamiltonian as a summation of the kinetic energy and the interaction operator, $H = T + U$. The diagonal block H_{ii} contains T_{ii} and U_{ii}, and the off-diagonal block is $H_{ij} = U_{ij}$, $(i \neq j)$. The interaction matrix U is illustrated in Table 3.

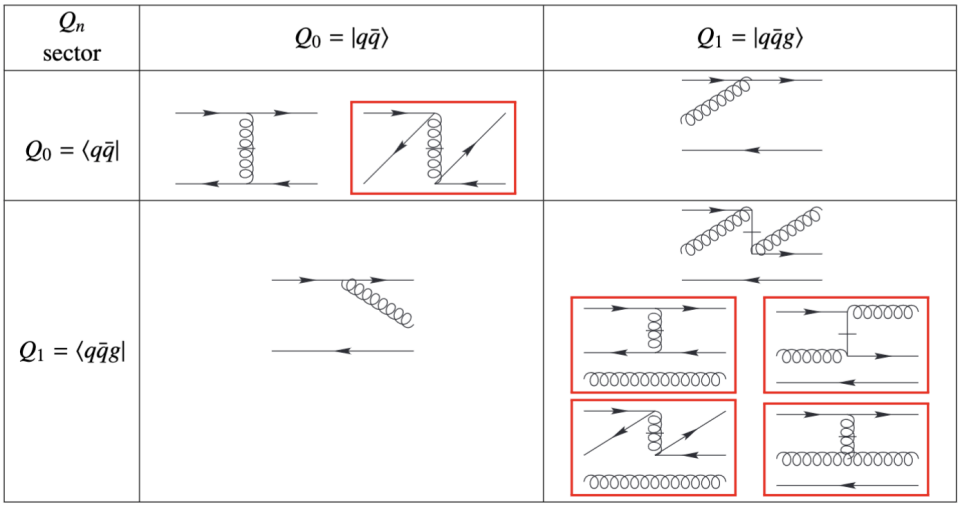


FIG. 3. The interaction matrix U for a meson in the Fock space $|q\bar{q}\rangle + |q\bar{q}g\rangle$. The matrix elements are represented by diagrams. For each diagram where the gluon couples to the quark, there also exists a corresponding diagram with the gluon coupling to the antiquark. Diagrams in the red frames are excluded by color factor or gauge cutoff, see details in the text.

We first focus on the denominator of the second term in Eq. (64). To maintain the gauge invariance in the truncated Fock space, we implement the "gauge cutoff" formulated by Tang, Brodsky, and Pauli [18], that is, the instantaneous parton graph is only retained if the corresponding propagating parton graph contributes in the truncated theory. As a consequence, some instantaneous interactions in U_{00} and U_{11} are excluded. In the U_{11} block, the diagrams in the red frames should not be considered since the corresponding $|q\bar{q}gg\rangle$

sector is absent in the truncated space. The second diagram in U_{00} block vanishes for another reason: zero color factor. We further adopt the approximation $\delta U(\omega) \approx 0$ in Eq. (63), i.e. $U_{11} \to 0$. In principle, this approximation can be improved systematically by performing an expansion in $\delta U(\omega)$ and retaining terms order-by-order in that expansion. The energy denominator now reduces to $T^* - T_{11}$.

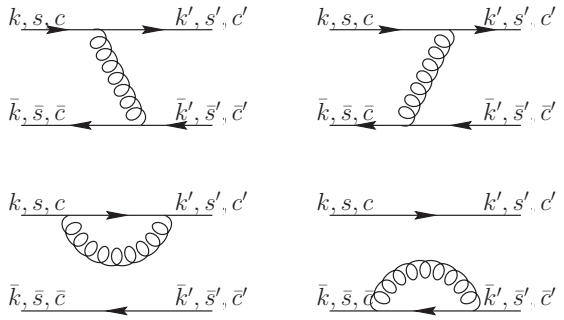


FIG. 4. Iterated interactions generated in the two-body effective interaction. The top two panels are the gluon-exchange diagrams. The bottom two panels are the fermion-self-energy contributions. Each fermion lines are labled by its momentum (k), spin (s) and color (c).

The first term in Eq. (64), H_{00}, contains an instantaneous gluon-exchange interaction, U_{00}. The second term, by stitching U_{01} and U_{10}, generates both fermion-self-energy loops and exchanges of gluons between the quark and the antiquark as shown in Fig. 4. We simplify the interaction by neglecting the self-energy terms in these investigations and we will adopt the strategy of using quark masses as adjustable parameters (called "constituent quarks"). The remaining one-gluon exchange can be combined together with the instantaneous contributions from U_{00} into one term, namely $V_{\rm OGE}$. In the Basis Light-Front Quantization (BLFQ) formalism of ref. [19], the one-gluon exchange term reads,

$$V_{\rm OGE} = -\frac{C_F 4\pi\alpha_s(q^2)}{q^2}\bar{u}_{s'}(k')\gamma_\mu u_s(k)\bar{v}_{\bar{s}}(\bar{k})\gamma^\mu v_{\bar{s}'}(\bar{k}') \,. \tag{65}$$

The energy denominator can now be interpreted as the average 4-momentum squared carried by the exchanged gluon, $q^2 = -(1/2)(k' - k)^2 - (1/2)(\bar{k}' - \bar{k})^2$. C_F is the color factor of the one-gluon exchange diagram, and its calculation follows the corresponding QCD vertices [20]. Here the initial and final quark-antiquark pairs are both in the color singlet configuration, thereby $C_F = 1/4(1/\sqrt{3}c'^\dagger T^\alpha c)(1/\sqrt{3}c^\dagger T^\alpha c') = 4/3$, where T^α ($\alpha = 1, \ldots, 8$) are the Gell-Mann matrices and c, c' = red, blue, green are the color vectors, their expressions can be found in Appendix A 3. The overall"-" sign in Eq. (65) results from the anti-communitation relation of the fermion fields in calculating the vertices, in analogy to the Coulomb potential between two opposite charges in electrodynamics. This term implements the short-distance physics between the quark and the antiquark, and determines the spin structure of the mesons. The eigenvalue equation of

Eq. (64) then reduces to

$$(T_{00} + V_{\rm OGE})\psi_0 = \omega_0\psi_0 \ . \tag{66}$$

The one-gluon exchange interaction $V_{\rm OGE}$ is identical to the one-photon exchange in quantum electrodynamics (QED), except for the color factor. The eigenvalue equation of Eq. (66) with the one-photon exchange has been applied to the positronium system in the basis function approach by Ref. [21]. In the relative coordinate presentation, the kinetic term can be written as $T_{00} = (\vec{k}_\perp^2 + m_q^2)/x + (\vec{k}_\perp^2 + m_{\bar{q}}^2)/(1-x)$. **[Exercise]** Derive this expression of T_{00} from the light-front QCD Hamiltonian in Eq. (38). Recall that $x = p_q^+/P^+$ is the longitudinal momentum fraction of the quark and $\vec{k}_\perp = \vec{k}_{q\perp} - x\vec{P}_\perp$ is the relative transverse momentum.

One can imagine that expanding the Q-space directly would introduce more interaction terms. Apart from the standard way of including interactions from a finite Q-space, there are also phenomenological approaches. Light-front holography constructs an effective Hamiltonian based on inspirations from string theory. It addresses confinement, an essential feature of QCD, by holographic mapping gravity in a higher-dimensional anti-de Sitter(AdS) space to light-front dynamics [22]. In the soft-wall model, a 2-dimensional soft-wall confinement originates from the gravitational background field [23]. Y. Li et al. further improved the confinement by including the longitudinal degree of freedom [19, 24],

$$V_{\rm confinement} = \kappa^4 x(1-x)r_\perp^2 - \frac{\kappa^4}{(m_q + m_{\bar{q}})^2}\partial_x(x(1-x)\partial_x) \ . \tag{67}$$

κ is the strength of the confinement, $r_\perp = |\vec{r}_{q\perp} - \vec{r}_{\bar{q}\perp}|$ is the transverse separation of the partons. This phenomenological confinement takes into account the long-distance physics, and provides another approximation to QCD. The eigenvalue equation provides a more extensive model of QCD by absorbing the confining potential,

$$(T_{00} + V_{\rm OGE} + V_{\rm confinement})\psi_0 = \omega\psi_0 \ . \tag{68}$$

Conventionally, all the contributions in the Hamiltonian excluding the kinetic energy are combined and the resulting interaction is referred to as the effective interaction, $V_{\rm eff} = V_{\rm OGE} + V_{\rm confinement}$. The mass spectrum and LFWFs are the direct solutions of the eigenvalue equation, and could be obtained as in BLFQ by diagonalizing the Hamiltonian in a basis representation.

III. TWO STUDIES OF MESON LIGHT-FRONT WAVEFUNCTIONS (0.25 H')

A. Basis Light-Front Quantization (BLFQ)

In the Hamiltonian approach of studying the meson states, the central task is to diagonalize the QCD Hamiltonian to solve the eigenvalue equation. The basis light-front quantization (BLFQ) has been developed as a flexible computational platform for such purpose, dealing with relativistic strong interaction many-body bound-state problems [10]. Based on the Hamiltonian formalism in light-front dynamics, which we have discussed previously, BLFQ adopts basis function representation. This key feature provides us considerable freedom in the choice of the orthonormal and complete set of basis functions with convenience and convergence rates.

The BLFQ approach is first applied to the heavy quarkonium system in Ref. [19, 24]. As we have already introduced in the proceeding section, the effective light-front Hamiltonian is constructed in the $|q\bar{q}\rangle$ space, consisting of the holographic QCD Hamiltonian and the one-gluon exchange, as in Eq. (68). There are two model parameters, κ and m_q, and they are determined by fitting the mass spectrum of the quarkonium system to experiments. In solving the eigenvalue equation of Eq. (68), the eigenfunctions of part of the Hamiltonian, $T_{00} + V_{\text{confinement}}$, are taken as the basis functions, which largely brings in numerical efficiency. The heavy quarkonium system is then solved in such basis representation, giving the spectrum and the meson LFWFs. Following the applications in the heavy meson system, BLFQ is further developed and applied to the heavy-light system [25], the light mesons [26, 27], and the nucleon [28, 29].

To understand and explain the meson systems from QCD first principles, a significant and more challenging step to take is including higher Fock sectors. The BLFQ study, Ref. [30], addresses the light meson systems in the $|q\bar{q}\rangle + |q\bar{q}g\rangle$ space.

B. Small-basis Light-Front Wavefunction (sLFWF) by design

In a standard light-front Hamiltonian formalism, the meson LFWFs are solved from the Schrödinger(-like) eigenvalue equations. A different path to obtain the meson LFWF is to model it directly or determine it from other formalisms. Works in this category include, the widely used boosted Gaussian [31–33], the LFWFs determined from the Dyson-Schwinger and Bethe-Salpeter approach [34–37], and LFWFs boosted from the NRQCD solution in the rest frame [38, 39].

As a complementary study to the existing modeled LFWFs, Ref. [40] proposed a method of designing the LFWFs of meson bound states with a simple-functional form. Such "by design" approach is apparently not first-principle, in which one chooses by hand to apply the constraints being considered for phenomeno-

logical applications of the wavefunctions. This involves a certain amount of judgement as to how many basis functions to include and which constraints to impose. One primary advantage of this approach is that the resulting LFWFs are analytically tractable and can be used to calculate a wide variety of physical observables.

The following briefs the basic idea of this method. Consider a meson state h consisting of a quark and an antiquark, with momentum $(P^+, \vec{P}_\perp)$, and expand its wavefunction on an orthonormal basis $\{\beta_1, \beta_2, \ldots, \beta_{N_\beta}\}$,

$$\psi_h(\vec{k}_\perp, x) = \sum_{i=1}^{N_\beta} C_{h,i}\beta_i(\vec{k}_\perp, x) \ , \tag{69}$$

where $C_{h,i}$ are the basis coefficients for h and N_β is the number of basis states. Here we are writing the wavefunction in a relative coordinate, where $x = p_q^+/P^+$ is the longitudinal momentum fraction of the quark and $\vec{k}_\perp = \vec{k}_{q\perp} - x\vec{P}_\perp$ is the relative transverse momentum.

The wavefunctions should satisfy the orthonormalization relation

$$\sum_{i=1}^{N_\beta} C_{h,i}C^*_{h',i} = \delta_{h,h'} \ . \tag{70}$$

Physical quantities and observables (O) such as decay widths and charge radius are functions (f_O) of the basis coefficients,

$$O_h = f_O(C_{h,i}) \ . \tag{71}$$

The constraints Eqs. (70) and (71) form a system of equations, and the unknowns are the basis coefficients $C_{h,i}$ and could also include parameters in the basis functions. The procedure of designing LFWFs is, in essence, solving such a system of equations.

In work [40], we modeled the LFWFs for four charmonium states, η_c, J/ψ, ψ', and $\psi(3770)$ as superpositions of orthonormal basis functions. We choose the basis functions as eigenfunctions of an effective Hamiltonian, which has a longitudinal confining potential in addition to the transverse confining potential from light-front holographic QCD, the same basis functions as in BLFQ [19]. We determine the basis function parameters and superposition coefficients by employing both guidance from the nonrelativistic description of the meson states and the experimental measurements of the meson decay widths. With the obtained wavefunctions, we study the features of those meson states, including charge radii and parton distribution functions. The obtained LFWFs have simple-functional forms and can be readily used to predict additional experimental observables.

Appendix A: Conventions

1. Light-Front coordinates

The contravariant four-vectors of position x^μ are written as $x^\mu = (x^+, x^-, x^1, x^2)$, where $x^+ = x^0 + x^3$ is the light-front time, $x^- = x^0 - x^3$ is the longitudinal coordinate, and $\vec{x}_\perp = (x^1, x^2)$ are the transverse coordinates. We sometimes write the transverse components with subscript x (y) in place of 1 (2), for example $\vec{r}_\perp = (r^x, r^y)$.

The covariant vectors are obtained by $x_\mu = g_{\mu\nu}x^\nu$, with the metric tensors $g_{\mu\nu}$ and $g^{\mu\nu}$. The nonzero components of the metric tensors are,

$$g^{+-} = g^{-+} = 2, \qquad g_{+-} = g_{-+} = \frac{1}{2}, \qquad g^{ii} = g_{ii} = -1 \ (i = 1, 2) \ . \tag{A1}$$

Scalar products are

$$a \cdot b = a^\mu b_\mu = a^+ b_+ + a^- b_- + a^1 b_1 + a^2 b_2 = \frac{1}{2}(a^+ b^- + a^- b^+) - \vec{a}_\perp \cdot \vec{b}_\perp \ . \tag{A2}$$

Derivatives are written as

$$\partial_+ = \frac{\partial}{\partial x^+} = \frac{\partial}{2\partial x_-} = \frac{1}{2}\partial^-, \quad \partial_- = \frac{\partial}{\partial x^-} = \frac{\partial}{2\partial x_+} = \frac{1}{2}\partial^+ \ . \tag{A3}$$

We define the integral operators

$$\frac{1}{\partial^+} f(x^-) = \frac{1}{4}\int_{-\infty}^{+\infty} \epsilon(x^- - y^-) f(y^-) \ , \tag{A4}$$

$$\left(\frac{1}{\partial^+}\right)^2 f(x^-) = \frac{1}{8}\int_{-\infty}^{+\infty} |x^- - y^-| f(y^-) \ . \tag{A5}$$

Here, the antisymmetric step function

$$\epsilon(x) = \theta(x) - \theta(-x) \ , \qquad \frac{\partial \epsilon(x)}{\partial x} = 2\delta(x) \ . \tag{A6}$$

with the step function $\theta(x) = 0(x < 0); 1(x > 0)$. It follows that $|x| = x\epsilon(x)$.

[Exercise] For the exponential function, check the following relation,

$$\frac{1}{i\partial^+} e^{-ikx} = \frac{1}{k^+} e^{-ikx} \ . \tag{A7}$$

The Levi-Civita tensor is

$$\epsilon^{\mu\nu\rho\sigma} = \frac{1}{\sqrt{|g|}} \begin{cases} +1, & \text{if } \mu, \nu, \rho, \sigma \text{ is an even permutation of } -, +, 1, 2 \\ -1, & \text{if } \mu, \nu, \rho, \sigma \text{ is an odd permutation of } -, +, 1, 2 \\ 0, & \text{other cases} \end{cases} \tag{A8}$$

in which $g \equiv \det g_{\mu\nu} = -\frac{1}{2}$.

The full four-dimensional integral is

$$\int \mathrm{d}^4x = \int \mathrm{d}x^0\,\mathrm{d}x^1\,\mathrm{d}x^2\,\mathrm{d}x^3 = \frac{1}{2}\int \mathrm{d}x^+\,\mathrm{d}x^-\,\mathrm{d}^2x_\perp = \int \mathrm{d}^3x\,\mathrm{d}x^+ \,, \tag{A9}$$

where we also define the volume integral as

$$\int \mathrm{d}^3x \equiv \int \mathrm{d}x_+\,\mathrm{d}^2x^\perp = \frac{1}{2}\int \mathrm{d}x^-\,\mathrm{d}^2x^\perp \,. \tag{A10}$$

In the momentum space, the Lorentz invariant integral is,

$$\begin{aligned}\int \frac{\mathrm{d}^4p}{(2\pi)^4}\theta(p^+)(2\pi)\delta(p^+p^- - \vec{p}_\perp^{\,2} - m^2) =& \frac{1}{2}\int \frac{\mathrm{d}p^+\,\mathrm{d}p^-\,\mathrm{d}^2p_\perp}{(2\pi)^4}\theta(p^+)(2\pi)\delta(p^+p^- - \vec{p}_\perp^{\,2} - m^2)\\ =& \int \frac{\mathrm{d}^2p_\perp\,\mathrm{d}p^+}{(2\pi)^3 2p^+}\theta(p^+)\end{aligned} \tag{A11}$$

The Fourier transform of a function $f(\vec{r}_\perp)$ and the inverse transform are defined as

$$f(\vec{r}_\perp) = \int \frac{\mathrm{d}^2p_\perp}{(2\pi)^2} e^{i\vec{p}_\perp\cdot\vec{r}_\perp}\tilde{f}(\vec{p}_\perp), \qquad \tilde{f}(\vec{p}_\perp) = \int \mathrm{d}^2\vec{r}_\perp e^{-i\vec{p}_\perp\cdot\vec{r}_\perp} f(\vec{r}_\perp) \,. \tag{A12}$$

The Dirac deltas read

$$\int \mathrm{d}^2\vec{r}_\perp e^{-i\vec{p}_\perp\cdot\vec{r}_\perp} = (2\pi)^2\delta^2(\vec{p}_\perp), \qquad \int \mathrm{d}^2\vec{p}_\perp e^{i\vec{p}_\perp\cdot\vec{r}_\perp} = (2\pi)^2\delta^2(\vec{r}_\perp) \,. \tag{A13}$$

2. γ matrices

The Dirac matrices are four unitary traceless 4×4 matrices:

$$\gamma^0 = \beta = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \gamma^+ = \begin{pmatrix} 0 & 0 \\ 2i & 0 \end{pmatrix}, \quad \gamma^- = \begin{pmatrix} 0 & -2i \\ 0 & 0 \end{pmatrix}, \quad \gamma^i = \begin{pmatrix} -i\hat{\sigma}^i & 0 \\ 0 & i\hat{\sigma}^i \end{pmatrix} . \tag{A14}$$

They are expressed in terms of the 2×2 Pauli matrices,

$$\hat{\sigma}^1 = \sigma^2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \hat{\sigma}^2 = -\sigma^1 = \begin{pmatrix} 0 & -1 \\ -1 & 0 \end{pmatrix} . \tag{A15}$$

Note that $\gamma^3 = \gamma^+ - \gamma^0$. It is also convenient to define $\gamma^R \equiv \gamma^1 + i\gamma^2$ and $\gamma^L \equiv \gamma^1 + i\gamma^2$. The chiral matrix is $\gamma^5 = i\gamma^0\gamma^1\gamma^2\gamma^3$. Some useful relations,

$$\gamma^1\gamma^+\gamma^1 = \gamma^2\gamma^+\gamma^2 = \gamma^+, \quad \gamma^1\gamma^+\gamma^2 = -\gamma^2\gamma^+\gamma^1 = i\gamma^+ \tag{A16}$$

$$\gamma^0\gamma^\mu = \gamma^{\mu\dagger}\gamma^0, \qquad \{\gamma^\mu, \gamma^\nu\} = 2g^{\mu\nu}I \tag{A17}$$

$$\alpha^\kappa = \gamma^0\gamma^\kappa, \quad (\alpha^1)^2 = (\alpha^2)^2 = I, \quad \alpha^1\alpha^2 = -\alpha^1\alpha^2 \tag{A18}$$

Combinations of Dirac matrices as projection operators,

$$\Lambda^\pm = \frac{1}{4}\gamma^\mp\gamma^\pm = \frac{1}{2}\gamma^0\gamma^\pm = \frac{1}{2}(I \pm \alpha^3)\ . \tag{A19}$$

They have the following properties,

$$\begin{aligned} &\Lambda^+ + \Lambda^- = I\,, \quad (\Lambda^\pm)^2 = \Lambda^\pm\,, \quad \Lambda^\pm\Lambda^\mp = 0\,, \quad (\Lambda^\pm)^\dagger = \Lambda^\pm\,, \\ &\alpha^i\Lambda^\pm = \Lambda^\mp\alpha^i\,, \quad \gamma^0\Lambda^\pm = \Lambda^\mp\gamma^0\ . \end{aligned} \tag{A20}$$

We use the following spinor representation, The u, v spinors are defined as,

$$\begin{aligned} u(p,\lambda=\frac{1}{2}) &= \frac{1}{\sqrt{p^+}}(p^+, 0, im_q, ip^x - p^y)^\intercal\ , \\ u(p,\lambda=-\frac{1}{2}) &= \frac{1}{\sqrt{p^+}}(0, p^+, -ip^x - p^y, im_q)^\intercal\ , \\ \bar{u}(p,\lambda=\frac{1}{2}) &= \frac{1}{\sqrt{p^+}}(m_q, p^x - ip^y, -ip^+, 0)\ , \\ \bar{u}(p,\lambda=-\frac{1}{2}) &= \frac{1}{\sqrt{p^+}}(-p^x - ip^y, m_q, 0, -ip^+)\ , \end{aligned} \tag{A21}$$

and

$$\begin{aligned} v(p,\lambda=\frac{1}{2}) &= \frac{1}{\sqrt{p^+}}(p^+, 0, -im_q, ip^x - p^y)^\intercal\ , \\ v(p,\lambda=-\frac{1}{2}) &= \frac{1}{\sqrt{p^+}}(0, p^+, -ip^x - p^y, -im_q)^\intercal\ , \\ \bar{v}(p,\lambda=\frac{1}{2}) &= \frac{1}{\sqrt{p^+}}(-m_q, p^x - ip^y, -ip^+, 0)\ , \\ \bar{v}(p,\lambda=-\frac{1}{2}) &= \frac{1}{\sqrt{p^+}}(-p^x - ip^y, -m_q, 0, -ip^+)\ . \end{aligned} \tag{A22}$$

The polarization vectors for gluon are defined as

$$e(k,\lambda=\pm1) = (0, \frac{2\epsilon_\lambda^\perp\cdot\vec{k}_\perp}{k^+}, \epsilon_\lambda^\perp) \tag{A23}$$

where $\epsilon_\pm^\perp = (1, \pm i)/\sqrt{2}$.

3. QCD color space

The specification of the quark state in the color space is by a three-element column vector c,

$$c = \begin{pmatrix}1\\0\\0\end{pmatrix} \text{ for red,} \quad \begin{pmatrix}0\\1\\0\end{pmatrix} \text{ for blue,} \quad \begin{pmatrix}0\\0\\1\end{pmatrix} \text{ for green.} \tag{A24}$$

We use the standard basis for the fundamental representation of SU(3), i.e. the Gell-Mann matrices,

$$T^1 = \frac{1}{2}\begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad T^2 = \frac{1}{2}\begin{pmatrix} 0 & -i & 0 \\ i & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad T^3 = \frac{1}{2}\begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{pmatrix},$$
$$T^4 = \frac{1}{2}\begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix}, \quad T^5 = \frac{1}{2}\begin{pmatrix} 0 & 0 & -i \\ 0 & 0 & 0 \\ i & 0 & 0 \end{pmatrix}, \quad T^6 = \frac{1}{2}\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}, \tag{A25}$$
$$T^7 = \frac{1}{2}\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -i \\ 0 & i & 0 \end{pmatrix}, \quad T^8 = \frac{1}{2\sqrt{3}}\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -2 \end{pmatrix}.$$

In the matrix notation, $A^\mu = T^a A^\mu_a$ with the gluon index $a = 1, \ldots, 8$. The color matrix element $A^\mu_{cc'} = T^a_{cc'} A^\mu_a$

$$A^\mu = \frac{1}{2}\begin{pmatrix} \frac{1}{\sqrt{3}}A^\mu_8 + A^\mu_3 & A^\mu_1 - iA^\mu_2 & A^\mu_4 - iA^\mu_5 \\ A^\mu_1 + iA^\mu_2 & \frac{1}{\sqrt{3}}A^\mu_8 - A^\mu_3 & A^\mu_6 - iA^\mu_7 \\ A^\mu_4 + iA^\mu_5 & A^\mu_6 + iA^\mu_7 & -\frac{2}{\sqrt{3}}A^\mu_8 \end{pmatrix}. \tag{A26}$$

[1] S. J. Brodsky, H.-C. Pauli, and S. S. Pinsky, Phys. Rept. **301**, 299 (1998), arXiv:hep-ph/9705477.

[2] A. Harindranath, in *International School on Light-Front Quantization and Non-Perturbative QCD (To be followed by the Workshop 3-14 Jun 1996)* (1996) arXiv:hep-ph/9612244.

[3] S. Weinberg, *The Quantum theory of fields. Vol. 1: Foundations* (Cambridge University Press, 2005).

[4] P. A. M. Dirac, Rev. Mod. Phys. **21**, 392 (1949).

[5] E. P. Biernat, W. Schweiger, K. Fuchsberger, and W. H. Klink, Phys. Rev. **C79**, 055203 (2009), arXiv:0902.2348 [nucl-th].

[6] E. P. Biernat, W. H. Klink, and W. Schweiger, *International Workshop on Relativistic Description of Two- and Three-body Systems in Nuclear Physics Trento, Italy, October 19-23, 2009*, Few Body Syst. **49**, 149 (2011), arXiv:1008.0244 [nucl-th].

[7] M. Gomez-Rocha and W. Schweiger, Phys. Rev. **D86**, 053010 (2012), arXiv:1206.1257 [hep-ph].

[8] T. Heinzl, *Methods of quantization. Proceedings, 39. Internationale Universitätswochen für Kern- und Teilchenphysik, IUKT 39: Schladming, Austria, February 26-March 4, 2000*, Lect. Notes Phys. **572**, 55 (2001), arXiv:hep-th/0008096 [hep-th].

[9] M. E. Peskin and D. V. Schroeder, *An Introduction to quantum field theory* (Addison-Wesley, Reading, USA, 1995).

[10] J. P. Vary, H. Honkanen, J. Li, P. Maris, S. J. Brodsky, A. Harindranath, G. F. de Teramond, P. Sternberg, E. G. Ng, and C. Yang, Phys. Rev. **C81**, 035205 (2010), arXiv:0905.1411 [nucl-th].

[11] I. Tamm, J. Phys.(USSR) **9**, 449 (1945).

[12] S. M. Dancoff, Phys. Rev. **78**, 382 (1950).

[13] H.-C. Pauli, Nucl. Phys. **A369**, 413 (1981).

[14] D. C. Zheng, J. P. Vary, and B. R. Barrett, Nucl. Phys. **A560**, 211 (1993), arXiv:nucl-th/9301013 [nucl-th].

[15] M. Krautgartner, H. C. Pauli, and F. Wolz, Phys. Rev. **D45**, 3755 (1992).

[16] T. Appelquist, R. M. Barnett, and K. D. Lane, Ann. Rev. Nucl. Part. Sci. **28**, 387 (1978).

[17] S. Godfrey and N. Isgur, Phys. Rev. **D32**, 189 (1985).

[18] A. C. Tang, S. J. Brodsky, and H. C. Pauli, Phys. Rev. **D44**, 1842 (1991).

[19] Y. Li, P. Maris, and J. P. Vary, Phys. Rev. **D96**, 016022 (2017), arXiv:1704.06968 [hep-ph].

[20] D. Griffiths, *Introduction to elementary particles* (2008).

[21] P. Wiecki, Y. Li, X. Zhao, P. Maris, and J. P. Vary, Phys. Rev. **D91**, 105009 (2015), arXiv:1404.6234 [nucl-th].

[22] S. J. Brodsky, G. F. de Teramond, H. G. Dosch, and J. Erlich, Phys. Rept. **584**, 1 (2015), arXiv:1407.8131 [hep-ph].

[23] A. Karch, E. Katz, D. T. Son, and M. A. Stephanov, Phys. Rev. **D74**, 015005 (2006), arXiv:hep-ph/0602229 [hep-ph].

[24] Y. Li, P. Maris, X. Zhao, and J. P. Vary, Phys. Lett. **B758**, 118 (2016), arXiv:1509.07212 [hep-ph].

[25] S. Tang, Y. Li, P. Maris, and J. P. Vary, Eur. Phys. J. C **80**, 522 (2020), arXiv:1912.02088 [nucl-th].

[26] S. Jia and J. P. Vary, Phys. Rev. **C99**, 035206 (2019), arXiv:1811.08512 [nucl-th].

[27] W. Qian, S. Jia, Y. Li, and J. P. Vary, Phys. Rev. C **102**, 055207 (2020), arXiv:2005.13806 [nucl-th].

[28] C. Mondal, S. Xu, J. Lan, X. Zhao, Y. Li, D. Chakrabarti, and J. P. Vary, PoS **LC2019**, 067 (2020), arXiv:2001.04414 [hep-ph].

[29] S. Xu, C. Mondal, J. Lan, X. Zhao, Y. Li, and J. P. Vary (BLFQ), Phys. Rev. D **104**, 094036 (2021), arXiv:2108.03909 [hep-ph].

[30] J. Lan, K. Fu, C. Mondal, X. Zhao, and j. P. Vary (BLFQ), Phys. Lett. B **825**, 136890 (2022), arXiv:2106.04954 [hep-ph].

[31] H. Kowalski, L. Motyka, and G. Watt, Phys. Rev. D **74**, 074016 (2006), arXiv:hep-ph/0606272.

[32] J. Nemchik, N. N. Nikolaev, and B. G. Zakharov, Phys. Lett. B **341**, 228 (1994), arXiv:hep-ph/9405355.

[33] J. Nemchik, N. N. Nikolaev, E. Predazzi, and B. G. Zakharov, Z. Phys. C **75**, 71 (1997), arXiv:hep-ph/9605231.

[34] C. Shi and I. C. Cloët, Phys. Rev. Lett. **122**, 082301 (2019), arXiv:1806.04799 [nucl-th].

[35] C. Mezrag, H. Moutarde, and J. Rodriguez-Quintero, Few Body Syst. **57**, 729 (2016), arXiv:1602.07722 [nucl-th].

[36] W. de Paula, E. Ydrefors, J. H. Alvarenga Nogueira, T. Frederico, and G. Salmè, Phys. Rev. D **103**, 014002 (2021), arXiv:2012.04973 [hep-ph].

[37] C. Shi, Y.-P. Xie, M. Li, X. Chen, and H.-S. Zong, Phys. Rev. D **104**, L091902 (2021), arXiv:2101.09910 [hep-ph].

[38] M. Krelina, J. Nemchik, and R. Pasechnik, Eur. Phys. J. C **80**, 92 (2020), arXiv:1909.12770 [hep-ph].

[39] T. Lappi, H. Mäntysaari, and J. Penttala, Phys. Rev. D **102**, 054020 (2020), arXiv:2006.02830 [hep-ph].

[40] M. Li, Y. Li, G. Chen, T. Lappi, and J. P. Vary, Eur. Phys. J. C **82**, 1045 (2022), arXiv:2111.07087 [hep-ph].