Mesons on the light front
Part 2: Electromagnetic transitions and form factors

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This lecture note is written for "Courses on Light-Cone Techniques applied to QCD", Nov 21-25, IGFAE. It is intended to provide the basic knowledge and selective perspectives on the application of light-front Hamiltonian approach to mesons in two 1.5-hour lectures. This is part 2(of 2). The content of this section is mainly based on Chapter 2 and Appendix D of Ref. [1] (the author's Ph.D. thesis), and Refs. [2–5]. Chapter 10 of Ref. [6] (QFT by Weinberg) could also be very helpful.

I. ELECTROMAGNETIC AND TRANSITION FORM FACTORS

The content of this section is mainly based on Chapter 2 and Appendix D of Ref. [1] (the author's Ph.D. thesis), Refs. [2–5]. Chapter 10 of Ref. [6] (QFT by Weinberg) could also be very helpful.

In quantum field theory, the electromagnetic (EM) elastic form factors (EFFs) characterize the structure of a bound state system, which generalize the multipole expansion of the charge and current densities in the nonrelativistic quantum mechanics. The physical process that determines the EFFs is $\psi_h(P) + \gamma^{(*)}(q = P' - P) \to \psi_h(P')$. The form factors are defined as the Lorentz invariants arising in the Lorentz structure decomposition of the hadron matrix element $\langle\psi_h(P')|J^\mu(x)|\psi_h(P)\rangle$. For a spin-j particle, assuming charge conjugation, parity and time reversal symmetries, there are $2j + 1$ independent Lorentz invariant form factors. Similarly, the EM transition form factors (TFFs) arise in the transition between two meson states via emission of a photon, $\psi_A(P') \to \psi_B(P) + \gamma(q = P' - P)$, and the corresponding hadron matrix element is $\langle\psi_B(P)|J^\mu(x)|\psi_A(P')\rangle$. Both EFFs and TFFs could help us in understanding the internal structure of mesons.

There is a connection between the experimentally measured decay width and the hadron matrix elements. In the physical process of $A \to B + \gamma$, the photon is on shell ($q^2 = 0$). The transition amplitude is

$$\mathcal{M}^\lambda_{m_j,m'_j} = I^\mu_{m_j,m'_j}\epsilon^*_{\mu,\lambda}(q)|_{q^2=0}\ , \qquad I^\mu_{m_j,m'_j} \equiv \langle B(P, j_B, m_j)|\, J^\mu(0)\, |A(P', j_A, m'_j)\rangle\ , \tag{1}$$

with the hadron matrix element $I^\mu_{m_j,m'_j}$ written out with the magnetic projections, and $\epsilon_{\mu,\lambda}$ the polarization vector of the final-state photon with its spin projection $\lambda = \pm 1$. The decay width is usually measured in the rest frame of the initial particle A, as such, the momenta of the initial meson, final meson, and the photon

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read (see Appendix A 1 for the convention of ordering the 4-vector components in light-front coordinates), $P' = (m_A, m_A, 0, 0)$, $P = (\sqrt{m_B^2 + k^2}, \sqrt{m_{\mathcal{P}}^2 + k^2}, k, 0)$, and $q = P' - P = (k, k, -k, 0)$. The momentum of the photon is determined by energy-momentum conservation, $|\vec{q}| = k = (m_A^2 - m_B^2)/2m_A$. The decay width of $A \to B + \gamma$ follows by averaging over the initial polarization and summing over the final polarization.

$$\Gamma(A \to B + \gamma) = \int \mathrm{d}\Omega_q \frac{1}{32\pi^2} \frac{|\vec{q}|}{m_A^2} \frac{1}{2j_A + 1} \sum_{m_j, m'_j, \lambda} |\mathcal{M}^{\lambda}_{m_j, m'_j}|^2 \ . \tag{2}$$

We proceed by first examining the Lorentz vector decomposition of the hadron matrix elements in Sec. I A, then writing out the matrix elements in the light-front wavefunction representation in Sec. I B, and finally we calculate the EFFs and TFFs in Secs. I D and I E.

A. Lorentz structure decomposition

According to spacetime translational invariance, the matrix element of the EM current operator satisfies

$$\langle \psi'_h(P', j', m'_j)| J^{\mu}(x) |\psi_h(P, j, m_j)\rangle = \langle \psi'_h(P', j', m'_j)| J^{\mu}(0) |\psi_h(P, j, m_j)\rangle \, e^{i(P-P')\cdot x} \ . \tag{3}$$

[Exercise] Check the above equation. You will see it again in Eq. (45).

The current conservation condition $\partial_\mu J^\mu = 0$ leads to

$$(P' - P)_\mu \langle \psi'_h(P', j', m'_j)| J^{\mu}(0) |\psi_h(P, j, m_j)\rangle = 0 \ . \tag{4}$$

The charge operator on the light front is defined as

$$Q \equiv \int \mathrm{d}x_+ \, \mathrm{d}^2 x_\perp \, J^+(x) \ . \tag{5}$$

The eigenvalue of Q on for a particle state is interpreted as the charge of that particle,

$$Q \, |\psi_h(P, j, m_j)\rangle = e_h \, |\psi_h(P, j, m_j)\rangle \ . \tag{6}$$

The evaluation of the charge operator on a particle state leads to a normalization relation at zero momentum transfer, i.e., $P' = P$,

$$\langle \psi_h(P', j, m_j)| \, Q \, |\psi_h(P, j, m_j)\rangle = \int \mathrm{d}x_+ \, \mathrm{d}^2 x_\perp \, \langle \psi_h(P', j, m_j)| \, J^+(x) \, |\psi_h(P, j, m_j)\rangle \ . \tag{7}$$

That is,

$$2e_h P^+ (2\pi)^3 \delta^3(P - P') = (2\pi)^3 \delta^3(P - P') \, \langle \psi_h(P', j, m_j)| \, J^+(0) \, |\psi_h(P, j, m_j)\rangle \ , \tag{8}$$

thus

$$\langle\psi_h(P,j,m_j)|\,J^+(0)\,|\psi_h(P,j,m_j)\rangle = 2e_hP^+ \ . \tag{9}$$

The form factors of particle transitions are those coefficients F_i of vectors V_i obtained by decomposing the hadron matrix element,

$$\langle\psi'_h(P',j',m'_j)|\,J^\mu(0)\,|\psi_h(P,j,m_j)\rangle = \sum_i^n F_iV_i^\mu \ . \tag{10}$$

1. *Spin* 0 *mesons*

In the quark model, a spin-0 ($J = 0$) meson could be either a scalar 0^+ or a pseudo-scalar 0^-. And C = 1 for quarkonium. The matrix element of the current reads

$$\langle h'_{q\bar{q}}(P',j'=m'_i=0)|\,J^\mu(0)\,|h_{q\bar{q}}(P,j=m_i=0)\rangle = e_q\mathscr{J}^\mu \ , \tag{11}$$

where $\mathscr{J}^\mu$ is a four-vector function of P'^μ and P^μ. Relevant scalars are $|P'|$, $|P|$ and $P'\cdot P$. The first two are fixed by the on shell conditions,

$$P'^\mu P'_\mu = m_{h'}^2, \qquad P^\mu P_\mu = m_h^2 \ . \tag{12}$$

Therefore the coefficients of vectors should only depend on $P'\cdot P$. Define

$$q^\mu \equiv P'^\mu - P^\mu, \qquad \bar{p}^\mu \equiv P'^\mu + P^\mu \ , \tag{13}$$

and decompose $\mathscr{J}^\mu$ into the form of

$$\mathscr{J}^\mu = q^\mu H(q^2) + \bar{p}^\mu F(q^2) \ , \tag{14}$$

The condition of current conservation in Equation (4) requires,

$$q_\mu\cdot q^\mu H(q^2) + q_\mu\cdot\bar{p}^\mu F(q^2) = 0 \ . \tag{15}$$

This means there is only one independent form factor,

$$H(q^2) = -\frac{q_\mu\cdot\bar{p}^\mu}{q_\mu\cdot q^\mu}F(q^2) = -\frac{m_{h'}^2-m_h^2}{q^2}F(q^2) \ . \tag{16}$$

It follows that

$$\langle h'_{q\bar{q}}(P')|\,J^\mu(0)\,|h_{q\bar{q}}(P)\rangle = e_q[\bar{p}^\mu - \frac{m_{h'}^2-m_h^2}{q^2}q^\mu]F(q^2) \ , \tag{17}$$

and $F(q^2)$ is the electromagnetic form factor. To satisfy hermiticity,

$$\langle h'_{q\bar{q}}(P')|\, J^\mu(0)\, |h_{q\bar{q}}(P)\rangle = \langle h_{q\bar{q}}(P)|\, J^\mu(0)\, |h'_{q\bar{q}}(P')\rangle^* \ , \tag{18}$$

$F(q^2)$ must be real.

For the elastic scattering, $h' = h$, thus $m_{h'} = m_h$,

$$\langle h_{q\bar{q}}(P')|\, J^\mu(0)\, |h_{q\bar{q}}(P)\rangle = e_q \bar{p}^\mu F(q^2) \ . \tag{19}$$

Compare with the normalization relation in Eq. (9), we get $F(0) = 1$.

Let us now analyze the symmetries of parity and charge conjugation (see Appendix B 2), and find out what kind of transitions are allowed. We first insert two complete sets of the parity operator to the matrix element,

$$\begin{aligned}\langle h'_{q\bar{q}}(P', \mathrm{P}_2)|\, J^\mu(0)\, |h_{q\bar{q}}(P, \mathrm{P}_1)\rangle =& \langle h'_{q\bar{q}}(P', \mathrm{P}_2)|\, \mathbb{P}\mathbb{P}^{-1} J^\mu(0) \mathbb{P}\mathbb{P}^{-1}\, |h_{q\bar{q}}(P, \mathrm{P}_1)\rangle \\ =& \mathrm{P}_2\mathrm{P}_1 \mathcal{P}^\mu_\nu \langle h'_{q\bar{q}}(\mathcal{P}\cdot P', \mathrm{P}_2)|\, J^\nu(0)\, |h_{q\bar{q}}(\mathcal{P}\cdot P, \mathrm{P}_1)\rangle \\ =& e_q \mathrm{P}_2\mathrm{P}_1 \mathcal{P}^\mu_\nu \mathcal{P}^\nu_\rho [\bar{p}^\rho - \frac{m^2_{h'} - m^2_h}{q^2} q^\rho] F(q^2) \\ =& e_q \mathrm{P}_2\mathrm{P}_1 [\bar{p}^\mu - \frac{m^2_{h'} - m^2_h}{q^2} q^\mu] F(q^2) \ .\end{aligned} \tag{20}$$

Compare with Eq (17), we arrive at

$$\mathrm{P}_2\mathrm{P}_1 = +1 \ . \tag{21}$$

This means the electromagnetic transitions of spin 0 particles preserves the parity. The allowed transition modes are $0^+ \to 0^+$ (scalar-to-scalar) and $0^- \to 0^-$ (pseudoscalar-to-pseudoscalar).

We then consider the charge conjugation of quarkonium.

$$\begin{aligned}\langle h'_{q\bar{q}}(P', \mathrm{C}_2)|\, J^\mu(0)\, |h_{q\bar{q}}(P, \mathrm{C}_1)\rangle =& \langle h'_{q\bar{q}}(P', \mathrm{C}_2)|\, \mathbb{C}\mathbb{C}^{-1} J^\mu(0) \mathbb{C}\mathbb{C}^{-1}\, |h_{q\bar{q}}(P, \mathrm{C}_1)\rangle \\ =& -\mathrm{C}_2\mathrm{C}_1 \langle h'_{q\bar{q}}(P', \mathrm{C}_2)|\, J^\mu(0)\, |h_{q\bar{q}}(P, \mathrm{C}_1)\rangle \ .\end{aligned} \tag{22}$$

Compare with Eq (17), we arrive at

$$\mathrm{C}_2\mathrm{C}_1 = -1 \ . \tag{23}$$

This means the electromagnetic transitions of quarkonium must change the charge conjugation. However, all the spin 0 quarkonium have the same parity conjugation $\mathrm{C} = +1$. Therefore the form factors for spin-0 quarkonium are zero.

2. Spin-0 ↔ spin-1 mesons

The matrix element of the transition between a spin-0 and a spin-1 meson reads

$$\langle h'_{q\bar{q}}(P', j' = 1, m'_j = 0, \pm 1)|\, J^\mu(0)\, |h_{q\bar{q}}(P, j = 0, m_j = 0)\rangle = e^{\alpha *}(P', m'_j)\Gamma^\mu_\alpha \ , \tag{24}$$

where e^* is the spin vector defined in Appendix B and Γ^μ_α is a 2nd-order tensor function of P^μ, P'^μ, $g^{\mu\nu}$ and $\epsilon^{\mu\nu\rho\sigma}$. Note that we did not write out the charge here for simplicity. All possible non-vanishing combinations are

$$\begin{aligned} P^\mu, P'^\mu \ &: P^\mu P_\alpha, P^\mu P'_\alpha, P'^\mu P'_\alpha, P'^\mu P_\alpha \ , \\ g_{\alpha\beta} \ &: g^\mu_\alpha \ , \\ \epsilon^{\mu\nu\rho\sigma} \ &: \epsilon^\mu_{\alpha\rho\sigma} P^\rho P'^\sigma \ . \end{aligned} \tag{25}$$

Contracting with the spin vectors in Eq. (24), and according to the Proca equation in B,

$$P_\beta e^\beta(P, m_j) = 0 \ , \qquad P'_\alpha e^{\alpha *}(P', m'_j) = 0 \ , \tag{26}$$

we get all possible non-vanishing vectors of $e^{\alpha *}(P', m'_j)\Gamma^\mu_\alpha$.

$$\begin{aligned} &P^\mu(e^*(P', m'_j) \cdot P), P'^\mu(e^*(P', m'_j) \cdot P) \ , \\ &e^{\mu *}(P', m'_j) \ , \\ &\epsilon^\mu_{\alpha\rho\sigma} P^\rho P'^\sigma e^{\alpha *}(P', m'_j) \ . \end{aligned} \tag{27}$$

Their coefficients are functions of $|P'|$, $|P|$ and $P' \cdot P$. The first two are fixed by on shell conditions,

$$P'^\mu P'_\mu = m^2_{h'}, \qquad P^\mu P_\mu = m^2_h \ . \tag{28}$$

Therefore those coefficients should only depend on $P' \cdot P$. For convenience, we define

$$q^\mu \equiv P'^\mu - P^\mu, \qquad \bar{p}^\mu \equiv P'^\mu + P^\mu \ . \tag{29}$$

The on shell condition now reads

$$q_\mu q^\mu = q^2 \ , \qquad q_\mu \bar{p}^\mu = m^2_{h'} - m^2_h \equiv \Delta_m \ . \tag{30}$$

We thereby write $e^{\alpha *}(P', m'_j)\Gamma^\mu_\alpha$ as a linear combination of the vectors we found,

$$\begin{aligned} e^{\alpha *}(P', m'_j)\Gamma^\mu_\alpha = &\bar{p}^\mu(e^*(P', m'_j) \cdot P)F_1 + q^\mu(e^*(P', m'_j) \cdot P)F_2 + e^{\mu *}(P', m'_j)F_3 \\ &+ \epsilon^\mu_{\alpha\rho\sigma}\bar{p}^\rho q^\sigma e^{\alpha *}(P', m'_j)F_4 \ . \end{aligned} \tag{31}$$

The condition of current conservation in Eq. (4) requires,

$$q_\mu e^{\alpha *}(P', m_j')\Gamma^\mu_\alpha = 0 \ . \tag{32}$$

That is,

$$0 =(e^*(P', m_j') \cdot P)[\Delta_m F_1 + q^2 F_2 - F_3] + \epsilon^\mu_{\alpha\rho\sigma}\bar{p}^\rho q^\sigma q_\mu e^{\alpha *}(P', m_j')F_4 \ . \tag{33}$$

F_4 survives since

$$\epsilon^\mu_{\alpha\rho\sigma}\bar{p}^\rho q^\sigma q_\mu = 0 \tag{34}$$

The other terms satisfy,

$$\Delta_m F_1 + q^2 F_2 - F_3 = 0 \tag{35}$$

We therefore rewrite the vector decomposition with new coefficients,

$$\begin{aligned} e^{\alpha *}(P', m_j')\Gamma^\mu_\alpha =&[\bar{p}^\mu(e^*(P', m_j') \cdot P) - q^\mu(e^*(P', m_j') \cdot P)\Delta_m/q^2]H_1 \\ &+ [e^{\mu *}(P', m_j') + q^\mu(e^*(P', m_j') \cdot P)/q^2]H_2 + \epsilon^\mu_{\alpha\rho\sigma}\bar{p}^\rho q^\sigma e^{\alpha *}(P', m_j')H_3 \ . \end{aligned} \tag{36}$$

Parity invariance requires that

$$\begin{aligned} &\langle h'_{q\bar{q}}(P', j' = 1, m_j', \mathrm{P}_2)|J^\mu(0)\,|h_{q\bar{q}}(P, j = 0, m_j, \mathrm{P}_1)\rangle \\ &\quad = \langle h'_{q\bar{q}}(P', j' = 1, m_j', \mathrm{P}_2)|\ \mathbb{P}\mathbb{P}^{-1}J^\mu(0)\mathbb{P}\mathbb{P}^{-1}\ |h_{q\bar{q}}(P, j = 0, m_j, \mathrm{P}_1)\rangle \\ &\quad =\mathrm{P}_2\mathrm{P}_1\mathcal{P}^\mu_\nu\ \langle h'_{q\bar{q}}(\mathcal{P} \cdot P', j' = 1, m_j', \mathrm{P}_2)|\ J^\nu(0)\,|h_{q\bar{q}}(\mathcal{P} \cdot P, j = 0, m_j, \mathrm{P}_1)\rangle \ . \end{aligned} \tag{37}$$

The matrix element under the parity transformation reads

$$\begin{aligned} &\langle h'_{q\bar{q}}(\mathcal{P} \cdot P', j' = 1, m_j', \mathrm{P}_2)|J^\nu(0)\,|h_{q\bar{q}}(\mathcal{P} \cdot P, j = 0, m_j, \mathrm{P}_1)\rangle \\ &\quad =\mathcal{P}^\nu_\kappa[\bar{p}^\kappa(e^*(\mathcal{P} \cdot P', m_j') \cdot (\mathcal{P} \cdot P)) - q^\kappa(e^*(\mathcal{P} \cdot P', m_j') \cdot (\mathcal{P} \cdot P))\Delta_m/q^2]H_1 \\ &\quad\quad - \mathcal{P}^\nu_\kappa[e^{\kappa *}(\mathcal{P} \cdot P', m_j') + \mathcal{P}^\nu_\kappa q^\kappa(e^*(\mathcal{P} \cdot P', m_j') \cdot (\mathcal{P} \cdot P))/q^2]H_2 \\ &\quad\quad - \mathcal{P}^\rho_{\kappa_1}\mathcal{P}^\sigma_{\kappa_2}\mathcal{P}^\alpha_{\kappa_3}\epsilon^\nu_{\alpha\rho\sigma}\bar{p}^{\kappa_1}q^{\kappa_2}e^{\kappa_3 *}(P', m_j')H_3 \ . \end{aligned} \tag{38}$$

By using the following transformation relations,

$$e^\mu(\mathcal{P} \cdot P, m_j) = -\mathcal{P}^\mu_\nu e^\nu(P, m_j) \ , \tag{39}$$

$$e^*(\mathcal{P} \cdot P', m_j') \cdot (\mathcal{P} \cdot P) = -\mathcal{P}^\nu_\kappa e^{\kappa *}(P', m_j')\mathcal{P}^\chi_\nu P_\chi = -e^*(P', m_j') \cdot P \ , \tag{40}$$

$$e^*(\mathcal{P} \cdot P', m_j') \cdot e(\mathcal{P} \cdot P, m_j) = \mathcal{P}^\nu_\kappa e^{\kappa *}(P', m_j')\mathcal{P}^\chi_\nu e^{\chi *}(P', m_j') = e^*(P', m_j') \cdot e(P, m_j) \ , \tag{41}$$

we get

$$
\begin{aligned}
&\langle h'_{q\bar{q}}(\mathcal{P}\cdot P', j'=1, m'_j, \mathrm{P}_2)|\bar{\Psi}\gamma^\nu\Psi\,|h_{q\bar{q}}(\mathcal{P}\cdot P, j=0, m_j, \mathrm{P}_1)\rangle \\
&\qquad = -\mathcal{P}^\nu_\kappa[\bar{p}^\kappa(e^*(P',m'_j)\cdot P) - q^\kappa(e^*(P',m'_j)\cdot P)\Delta_m/q^2]H_1 \\
&\qquad\quad -\mathcal{P}^\nu_\kappa[e^{\kappa *}(P',m'_j) + \mathcal{P}^\nu_\kappa q^\kappa(e^*(P',m'_j)\cdot P)/q^2]H_2 \\
&\qquad\quad +\mathcal{P}^\nu_\kappa \epsilon^\kappa_{\kappa_3\kappa_1\kappa_2}\bar{p}^{\kappa 1} q^{\kappa 2} e^{\kappa 3 *}(P',m'_j)H_3\ .
\end{aligned}
\tag{42}
$$

Plugging it back into Eq. (37), we find

$$
\mathrm{P}_2\mathrm{P}_1 = \begin{cases} +1 & \to H_1, H_2 = 0 \\ -1 & \to H_3 = 0 \end{cases} .
\tag{43}
$$

H_1, H_2 are form factors of parity flipped transition, and H_3 are form factors of parity conserved transition.

To summarize, there are two classes of allowed transitions, (1):

$$
0^{++}(\text{scalar}) \to 1^{--}(\text{vector})\,, \qquad 0^{-+}(\text{ pseudoscalar }) \to 1^{+-}(\text{axial-vector }).
$$

and the transition form factors are H_1 and H_2; (2):

$$
0^{++}(\text{scalar}) \to 1^{+-}(\text{axial-vector})\,, \qquad 0^{-+}(\text{ pseudoscalar }) \to 1^{--}(\text{vector })
$$

and the transition form factor is H_3. The transitions between pseudoscalar 0^{-+} and axial-vector 1^{++}, between scalar 0^{++} and axial-vector 1^{++} are forbidden due to charge conjugation.

B. The hadron matrix element

The electromagnetic transition between two hadron states ψ_A and ψ_B is governed by the matrix element $\langle\psi_B(P, j, m_j)|\,J^\mu(x)\,|\psi_A(P', j', m'_j)\rangle$. The elastic process is a special case where $\psi_A = \psi_B$. In this section, we derive the light-front wavefunction representation of the hadron matrix element, which we will use later in calculating the elastic form factor and the transition form factor.

The EM current operator is defined as $J^\mu = \bar{\Psi}\gamma^\mu\Psi$. In the light-front representation,

$$
\begin{aligned}
J^\mu(x) = &\sum_{\lambda_1,\lambda_2}\sum_{c_1,c_2}\int\frac{\mathrm{d}^2 p_{1\perp}\,\mathrm{d}p_1^+}{(2\pi)^3 2p_1^+}\int\frac{\mathrm{d}^2 p_{2\perp}\,\mathrm{d}p_2^+}{(2\pi)^3 2p_2^+}\left[b^\dagger_{\lambda_2 c_2}(p_2)\bar{u}_{\lambda_2}(p_2)e^{ip_2\cdot x} + d_{\lambda_2 c_2}(p_2)\bar{v}_{\lambda_2}(p_2)e^{-ip_2\cdot x}\right] \\
&\gamma^\mu\left[b_{\lambda_1 c_1}(p_1)u_{\lambda_1}(p_1)e^{-ip_1\cdot x} + d^\dagger_{\lambda_1 c_1}(p_1)v_{\lambda_1}(p_1)e^{ip_1\cdot x}\right]\ .
\end{aligned}
\tag{44}
$$

By spacetime translation invariance,

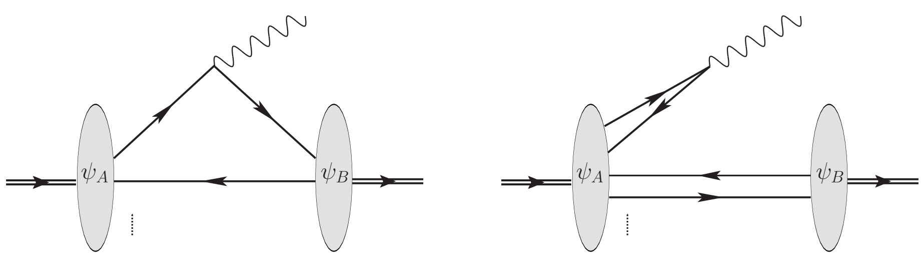
$$
\begin{aligned}
\langle\psi_B(P, j, m_j)|\,J^\mu(x)\,|\psi_A(P', j', m'_j)\rangle &= \langle\psi_B(P, j, m_j)|\,e^{-i\hat{p}x}\bar{\Psi}(0)e^{i\hat{p}x}\gamma^\mu e^{-i\hat{p}x}\Psi(0)e^{i\hat{p}x}\,|\psi_A(P', j', m'_j)\rangle \\
&= \langle\psi_B(P, j, m_j)|\,J^\mu(0)\,|\psi_A(P', j', m'_j)\rangle\,e^{i(P'-P)x}\ .
\end{aligned}
\tag{45}
$$

The argument x only results in an overall phase factor, so in the literature one usually takes $J^\mu(0)$ in calculating the matrix element.

We have shown in Part 1 that the meson state vector $|\psi_h(P, j, m_j)\rangle$ can be expanded in the light-front Fock space. The coefficients of the Fock expansion are the complete set of n-particle light-front wavefunctions, $\{\psi^{(m_j)}_{n/h}(x_i, \vec{k}_{i\perp}, s_i)\}$. $x_i \equiv \kappa_i^+/P^+$ is the longitudinal momentum fraction of the i-th parton, and $\vec{k}_{i\perp} \equiv \vec{\kappa}_{i\perp} - x\vec{P}_\perp$ is the relative transverse momenta, with κ_i being the momenta of the corresponding parton. s is the spin of the parton. The electromagnetic current matrix element is in general given by the sum of the diagonal $n \to n$ and off-diagonal $n + 2 \to n$ transitions, as shown in Fig. 1.

$$\langle\psi_B|\, J^\mu \,|\psi_A\rangle = \langle\psi_B|\, J^\mu \,|\psi_A\rangle_{n\to n} + \langle\psi_B|\, J^\mu \,|\psi_A\rangle_{n+2\to n} \ . \tag{46}$$

In the former case, the external photon is coupled to a quark or an antiquark. In the latter case, a quark-antiquark pair is annihilated into the external photon.



(a) $n \to n$ transition (b) $n + 2 \to n$ transition

FIG. 1. Light-front wavefunction representation of the hadron matrix element. The double-lines represents the hadrons. The solid lines represent the partons. The wavy lines represent the external photon. The shaded areas represent the light-front wavefunctions. These diagrams are ordered by light-front time x^+, which flows from left to right. In (a), the $n \to n$ transition, parton number is conserved, whereas in (b), the $n+2 \to n$ transition, parton number is reduced by 2 due to pair annihilation. (Figure adapted from Ref. [3].)

Let us make the derivation of the $2 \to 2$ transition explicitly. In the case where the hadrons are solved in the $|q\bar{q}\rangle$ Fock sector, only the $n \to n$ ($n = 2$) term would contribute to the transition,

$$\begin{aligned}
&\langle\psi_{q\bar{q}/B}(P, j, m_j)|J^\mu(0)\,|\psi_{q\bar{q}/A}(P', j', m'_j)\rangle \\
&=\frac{1}{N_c}\sum_{i,j=1}^{N_c}\langle 0|\sum_{s,\bar{s}}\int_0^1 \frac{\mathrm{d}x}{2x(1-x)}\int\frac{\mathrm{d}^2k_\perp}{(2\pi)^3}\psi^{(m_j)*}_{s\bar{s}/B}(\vec{k}_\perp, x) \\
&d_{j\bar{s}}((1-x)P^+, -\vec{k}_\perp + (1-x)\vec{P}_\perp)b_{js}(xP^+, \vec{k}_\perp + x\vec{P}_\perp) \\
&\sum_{c_1,c_2}\sum_{\lambda_1,\lambda_2}\int\frac{\mathrm{d}^2\vec{p}^{\,1}_\perp\,\mathrm{d}p_1^+}{(2\pi)^3 2p_1^+}\int\frac{\mathrm{d}^2\vec{p}^{\,2}_\perp\,\mathrm{d}p_2^+}{(2\pi)^3 2p_2^+}\left[b^\dagger_{\lambda_2c_2}(p_2)\bar{u}_{\lambda_2}(p_2) + d_{\lambda_2c_2}(p_2)\bar{v}_{\lambda_2}(p_2)\right] \\
&\gamma^\mu\left[b_{\lambda_1c_1}(p_1)u_{\lambda_1}(p_1) + d^\dagger_{\lambda_1c_1}(p_1)v_{\lambda_1}(p_1)\right] \\
&\times\sum_{s',\bar{s}'}\int_0^1\frac{\mathrm{d}x'}{2x'(1-x')}\int\frac{\mathrm{d}^2\vec{k}'_\perp}{(2\pi)^3}\psi^{(m'_j)}_{s'\bar{s}'/A}(\vec{k}'_\perp, x') \\
&b^\dagger_{is'}(x'P'^+, \vec{k}'_\perp + x'\vec{P}'_\perp)d^\dagger_{i\bar{s}'}((1-x')P', -\vec{k}'_\perp + (1-x')\vec{P}'_\perp)\,|0\rangle\ .
\end{aligned}\tag{47}$$

There are two non-vanishing terms as we pair up the creation and annihilation operators. One is the contribution from the quark radiation and the other from the antiquark. We will use J^μ_q ($J^\mu_{\bar{q}}$) as the operator acting on the quark (antiquark).

[Exercise] Why do the other two terms db and $d^\dagger b^\dagger$ vanish? What physical process do they describe? See Fig. 1 .

Contracting the creation and annihilation operators,

$$\begin{aligned}
&\langle\psi_{q\bar{q}/B}(P, j, m_j)|\,J^\mu_q(0)\,|\psi_{q\bar{q}/A}(P', j', m'_j)\rangle \\
&=\frac{1}{N_c}\sum_{i,j=1}^{N_c}\sum_{\lambda_1,\lambda_2}\sum_{c_1,c_2}\sum_{s',\bar{s}'}\sum_{s,\bar{s}}\int\frac{\mathrm{d}^2\vec{p}^{\,1}_\perp\,\mathrm{d}p_1^+}{(2\pi)^3 2p_1^+}\int\frac{\mathrm{d}^2\vec{p}^{\,2}_\perp\,\mathrm{d}p_2^+}{(2\pi)^3 2p_2^+}\int_0^1\frac{\mathrm{d}x'}{2x'(1-x')} \\
&\int\frac{\mathrm{d}^2\vec{k}'_\perp}{(2\pi)^3}\int_0^1\frac{\mathrm{d}x}{2x(1-x)}\int\frac{\mathrm{d}^2k_\perp}{(2\pi)^3}\psi^{(m'_j)}_{s'\bar{s}'/A}(\vec{k}'_\perp, x')\psi^{(m_j)*}_{s\bar{s}/B}(\vec{k}_\perp, x) \\
&2p_2^+\theta(p_2^+)(2\pi)^3\delta(p_2^+ - xP^+)\delta^2(\vec{p}^{\,2}_\perp - \vec{k}_\perp - x\vec{P}_\perp)\delta_{j,c_2}\delta_{s,\lambda_2} \\
&2p_1^+\theta(p_1^+)(2\pi)^3\delta(p_1^+ - x'P'^+)\delta^2(\vec{p}^{\,1}_\perp - \vec{k}'_\perp - x'\vec{P}'_\perp)\delta_{i,c_1}\delta_{s',\lambda_1} \\
&2(1-x)P^+\theta(P^+)(2\pi)^3\delta((1-x')P'^+ - (1-x)P^+) \\
&\delta^2(-\vec{k}'_\perp + (1-x')\vec{P}'_\perp + \vec{k}_\perp - (1-x)\vec{P}_\perp)\delta_{j,i}\delta_{\bar{s}',\bar{s}} \\
&\bar{u}_{\lambda_2}(p_2)\gamma^\mu u_{\lambda_1}(p_1)\ .
\end{aligned}\tag{48}$$

We could first integrate over x and $\vec{k}_\perp$ by the last two delta functions and get,

$$x = 1 - (1-x')P'^+/P^+, \qquad \vec{k}_\perp = \vec{k}'_\perp - (1-x')\vec{P}'_\perp + (1-x)\vec{P}_\perp\ . \tag{49}$$

Integrate over p_1, p_2, we get

$$\begin{aligned}&\langle\psi_{q\bar{q}/B}(P,j,m_j)|\,J_q^\mu(0)\,|\psi_{q\bar{q}/A}(P',j',m_j')\rangle\\&=\sum_{s,\bar{s}}\int_{\max(0,1-P^+/P'^+)}^1\frac{\mathrm{d}x'}{2x'(1-x')}\int\frac{\mathrm{d}^2k'_\perp}{(2\pi)^3}\frac{1}{x}\sum_{s'}\psi^{(m_j')}_{s'\bar{s}/A}(\vec{k}'_\perp,x')\psi^{(m_j)*}_{s\bar{s}/B}(\vec{k}_\perp,x)\\&\times\bar{u}_s(xP^+,\vec{k}_\perp+x\vec{P}_\perp)\gamma^\mu u_{s'}(x'P'^+,\vec{k}'_\perp+x'\vec{P}'_\perp)\ .\end{aligned}\tag{50}$$

Note that the lower bound of the integral over x is not 0 when $P^+ < P'^+$, which results from the condition $x\in[0,1]$. However, when evaluating the x'-integral numerically, it would be more convenient to have the integral range as $[0,1]$. This is actually possible by integrating over x' and $\vec{k}'_\perp$ instead in Eq. (48),

$$\begin{aligned}&\langle\psi_{q\bar{q}/B}(P,j,m_j)|\,J_q^\mu(0)\,|\psi_{q\bar{q}/A}(P',j',m_j')\rangle\\&=\sum_{s,\bar{s}}\int_{\max(0,1-P'^+/P^+)}^1\frac{\mathrm{d}x}{2x(1-x)}\int\frac{\mathrm{d}^2\vec{k}_\perp}{(2\pi)^3}\frac{1}{x'}\sum_{s'}\psi^{(m_j')}_{s'\bar{s}/A}(\vec{k}'_\perp,x')\psi^{(m_j)*}_{s\bar{s}/B}(\vec{k}_\perp,x)\\&\times\bar{u}_s(xP^+,\vec{k}_\perp+x\vec{P}_\perp)\gamma^\mu u_{s'}(x'P'^+,\vec{k}'_\perp+x'\vec{P}'_\perp)\ ,\end{aligned}\tag{51}$$

where

$$x'=1-(1-x)P^+/P'^+,\qquad \vec{k}'_\perp=\vec{k}_\perp-(1-x)\vec{P}_\perp+(1-x')\vec{P}'_\perp\ .\tag{52}$$

As expected, the lower bound of the integral over x is 0 when $P^+ < P'^+$. Eqs. (50) and (51) are equivalent, and one could choose the one that facilitates the numerical calculations. If one considers the process $\psi_A(P')\to\psi_B(P)+\gamma^{(*)}(q=P'-P)$ where $P^+<P'^+$, it is more convenient to use the expression in Eq. (51). However, if one considers the process $\psi_A(P')+\gamma^{(*)}(q=P-P')\to\psi_B(P)$, which is usually the case in calculating the elastic form factor, where $P^+>P'^+$, it would be more convenient to use the expression in Eq. (50). In analogy, we get the hadron matrix element of the antiquark current,

$$\begin{aligned}&\langle\psi_{q\bar{q}/B}(P,j,m_j)|\,J_{\bar{q}}^\mu(0)\,|\psi_{q\bar{q}/A}(P',j',m_j')\rangle\\&=-\sum_{s,\bar{s}}\int_0^{\min(1,P^+/P'^+)}\frac{\mathrm{d}x'}{2x'(1-x')}\int\frac{\mathrm{d}^2k'_\perp}{(2\pi)^3}\frac{1}{1-x}\sum_{\bar{s}'}\psi^{(m_j)*}_{s\bar{s}/B}(\vec{k}_\perp,x)\psi^{(m_j')}_{s\bar{s}'/A}(\vec{k}'_\perp,x')\\&\times\bar{v}_{\bar{s}'}((1-x')P'^+,-\vec{k}'_\perp+(1-x')\vec{P}'_\perp)\gamma^\mu v_{\bar{s}}((1-x)P^+,-\vec{k}_\perp+(1-x)\vec{P}_\perp)\ ,\end{aligned}\tag{53}$$

where

$$x=x'P'^+/P^+\ ,\qquad \vec{k}_\perp=\vec{k}'_\perp+x'(P^+\vec{P}'_\perp-P'^+\vec{P}_\perp)/P^+\ .\tag{54}$$

[Exercise] What does the "-" sign in Eq. (53) imply physically? Think about the charge.

To have a more explicit form for the purpose of calculation, let us put in the expressions of spinors as in Appendix. B.

[Exercise] Write out the expressions$\bar{u}_{s_2}(p_2)\gamma^\mu u_{s_1}(p_1)$ for $\mu=+,-,x,y$, and $s_1, s_2=\pm 1/2$.

C. Frames and kinematics

Considering the process $\psi_A(P') \to \psi_B(P) + X(q = P' - P)$ or $\psi_B(P) + X(q = P' - P) \to \psi_A(P')$, the Lorentz invariant momentum transfer q^2 can be written as a function of two boost invariants [3, 7] according to the four-momentum conservation $q^2 = (P' - P)^2$,

$$q^2 = zm_A^2 - \frac{z}{1-z}m_B^2 - \frac{1}{1-z}\vec{\Delta}_\perp^2 \ . \tag{55}$$

where,

$$z \equiv (P'^+ - P^+)/P'^+, \qquad \vec{\Delta}_\perp \equiv \vec{q}_\perp - z\vec{P}'_\perp \ .$$

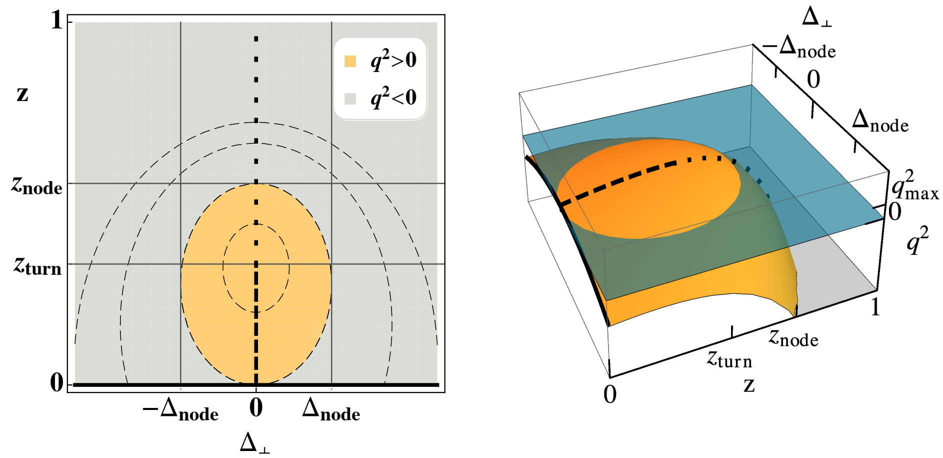
Both z and $\vec{\Delta}_\perp$ are invariant under the transverse Lorentz boost specified by the velocity vector $\vec{\beta}_\perp$,

$$v^+ \to v^+, \quad \vec{v}_\perp \to \vec{v}_\perp + v^+\vec{\beta}_\perp \ . \tag{56}$$

z can be interpreted as the relative momentum transfer in the longitudinal direction, and $\vec{\Delta}_\perp$ describes the momentum transfer in the transverse direction. Note that z is restricted to $0 \leq z < 1$ by definition. For each possible value of q^2, the values of the pair $(z, \vec{\Delta}_\perp)$ are not unique, and those different choices correspond to different reference frames (up to longitudinal and transverse light-front boost transformations). Fig. 2 should help visualize the functional form of $q^2(z, \vec{\Delta}_\perp)$. Since q^2 is relevant to the magnitude of $\vec{\Delta}_\perp$ but not its angle, we plot it in the $\arg\vec{\Delta}_\perp = 0, \pi$ plane. Form factors evaluated at different $(z, \vec{\Delta}_\perp)$ but at the same q^2 could reveal the frame dependence. In particular, we introduce two special frames for detailed consideration.

- Drell-Yan frame ($z = 0$) : $q^+ = 0$, $\vec{\Delta}_\perp = \vec{q}_\perp$ and $q^2 = -\vec{\Delta}_\perp^2$. This frame is shown as a single thick solid line in each panel of Fig. 2. The Drell-Yan frame is conventionally used together with the plus current J^+ to calculate the electromagnetic form factors. This choice, on the one hand, avoids spurious effects related to the orientation of the null hyperplane where the light-front wavefunction is defined and, on the other hand, it suppresses the contributions from the often-neglected pair creation process, at least for pseudoscalar mesons [8–13]. For the transition form factor, this is only true if zero-mode contributions are neglected. The transition form factor obtained in the Drell-Yan frame is significantly restricted in the space-like region, i.e. $q^2 \leq 0$. Although one could analytically continuate the form factor to the time-like region by changing $\vec{q}_\perp$ to $i\vec{q}_\perp$ [14–16], we elect to calculate transition form factors directly from wavefunctions.

- longitudinal frame ($\vec{\Delta}_\perp = 0$): $q^2 = zm_A^2 - zm_B^2/(1 - z)$. Note that we use the same definition for the longitudinal frame as in Ref. [3, 7], which is different from those in the literature where $\vec{q}_\perp = 0$ is



(a) Regional plot of $q^2(\Delta_\perp, z)$

(b) 3D plot of $q^2(\Delta_\perp, z)$

FIG. 2. Visualization of the Lorentz invariant momentum transfer squared q^2 as a function of z and $\vec{\Delta}_\perp$ at $\arg \vec{\Delta}_\perp = 0, \pi$. (a): regional plot of q^2. The time-like region ($q^2 > 0$) is the orange oval shape, bounded by $\Delta_{\text{node}} = (m_A^2 - m_B^2)/2m_A$ and $z_{\text{node}} = 1 - m_B^2/m_A^2$. The space-like region ($q^2 < 0$) is in light gray. Contour lines of q^2 are indicated with thin dashed curves. The maximal value $q^2_{\text{max}} = (m_A - m_B)^2$ occurs at ($z_{\text{turn}} = 1 - m_B/m_A, \Delta_\perp = 0$). (b): 3D plot of q^2 showing a convex shape in the $(z, \Delta_\perp)$ representation. The blue flat plane is the reference plane of $q^2 = 0$. In each figure, the Drell-Yan frame is shown as a thick solid line, and the longitudinal I and II frames are shown as thick dotted and thick dashed lines respectively. (Figure adapted from Ref. [3].)

called the longitudinal frame [10, 16–18]. In this frame, we have access to the kinematic region up to $q^2_{\text{max}} = (m_A - m_B)^2$, the point where the final meson does not recoil. This maximal value occurs at $z = 1 - m_B/m_A \equiv z_{\text{turn}}$. For a given q^2, there are two solutions for z, corresponding to either the positive or the negative recoil direction of the final meson relative to the initial meson, namely,

- longitudinal-I: $z = \left[m_A^2 - m_B^2 + q^2 + \sqrt{(m_A^2 - m_B^2 + q^2)^2 - 4m_A^2 q^2}\,\right]/(2m_A^2)$. $z_{\text{turn}} \leq z < 1$. This branch joins the second branch at $q^2 = q^2_{\text{max}}$ with $z = z_{\text{turn}}, \vec{\Delta}_\perp = 0$. The time-like region is accessed at $z_{\text{turn}} \leq z < z_{\text{node}}$, and the space-like region is at $z_{\text{node}} \leq z < 1$, where $z_{\text{node}} \equiv 1 - m_B^2/m_A^2$. The longitudinal-I frame is shown as thick dotted lines in Fig. 2.

- longitudinal-II: $z = \left[m_A^2 - m_B^2 + q^2 - \sqrt{(m_A^2 - m_B^2 + q^2)^2 - 4m_A^2 q^2}\,\right]/(2m_A^2)$. $0 \leq z \leq z_{\text{turn}}$. This second branch only exists in the time-like region, and it joins the Drell-Yan frame at $q^2 = 0$ with $z = 0, \vec{\Delta}_\perp = 0$. The longitudinal-II frame is shown as thick dashed lines in Fig. 2.

D. Elastic form factor of the spin-0 meson

The elastic form factor of a (pseudo)scalar ψ_h is the charge form factor $F(q^2)$, defined as

$$\langle\psi_h(P')|J^\mu(0)|\psi_h(P)\rangle = (P+P')^\mu F(q^2) \ , \tag{57}$$

as we have derived in Eq. (19). The charge form factor $F(q^2)$ is interpreted as the Fourier transformation of the charge density in the system. For quarkonium, the physical from factor vanishes due to charge conjugation symmetry (see also the discussion on charge conjugation in Sec. I A 1), so what being calculated is actually the fictitious form factor from the quark current, J_q^μ. In the light-front wavefunction representation of the valence Fock sector, the hadron matrix element reads

$$\begin{aligned}\langle\psi_h(P',m_j')|J_q^\mu(0)|\psi_h(P,m_j)\rangle = \sum_{s,s',\bar{s}}\int_0^1 \frac{\mathrm{d}x}{2x(1-x)}\int\frac{\mathrm{d}^2k_\perp}{(2\pi)^3}\frac{1}{x'}\psi^{(m_j)}_{s\bar{s}/h}(\vec{k}_\perp,x)\psi^{(m_j')*}_{s'\bar{s}/h}(\vec{k}'_\perp,x') \\ \times\bar{u}_{s'}(x'P'^+,\vec{k}'_\perp+x'\vec{P}'_\perp)\gamma^\mu u_s(xP^+,\vec{k}_\perp+x\vec{P}_\perp) \ ,\end{aligned} \tag{58}$$

where $x'=(P'^+-(1-x)P^+)/P'^+$ and $\vec{k}'_\perp=\vec{k}_\perp+(1-x)(P^+\vec{P}'_\perp-P'^+\vec{P}_\perp)/P'^+$. This is essentially the same as Eq. (50). We rewrite x' and $\vec{k}'_\perp$ in terms of the two boost invariants we have defined in Section I C, z and $\vec{\Delta}_\perp$, as

$$x' = x+z(1-x), \qquad \vec{k}'_\perp=\vec{k}_\perp+(1-x)\vec{\Delta}_\perp \ .$$

The transferred momentum square q^2 can be written according to Eq. (55) with $m_A=m_B=m_h$,

$$q^2 = -(z^2m_h^2+\vec{\Delta}_\perp^2)/(1-z) \ . \tag{59}$$

Note that $q^2\leq 0$.

One could extract the form factor with different current components. The $+$, $\perp$ and $-$ hadron matrix elements should be related through the transverse Lorentz boost specified by the velocity vector $\vec{\beta}_\perp$,

$$v^+\to v^+, \quad \vec{v}_\perp\to\vec{v}_\perp+v^+\vec{\beta}_\perp, \quad v^-\to v^-+2\vec{\beta}_\perp\cdot\vec{v}_\perp+\vec{\beta}_\perp^2v^+ \ . \tag{60}$$

The hadron matrix elements are thereby related through,

$$\begin{aligned}&\langle\psi_h(P'^+,\vec{P}'_\perp+P'^+\vec{\beta}_\perp)|\vec{J}_\perp|\psi_h(P^+,\vec{P}_\perp+P^+\vec{\beta}_\perp)\rangle \\ &\qquad=\langle\psi_h(P')|\vec{J}_\perp|\psi_h(P)\rangle+\vec{\beta}_\perp\langle\psi_h(P')|J^+|\psi_h(P)\rangle \ , \\ &\langle\psi_h(P'^+,\vec{P}'_\perp+P'^+\vec{\beta}_\perp)|J^-|\psi_h(P^+,\vec{P}_\perp+P^+\vec{\beta}_\perp)\rangle \\ &\qquad=\langle\psi_h(P')|J^-|\psi_h(P)\rangle+2\vec{\beta}_\perp\cdot\langle\psi_h(P')|\vec{J}_\perp|\psi_h(P)\rangle+\vec{\beta}_\perp^2\langle\psi_h(P')|J^+|\psi_h(P)\rangle \ .\end{aligned} \tag{61}$$

This relation implies that the form factors extracted from different current components should be equivalent. One can verify it by substituting Eq. (57) into Eq. (61). We would like to know if this is still true in the valence Fock sector, and write out the form factor with different current components in the valence light-front wavefunction representation.

1. the plus current

$$
\begin{aligned}
F(q^2)\Big|_{J^+} =& \langle\psi_h(P')|J_q^+(0)|\psi_h(P)\rangle \,/(P^+ + P'^+) \\
=& \sum_{s,\bar{s}} \int_0^1 \frac{\mathrm{d}x}{2x(1-x)} \int \frac{\mathrm{d}^2k_\perp}{(2\pi)^3} \frac{1}{x'} \psi_{s\bar{s}/h}(\vec{k}_\perp, x)\psi^*_{s\bar{s}/h}(\vec{k}'_\perp, x') 2\sqrt{x'P'^+xP^+}/(P^+ + P'^+) \\
=& \sum_{s,\bar{s}} \int_0^1 \frac{\mathrm{d}x}{2x(1-x)} \int \frac{\mathrm{d}^2k_\perp}{(2\pi)^3} \frac{2}{2-z} \sqrt{\frac{x(1-z)}{x+z(1-x)}} \psi_{s\bar{s}/h}(\vec{k}_\perp, x)\psi^*_{s\bar{s}/h}(\vec{k}'_\perp, x')\ .
\end{aligned}
\tag{62}
$$

In the second line, the form factor is written as a function of $(z, \vec{\Delta}_\perp)$, dependence on P or P' is eliminated. The normalization of the form factor at $q^2 = 0$ follows as the result of the normalization of the hadron wavefunction,

$$
F(0)\Big|_{J^+} = \sum_{s,\bar{s}} \int_0^1 \frac{\mathrm{d}x}{2x(1-x)} \int \frac{\mathrm{d}^2k_\perp}{(2\pi)^3} \psi_{s\bar{s}/h}(\vec{k}_\perp, x)\psi^*_{s\bar{s}/h}(\vec{k}_\perp, x) = 1\ . \tag{63}
$$

2. the transverse current

Now we turn to the transverse current. Assuming that the rotational symmetry on the transverse plane is preserved, using J^x or J^y component or linear combinations of the two should be equivalent. Here we use $J^R \equiv J^x + iJ^y$ and $J^L \equiv J^x - iJ^y$ as the transverse currents. For any transverse vector $\vec{k}_\perp$, which is expressed as (k^x, k^y) in the Cartesian coordinate or $(k_\perp, \theta)$ in the polar coordinate, we will write its complex form as $k^R \equiv k^x + ik^y = k_\perp e^{i\theta}$ and $k^L \equiv k^x - ik^y = k_\perp e^{-i\theta}$. The elastic form factor extracted from the J^R current reads,

$$
\begin{aligned}
F(q^2)\Big|_{J^R} =& \langle\psi_h(P')|J_q^R(0)|\psi_h(P)\rangle \,/(P^R + P'^R) \\
=& F(q^2)\Big|_{J^+} + \frac{1}{P^R + P'^R} \sum_{s\bar{s}} \int_0^1 \frac{\mathrm{d}x}{2x(1-x)} \int \frac{\mathrm{d}^2k_\perp}{(2\pi)^3} \psi_{s\bar{s}/h}(\vec{k}_\perp, x)\psi^*_{s\bar{s}/h}(\vec{k}'_\perp, x') \\
&\times \frac{1}{\sqrt{x(1-z)[x+z(1-x)]^3}} \Big\{[z + 2x(1-z)]k^R + \frac{x}{2-z}(2 - 2x - 3z + 2xz)\Delta^R\Big\}\ .
\end{aligned}
\tag{64}
$$

We have applied the symmetry among different spin components of spin-0 particle h_0,

$$
\psi_{\uparrow\uparrow/h_0}(\vec{k}_\perp, x) = \psi_{\downarrow\downarrow/h_0}(\vec{k}_\perp, x), \qquad \psi_{\uparrow\downarrow/h_0}(\vec{k}_\perp, x) = -\psi_{\downarrow\uparrow/h_0}(\vec{k}_\perp, x)\ . \tag{65}
$$

We see that $F(q^2)\big|_{J^R}$ and $F(q^2)\big|_{J^+}$ are different by the second term in the last line of Eq. (64). Moreover, this second term depends on $P^R + P'^R$ in the $(z, \vec{\Delta}_\perp)$ parameter space. This indicates that fixing $(z, \vec{\Delta}_\perp)$ is not sufficient to unambiguously determine a frame in this case. However, in the Drell-Yan and the longitudinal frames, it can be proved that this term actually vanishes, leaving $F(q^2)\big|_{J^R} = F(q^2)\big|_{J^+}$.

$$\begin{aligned} F(q^2)\big|_{J^R,\mathrm{DY}} =& \langle\psi_h(P')|J_q^R(0)|\psi_h(P)\rangle / (P^R + P'^R) \\ =& F(q^2)\big|_{J^+,\mathrm{DY}} \\ &+ \frac{1}{P^R + P'^R} \sum_{s\bar{s}} \int_0^1 \frac{\mathrm{d}x}{2x(1-x)} \int \frac{\mathrm{d}^2 k_\perp}{(2\pi)^3} \psi_{s\bar{s}/h}(\vec{k}_\perp, x) \psi^*_{s\bar{s}/h}(\vec{k}'_\perp, x') \frac{1}{x} [2k^R + (1-x)q^R] \\ =& F(q^2)\big|_{J^+,\mathrm{DY}} \,. \end{aligned} \tag{66}$$

The second term vanishes under the transverse integral with $\vec{k}'_\perp = \vec{k}_\perp + (1-x)q^R$ in the Drell-Yan frame. Now, in the longitudinal frame:

$$\begin{aligned} F(q^2)\big|_{J^R,\mathrm{long}} =& \langle\psi_h(P')|J_q^R(0)|\psi_h(P)\rangle / (P^R + P'^R) \\ =& F(q^2)\big|_{J^+,\mathrm{long}} + \frac{1}{P^R + P'^R} \sum_{s\bar{s}} \int_0^1 \frac{\mathrm{d}x}{2x(1-x)} \int \frac{\mathrm{d}^2 k_\perp}{(2\pi)^3} \psi_{s\bar{s}/h}(\vec{k}_\perp, x) \psi^*_{s\bar{s}/h}(\vec{k}'_\perp, x') \\ &\times \frac{[z + 2x(1-z)]k^R}{\sqrt{x(1-z)[x + z(1-x)]^3}} \\ =& F(q^2)\big|_{J^+,\mathrm{long}} \,. \end{aligned} \tag{67}$$

Note that $\vec{k}'_\perp = \vec{k}_\perp$ in the longitudinal frame, thus the second term vanishes since the angular integral is zero. As with the J^+ current, $F(0)|_{J^R} = 1$ is guaranteed by the normalization of the hadron wavefunction. At $q^2 = 0$, the terms proportional to k^R in the integral would vanish since the angular integration would be 0.

3. the minus current

Using the J^- current,

$$\begin{aligned} F(q^2)\big|_{J^-} =& \langle\psi_h(P')|J_q^-(0)|\psi_h(P)\rangle / (P^- + P'^-) \\ =& \frac{1}{P_\perp^2 + m_h^2 + (1-z)(P'^2_\perp + m_h^2)} \sum_{s\bar{s}} \int_0^1 \frac{\mathrm{d}x}{2x(1-x)} \int \frac{\mathrm{d}^2 k_\perp}{(2\pi)^3} \psi_{s\bar{s}/h}(\vec{k}_\perp, x) \psi^*_{s\bar{s}/h}(\vec{k}'_\perp, x') \\ &\times 2\sqrt{\frac{1-z}{x[x + z(1-x)]}} [m_q^2 + (\vec{k}_\perp + x\vec{P}_\perp) \cdot (\vec{k}'_\perp + x'\vec{P}'_\perp)] \,. \end{aligned} \tag{68}$$

In deriving Eq. (68), the spin flip terms vanish by exact cancellations among different spin components. The normalization of the elastic form factor ($F(0) = 1$) with J^- has a nontrivial requirement on the wavefunctions, and this is referred to as a type of Virial theorem [19]. We can see this explicitly in Eq. (69),

$$F(0)\Big|_{J^-} = \langle \psi_h(P')|J_q^-(0)|\psi_h(P)\rangle\,/(P^- + P'^-)$$
$$= \frac{1}{2(P_\perp^2 + m_h^2)} \sum_{s\bar{s}} \int_0^1 \frac{\mathrm{d}x}{2x(1-x)} \int \frac{\mathrm{d}^2 k_\perp}{(2\pi)^3} \psi_{s\bar{s}/h}(\vec{k}_\perp, x)\psi^*_{s\bar{s}/h}(\vec{k}_\perp, x)\frac{2}{x}[m_q^2 + (\vec{k}_\perp + x\vec{P}_\perp)^2]\ . \tag{69}$$

In the truncated Fock space, the light-front J^- current is not conserved and it violates the Ward-Takahashi identity [20, 21]. The valence Fock sector is not sufficient to extract the elastic form factor with the J^- current.

The work by H.M. Choi, H.Y. Ryu and C.R. Ji [22] implemented a replacement of the meson mass m_h by the invariant mass $m_0^2 = (m_q^2 + \vec{k}_\perp^2)/x + (m_q^2 + \vec{k}_\perp^2)/(1-x)$ in studying the $(\pi^0, \eta, \eta' \to \gamma^*\gamma^*)$ transitions with a manifestly covariant model. Following the format of this treatment, we see that restoring $F(0) = 1$ in Eq. (69) would require a replacement of $m_h^2 \to (m_q^2 + \vec{k}_\perp^2)/x - (1-x)\vec{P}_\perp^2$. In the meson rest frame where $\vec{P}_\perp = \vec{0}_\perp$, the expression reduces to $m_h^2 \to (m_q^2 + \vec{k}_\perp^2)/x$, suggesting to replace the meson mass by the invariant mass of the quark, or half of the invariant mass of the meson.

To conclude, the J^+ and $\vec{J}_\perp$ current components could guarantee the normalization of the elastic form factor in the valence Fock sector, but the J^- component could not. Though the elastic form factors extracted from the J^+ and the $\vec{J}_\perp$ components are expected to be the same through a transverse boost, the valence light-front wavefunction representation shows that the two are the same only in the Drell-Yan and the longitudinal frames. In a practical calculation, J^+ and the Drell-Yan frame is often preferred, and the main advantage of this choice is that vacuum pair production/ annihilation is suppressed [10, 23, 24]. A study on the frame dependence of the elastic form factor of pseudoscalars using the J^+ current can be found in Ref. [7].

In nonrelativistic quantum mechanics, the root-mean-square charge (mass) radius is the expectation value of the displacement operator that characterizes the charge (mass) distribution of the system. In quantum field theory, no such local position operator is allowed and, instead, the charge (mass) radius of the hadron is defined from the charge (gravitational) form factor at small momentum transfer:

$$\langle r_h^2 \rangle = \lim_{q^2 \to 0} -6\frac{\partial}{\partial q^2}F(q^2)\ . \tag{70}$$

In the Drell-Yan frame, $q^+ = 0$ and $q^2 = -|\vec{q}_\perp|^2$. We can write $\vec{q}_\perp$ in the polar coordinate $\{q, \theta\}$. With a change of variable, $t = q^2$,

$$\begin{aligned}\nabla^2_{\vec{q}_\perp} =& \frac{\partial^2}{\partial q^2} + \frac{1}{q}\frac{\partial}{\partial q} + \frac{1}{q^2}\frac{\partial^2}{\partial \theta^2} \\ =& \frac{\partial^2}{\partial t^2}\Big(\frac{\partial t}{\partial q}\Big)^2 + \frac{\partial}{\partial t}\Big(\frac{\partial^2 t}{\partial q^2}\Big) + \frac{1}{\sqrt{t}}\frac{\partial}{\partial t}\frac{\partial t}{\partial q} + \frac{1}{t}\frac{\partial^2}{\partial \theta^2} \\ =& 4t\frac{\partial^2}{\partial t^2} + 4\frac{\partial}{\partial t} + \frac{1}{t}\frac{\partial^2}{\partial \theta^2} \ .\end{aligned} \tag{71}$$

At the limit of $q^2 \to 0$, the first term vanishes. Since the form factor does not have angular dependence, the third term vanishes as well. It follows that

$$\frac{\partial}{\partial t}\bigg|_{t=0} = \frac{1}{4}\nabla^2_{\vec{q}_\perp} \ . \tag{72}$$

We can thereby rewrite the charge radius in Eq. (70) in terms of the two-dimensional Laplacian of the charge form factor,

$$\langle r_h^2 \rangle = -\frac{3}{2}\nabla^2_{\vec{q}_\perp} F(q^2)\bigg|_{q^2=0} \ . \tag{73}$$

We have already mentioned that the physical form factors of a hadron should receive contributions from each constituent, $F(q^2) = \sum_f e_f F_f(q^2)$, where f is the constituent (anti)quark with charge e_f. Though for quarkonium, the physical form factor vanishes due to charge conjugation and we calculate the fictitious form factor contributed from the quark only. For a charged hadron, such as $\pi^\pm$ and proton, one should consider its physical form factor that sums over the contributions of all constituent partons. In the following, we will derive the contributions of the quark and the antiquark separately. As an example, the charge radius of π^+ sums over the contributions from u and $\bar{d}$.

$$\begin{aligned}\langle r^2_{\pi^+} \rangle =& Q_u \langle r^2_{\pi^+} \rangle_u + Q_{\bar{d}} \langle r^2_{\pi^+} \rangle_{\bar{d}} \\ =& \frac{3}{2}\sum_{s,\bar{s}} \int_0^1 \frac{\mathrm{d}x}{4\pi} \int \mathrm{d}^2 r_\perp \Big[\frac{2}{3}(1-x)^2 + \frac{1}{3}x^2\Big] \vec{r}^{\,2}_\perp \tilde{\psi}_{s,\bar{s}/\pi^+}(\vec{r}_\perp, x) \tilde{\psi}^*_{s,\bar{s}/\pi^+}(\vec{r}_\perp, x) \ .\end{aligned} \tag{74}$$

The dimensionless fractional charge of the quark is, $Q_u = +2/3$ for the up quark and $Q_{\bar{d}} = +1/3$ for the anti-down quark.

E. Radiative transition between a spin-0 and a spin-1 mesons

The electromagnetic (EM) transition between meson states, which occurs via emission of a photon, $\psi_A \to \psi_B \gamma$, offers insights into the internal structure and the dynamics of such systems. The magnetic dipole (M1) transition, which takes place between pseudoscalar and vector mesons ($\psi_A, \psi_B = \mathcal{V}, \mathcal{P}$ or $\mathcal{P}, \mathcal{V}$), has been detected with strong signals [25] and stimulates various theoretical investigations [26–30].

TABLE I. The formulas of extracting the transition form factor $V(q^2)$ from different current components and different m_j states of the vector meson. The five independent extractions in a truncated Fock space are indicated in five different colors: orange, green, red, blue and brown. Table adapted from Table I in Ref. [5]. See detailed derivation in Ref. [1].

$\frac{2V(q^2)}{m_{\mathcal{P}}+m_{\mathcal{V}}}$	$m_j=0$	$m_j=1$	$m_j=-1$
J^+	-	$\frac{i\sqrt{2}I_1^+}{P'^+\Delta^R}$	$\frac{-i\sqrt{2}I_{-1}^+}{P'^+\Delta^L}$
J^R	$\frac{-iI_0^R}{m_{\mathcal{V}}\Delta^R}$	$\frac{i\sqrt{2}I_1^R}{P'^R\Delta^R}$	$\frac{i\sqrt{2}(1-z)I_{-1}^R}{(m_{\mathcal{P}}^2-(1-z)^2m_{\mathcal{V}}^2-P^R\Delta^L)}$
J^L	$\frac{iI_0^L}{m_{\mathcal{V}}\Delta^L}$	$\frac{-i\sqrt{2}(1-z)I_1^L}{(m_{\mathcal{P}}^2-(1-z)^2m_{\mathcal{V}}^2-P^L\Delta^R)}$	$\frac{-i\sqrt{2}I_{-1}^L}{P'^L\Delta^L}$
J^-	$\frac{-iP^+I_0^-}{m_{\mathcal{V}}(\Delta^RP^L-\Delta^LP^R)}$	$\frac{-i\sqrt{2}P^+P'^+I_1^-}{P'^+P'^R(m_{\mathcal{P}}^2-P^L\Delta^R)-P^+P^Rm_{\mathcal{V}}^2}$	$\frac{i\sqrt{2}P^+P'^+I_{-1}^-}{P'^+P'^L(m_{\mathcal{P}}^2-P^R\Delta^L)-P^+P^Lm_{\mathcal{V}}^2}$

The Lorentz covariant decomposition for the electromagnetic transition matrix element between a vector meson ($\mathcal{V}$) and a pseudoscalar ($\mathcal{P}$) is [31], as we have derived in Sec. I A 2,

$$I^\mu_{m_j} \equiv \langle \mathcal{P}(P)|\, J^\mu(0)\, |\mathcal{V}(P', m_j)\rangle = \frac{2V(q^2)}{m_{\mathcal{P}}+m_{\mathcal{V}}}\epsilon^{\mu\alpha\beta\sigma}P_\alpha P'_\beta e_\sigma(P', m_j)\ , \tag{75}$$

where $q^\mu = P'^\mu - P^\mu$ represents the momentum transfer between the two mesons. $V(q^2)$ is the transition form factor. $m_{\mathcal{P}}$ and $m_{\mathcal{V}}$ are the masses of the pseudoscalar and the vector, respectively. e_σ is the polarization vector of the vector meson, and $m_j = 0, \pm 1$ is the magnetic projection. Writing out all possible formulas of extracting the transition form factor $V(q^2)$ from different current components and different m_j states of the vector meson, one would get Table. I. To simplify the expression, we take the two variables defined in Sec. I C, $z \equiv (P'^+ - P^+)/P'^+$ and $\vec{\Delta}_\perp \equiv \vec{q}_\perp - z\vec{P}'_\perp$. The five independent extractions in a truncated Fock space are indicated by five different colors in Table. I.

In the valence Fock sector, the five independent hadron matrix elements overdetermine the transition form factor. In practice, the different prescriptions of extracting the same transition form factor could provide a test of violation of the Lorentz symmetry in the calculation. But more importantly, we would like to know if there is a preferred choice such that the result is closer to the true result that would emerge from a full Fock space basis.

Working in the valence Fock sector, we take the impulse approximation, in which the interaction of the external current with the meson is the summation of its coupling to the quark and to the antiquark. The vertex dressing as well as pair creation/annihilation from higher order diagrams are neglected. The hadron matrix element can be written accordingly as a sum of the quark term and the antiquark term:

$$\langle \mathcal{P}(P')|\, J^\mu(0)\,|\mathcal{V}(P,m_j)\rangle = eQ_f\,\langle \mathcal{P}(P')|\, J^\mu_q(0)\,|\mathcal{V}(P,m_j)\rangle - eQ_f\,\langle \mathcal{P}(P')|\, J^\mu_{\bar{q}}(0)\,|\mathcal{V}(P,m_j)\rangle\ . \tag{76}$$

Bu restoring the quark charges, the current operator reads $J^\mu(x) = e\sum_f Q_f \overline{\psi}_f(x)\gamma^\mu\psi_f(x)$ where $\psi_f(x)$ is the quark field operator with flavor f ($f = u, d, s, c, b, t$). J_q and $J_{\bar{q}}$ are the normal ordered pure quark ($b^\dagger b$) and antiquark ($d^\dagger d$) part of J^μ, respectively, where b (d) is the quark (antiquark) annihilation operator. The dimensionless fractional charge of the quark is, $Q_f = Q_c = +2/3$ for the charm quark and $Q_f = Q_b = -1/3$ for the bottom quark. The electric charge $e = \sqrt{4\pi\alpha_{\rm EM}}$. For quarkonium, due to the charge conjugation symmetry, the antiquark gives the same contribution as the quark to the total hadronic current. So, for our purpose, we calculate the hadron matrix element for the quark part. As such, we compute $\hat{V}(q^2)$ which is related to $V(q^2)$ by $V(q^2) = 2eQ_f\hat{V}(q^2)$.

There are five groups of combinations of the current component and the magnetic projection according to Table I.

1. J^+ and $m_j = \pm 1$

   The light-front wavefunction representation of the transition form factor reads,

$$\begin{aligned}\hat{V}|_{J^+,m_j=1}(q^2) =& \frac{i(m_\mathcal{V} + m_\mathcal{P})}{\sqrt{2}P'^+\Delta^R}\,\langle \mathcal{P}(P)|\, J^+_q(0)\,|\mathcal{V}(P', m_j = 1)\rangle \\ =& \frac{i\sqrt{2}(m_\mathcal{V} + m_\mathcal{P})}{\Delta^R}\sum_{s,\bar{s}}\int_0^1 \frac{\mathrm{d}x}{2x(1-x)}\int \frac{\mathrm{d}^2\vec{k}_\perp}{(2\pi)^3}\sqrt{\frac{x(1-z)}{x+z(1-x)}} \\ &\times \psi^*_{s\bar{s}/\mathcal{P}}(\vec{k}_\perp, x)\psi^{(m_j=1)}_{s\bar{s}/\mathcal{V}}(\vec{k}'_\perp, x')\ .\end{aligned} \tag{77}$$

   Note that the $m_j = -1$ state would lead to the same result, considering the symmetry of the $m_j = \pm 1$ light-front wavefunctions. According to Eq. (77), the transition form factor can be evaluated as a function of z and $\Delta_\perp$. It is evident from this expression that the overlapped spin components of the two wavefunctions indicate no spin-flip (between spin-triplet and spin-singlet), which may appear counter-intuitive for the M1 transition.

2. $J^{R/L}$ and $m_j = 0$

   Using J^R and J^L current components should give the same result with the $m_j = 0$ state of the vector meson. Here we present the expression derived from J^R,

$$
\begin{aligned}
&\hat{V}|_{J^R,m_j=0}(q^2) \\
=& - i\frac{m_\mathcal{V}+m_\mathcal{P}}{2m_\mathcal{V}\Delta^R}\langle\mathcal{P}(P)|\,J^R_q(0)\,|\mathcal{V}(P',m_j=0)\rangle \\
=& - i\frac{m_\mathcal{V}+m_\mathcal{P}}{2m_\mathcal{V}\Delta^R}\sum_{\bar{s}}\int_0^1\frac{\mathrm{d}x}{2x(1-x)}\int\frac{\mathrm{d}^2\vec{k}_\perp}{(2\pi)^3} \\
&\times\Big\{\psi^*_{\uparrow\bar{s}/\mathcal{P}}(\vec{k}_\perp,x)\psi^{(m_j=0)}_{\uparrow\bar{s}/\mathcal{V}}(\vec{k}'_\perp,x')\Big\{\frac{2\sqrt{x(1-z)}}{\sqrt{[x+z(1-x)]^3}}(k^R-\frac{x}{z}\Delta^R)+\frac{2}{z}\sqrt{\frac{x(1-z)}{x+z(1-x)}}q^R\Big\} \\
&+\psi^*_{\uparrow\bar{s}/\mathcal{P}}(\vec{k}_\perp,x)\psi^{(m_j=0)}_{\downarrow\bar{s}/\mathcal{V}}(\vec{k}'_\perp,x')\frac{2m_q z}{\sqrt{x(1-z)[x+z(1-x)]^3}} \\
&+\psi^*_{\downarrow\bar{s}/\mathcal{P}}(\vec{k}_\perp,x)\psi^{(m_j=0)}_{\downarrow\bar{s}/\mathcal{V}}(\vec{k}'_\perp,x')\Big\{\frac{2}{\sqrt{x(1-z)[x+z(1-x)]}}(k^R-\frac{x}{z}\Delta^R)+\frac{2}{z}\sqrt{\frac{x(1-z)}{x+z(1-x)}}q^R\Big\}\Big\}
\end{aligned}
\tag{78}
$$

We can further simplify the expression by taking advantage of the symmetries in the light-front wavefunctions,

$$
\begin{aligned}
&\psi^{(m_j=0)}_{\uparrow\uparrow/\mathcal{V}}(\vec{k}_\perp,x)=-\psi^{(m_j=0)*}_{\downarrow\downarrow/\mathcal{V}}(\vec{k}_\perp,x), \qquad \psi_{\uparrow\uparrow/\mathcal{P}}(\vec{k}_\perp,x)=\psi^*_{\downarrow\downarrow/\mathcal{P}}(\vec{k}_\perp,x), \\
&\psi^{(m_j=0)}_{\uparrow\downarrow/\mathcal{V}}(\vec{k}_\perp,x)=\psi^{(m_j=0)}_{\downarrow\uparrow/\mathcal{V}}(\vec{k}_\perp,x), \qquad \psi_{\uparrow\downarrow/\mathcal{P}}(\vec{k}_\perp,x)=-\psi_{\downarrow\uparrow/\mathcal{P}}(\vec{k}_\perp,x)\ .
\end{aligned}
\tag{79}
$$

This leads to a partial cancellation of the first and the third terms in Eq. (78) and reduces it to,

$$
\begin{aligned}
&\hat{V}|_{J^R,m_j=0}(q^2) \\
=& - i\frac{m_\mathcal{V}+m_\mathcal{P}}{2m_\mathcal{V}\Delta^R}\sum_{\bar{s}}\int_0^1\frac{\mathrm{d}x}{2x(1-x)}\int\frac{\mathrm{d}^2\vec{k}_\perp}{(2\pi)^3}\frac{2}{\sqrt{x(1-z)[x+z(1-x)]^3}} \\
&\times\Big[\psi^*_{\uparrow\bar{s}/\mathcal{P}}(\vec{k}_\perp,x)\psi^{(m_j=0)}_{\uparrow\bar{s}/\mathcal{V}}(\vec{k}'_\perp,x')(zk^R-x\Delta^R)+\psi^*_{\uparrow\bar{s}/\mathcal{P}}(\vec{k}_\perp,x)\psi^{(m_j=0)}_{\downarrow\bar{s}/\mathcal{V}}(\vec{k}'_\perp,x')m_q z\Big] \\
=& - i\frac{m_\mathcal{V}+m_\mathcal{P}}{2m_\mathcal{V}\Delta^R}\int_0^1\frac{\mathrm{d}x}{2x(1-x)}\int\frac{\mathrm{d}^2\vec{k}_\perp}{(2\pi)^3}\frac{2}{\sqrt{x(1-z)[x+z(1-x)]^3}} \\
&\times\Big[[\frac{1}{2}\psi^*_{\uparrow\downarrow-\downarrow\uparrow/\mathcal{P}}(\vec{k}_\perp,x)\psi^{(m_j=0)}_{\uparrow\downarrow+\downarrow\uparrow/\mathcal{V}}(\vec{k}'_\perp,x')+\psi^*_{\uparrow\uparrow/\mathcal{P}}(\vec{k}_\perp,x)\psi^{(m_j=0)}_{\uparrow\uparrow/\mathcal{V}}(\vec{k}'_\perp,x')](zk^R-x\Delta^R) \\
&+\frac{1}{\sqrt{2}}[\psi^*_{\uparrow\uparrow/\mathcal{P}}(\vec{k}_\perp,x)\psi^{(m_j=0)}_{\uparrow\downarrow+\downarrow\uparrow/\mathcal{V}}(\vec{k}'_\perp,x')+\psi^*_{\uparrow\downarrow-\downarrow\uparrow/\mathcal{P}}(\vec{k}_\perp,x)\psi^{(m_j=0)}_{\downarrow\downarrow/\mathcal{V}}(\vec{k}'_\perp,x')]m_q z\Big]\ .
\end{aligned}
\tag{80}
$$

In the second equality, we adopt the notations of spin configurations as $\psi_{\uparrow\downarrow\pm\downarrow\uparrow}\equiv(\psi_{\uparrow\downarrow}\pm\psi_{\downarrow\uparrow})/\sqrt{2}$. This would be convenient to study the non-relativistic limit. According to Eq. (80), the transition form factor can be evaluated as a function of z and $\Delta_\perp$.

3. $J^{R/L}$ and $m_j=\pm1$

According to our discussion, these four choices should give the same result based on the symmetry in the transverse plane. However, this "equivalence" is not very explicit in the light-front wavefunction representation, and we see two pairs of choices.

The first pair contains these two extractions: (J^R and $m_j = 1$) and (J^L and $m_j = -1$).

$$\begin{aligned}
&\hat{V}|_{J^R,m_j=1}(q^2)\\
=&\frac{i(m_{\mathcal{V}}+m_{\mathcal{P}})}{\sqrt{2}P'^R\Delta^R}\langle\mathcal{P}(P)|\,J_q^R(0)\,|\mathcal{V}(P',m_j=1)\rangle\\
=&\frac{i(m_{\mathcal{V}}+m_{\mathcal{P}})}{\sqrt{2}P'^R\Delta^R}\sum_{\bar{s}}\int_0^1\frac{\mathrm{d}x}{2x(1-x)}\int\frac{\mathrm{d}^2\vec{k}_\perp}{(2\pi)^3}\\
&\times\Bigg\{\psi^*_{\uparrow\bar{s}/\mathcal{P}}(\vec{k}_\perp,x)\psi^{(m_j=1)}_{\uparrow\bar{s}/\mathcal{V}}(\vec{k}'_\perp,x')\frac{2}{\sqrt{x(1-z)[x+z(1-x)]}}(k^R+(1-x)\Delta^R+[x+z(1-x)]P'^R)\\
&+\psi^*_{\downarrow\bar{s}/\mathcal{P}}(\vec{k}_\perp,x)\psi^{(m_j=1)}_{\uparrow\bar{s}/\mathcal{V}}(\vec{k}'_\perp,x')\frac{2m_qz}{\sqrt{x(1-z)[x+z(1-x)]^3}}\\
&+\psi^*_{\downarrow\bar{s}/\mathcal{P}}(\vec{k}_\perp,x)\psi^{(m_j=1)}_{\downarrow\bar{s}/\mathcal{V}}(\vec{k}'_\perp,x')\frac{2\sqrt{x(1-z)}}{\sqrt{[x+z(1-x)]^3}}(k^R+x(1-z)P'^R-x\Delta^R)\Bigg\}\,.
\end{aligned}\tag{81}$$

Unlike extracting the transition from factor with the first two choices as in Eqs. (77) and (80), fixing the values of z and $\Delta_\perp$ could not uniquely determine the transition form factor in Eq. (81). There is an extra dependence on the transverse momentum, $\vec{P}_\perp$, or equivalently on $\vec{P'}_\perp$ or $\vec{q}_\perp$. This implies that the transition form factor extracted this way is not invariant under the transverse boost.

The second pair contains (J^R and $m_j = -1$) and (J^L and $m_j = 1$).

$$\begin{aligned}
&\hat{V}|_{J^R,m_j=-1}(q^2)\\
=&\frac{i\sqrt{2}(1-z)}{m^2_{\mathcal{P}}-(1-z)^2m^2_{\mathcal{V}}-P^R\Delta^L}\langle\mathcal{P}(P)|\,J_q^R(0)\,|\mathcal{V}(P',m_j=-1)\rangle\\
=&\frac{i\sqrt{2}(1-z)}{m^2_{\mathcal{P}}-(1-z)^2m^2_{\mathcal{V}}-((1-z)P'^R-\Delta^R)\Delta^L}\sum_{\bar{s}}\int_0^1\frac{\mathrm{d}x}{2x(1-x)}\int\frac{\mathrm{d}^2\vec{k}_\perp}{(2\pi)^3}\\
&\times\Bigg\{\psi^*_{\uparrow\bar{s}/\mathcal{P}}(\vec{k}_\perp,x)\psi^{(m_j=-1)}_{\uparrow\bar{s}/\mathcal{V}}(\vec{k}'_\perp,x')(k^R+(1-x)\Delta^R+[x+z(1-x)]P'^R)\\
&+\psi^*_{\downarrow\bar{s}/\mathcal{P}}(\vec{k}_\perp,x)\psi^{(m_j=-1)}_{\uparrow\bar{s}/\mathcal{V}}(\vec{k}'_\perp,x')\frac{2m_qz}{\sqrt{x(1-z)[x+z(1-x)]^3}}\\
&+\psi^*_{\downarrow\bar{s}/\mathcal{P}}(\vec{k}_\perp,x)\psi^{(m_j=-1)}_{\downarrow\bar{s}/\mathcal{V}}(\vec{k}'_\perp,x')(k^R+x(1-z)P'^R-x\Delta^R)\Bigg\}
\end{aligned}\tag{82}$$

This formalism of extracting the transition form factor also has the problematic dependence on P'^R, and is therefore not used in practice.

4. J^- and $m_j = 0$

In this combination, the light-front wavefunction representation of the transition form factor reads,

$$\begin{aligned}\hat{V}|_{J^-,m_j=0}(q^2) =&\frac{-iP^+}{m_{\mathcal{V}}(\Delta^R P^L - \Delta^L P^R)}\sum_{\bar{s}}\int_0^1 \frac{\mathrm{d}x}{2x(1-x)}\int\frac{\mathrm{d}^2\vec{k}_\perp}{(2\pi)^3}\frac{1}{x'}\frac{2}{\sqrt{x'P'^+xP^+}}\\
&\times\Big[\psi^*_{\uparrow\bar{s}/\mathcal{P}}(\vec{k}_\perp,x)\psi^{(m_j=0)}_{\uparrow\bar{s}/\mathcal{V}}(\vec{k}'_\perp,x')[m_q^2+(k'^R+x'P'^R)(k^L+xP^L)]\\
&+\psi^*_{\uparrow\bar{s}/\mathcal{P}}(\vec{k}_\perp,x)\psi^{(m_j=0)}_{\downarrow\bar{s}/\mathcal{V}}(\vec{k}'_\perp,x')m_q(k^L+xP^L-k'^L-x'P'^L)\\
&+\psi^*_{\downarrow\bar{s}/\mathcal{P}}(\vec{k}_\perp,x)\psi^{(m_j=0)}_{\uparrow\bar{s}/\mathcal{V}}(\vec{k}'_\perp,x')m_q(k'^R+x'P'^R-k^R-xP^R)\\
&+\psi^*_{\downarrow\bar{s}/\mathcal{P}}(\vec{k}_\perp,x)\psi^{(m_j=0)}_{\downarrow\bar{s}/\mathcal{V}}(\vec{k}'_\perp,x')[m_q^2+(k'^L+x'P'^L)(k^R+xP^R)]\Big],\end{aligned}\tag{83}$$

We can simplify this expression by applying the symmetries in the light-front wavefunctions in Eq. (79).

$$\begin{aligned}\hat{V}|_{J^-,m_j=0}(q^2) =&\frac{-i}{m_{\mathcal{V}}(\Delta^R P^L - \Delta^L P^R)}\sum_{\bar{s}}\int_0^1 \frac{\mathrm{d}x}{2x(1-x)}\int\frac{\mathrm{d}^2\vec{k}_\perp}{(2\pi)^3}\frac{2\sqrt{1-z}}{\sqrt{x[x+z(1-x)]^3}}\\
&\times\Big[\psi^*_{\uparrow\bar{s}/\mathcal{P}}(\vec{k}_\perp,x)\psi^{(m_j=0)}_{\uparrow\bar{s}/\mathcal{V}}(\vec{k}'_\perp,x')2i[(k'^y+x'P'^y)(k^x+xP^x)-(k'^x+x'P'^x)(k^y+xP^y)]\\
&+\psi^*_{\uparrow\bar{s}/\mathcal{P}}(\vec{k}_\perp,x)\psi^{(m_j=0)}_{\downarrow\bar{s}/\mathcal{V}}(\vec{k}'_\perp,x')(-2i)m_q(k^y+xP^y-k'^y-x'P'^y)\Big],\end{aligned}\tag{84}$$

As in Eq. (81), fixing the values of z and $\Delta_\perp$ could not uniquely determine the transition form factor in Eq. (81). There is an extra dependence on the transverse momentum of the initial state, $\vec{P}'_\perp$. This implies that the transition form factor extracted this way is not invariant under the transverse boost.

5. J^- and $m_j = \pm 1$

With this combination, we see the extra dependence of the transition form factor on the transverse momentum, again,

$$\begin{aligned}\hat{V}|_{J^-,m_j=1}(q^2) =&\frac{-i\sqrt{2}P^+P'^+}{P'^+P'^R(m_{\mathcal{P}}^2-P^L\Delta^R)-P^+P^Rm_{\mathcal{V}}^2}\sum_{\bar{s}}\int_0^1 \frac{\mathrm{d}x}{2x(1-x)}\int\frac{\mathrm{d}^2\vec{k}_\perp}{(2\pi)^3}\frac{1}{x'}\frac{2}{\sqrt{x'P'^+xP^+}}\\
&\times\Big[\psi^*_{\uparrow\bar{s}/\mathcal{P}}(\vec{k}_\perp,x)\psi^{(m_j=1)}_{\uparrow\bar{s}/\mathcal{V}}(\vec{k}'_\perp,x')[m_q^2+(k'^R+x'P'^R)(k^L+xP^L)]\\
&+\psi^*_{\uparrow\bar{s}/\mathcal{P}}(\vec{k}_\perp,x)\psi^{(m_j=1)}_{\downarrow\bar{s}/\mathcal{V}}(\vec{k}'_\perp,x')m_q(k^L+xP^L-k'^L-x'P'^L)\\
&+\psi^*_{\downarrow\bar{s}/\mathcal{P}}(\vec{k}_\perp,x)\psi^{(m_j=1)}_{\uparrow\bar{s}/\mathcal{V}}(\vec{k}'_\perp,x')m_q(k'^R+x'P'^R-k^R-xP^R)\\
&+\psi^*_{\downarrow\bar{s}/\mathcal{P}}(\vec{k}_\perp,x)\psi^{(m_j=1)}_{\downarrow\bar{s}/\mathcal{V}}(\vec{k}'_\perp,x')[m_q^2+(k'^L+x'P'^L)(k^R+xP^R)]\Big],\end{aligned}\tag{85}$$

To summarize, only two combinations of the current component and the magnetic projection of the vector meson could unambiguously extract the transition form factor from the valence hadron matrix element: they are $\hat{V}|_{J^{R/L},m_j=0}(q^2)$ in Eq. (78) and $\hat{V}|_{J^+,m_j=\pm1}(q^2)$ in Eq. (77). The other choices are not invariant under the transverse boost, and are therefore not very useful for calculating the transition form factor. The work in Ref. [2] compared the two choices, and found that $\hat{V}|_{J^{R/L},m_j=0}(q^2)$ is preferred, at least for heavy mesons, since it employs the dominant spin components of the light-front wavefunctions and is more robust in practical calculations. For the study on the frame dependence of the transition form factor, one might find Ref. [3] interesting.

Appendix A: Conventions

1. Light-Front coordinates

The contravariant four-vectors of position x^μ are written as $x^\mu = (x^+, x^-, x^1, x^2)$, where $x^+ = x^0 + x^3$ is the light-front time, $x^- = x^0 - x^3$ is the longitudinal coordinate, and $\vec{x}_\perp = (x^1, x^2)$ are the transverse coordinates. We sometimes write the transverse components with subscript x (y) in place of 1 (2), for example $\vec{r}_\perp = (r^x, r^y)$. For an arbitrary transverse vector $\vec{k}_\perp(\vec{k}^*_\perp)$, define its complex representation as $k^R = k^x + ik^y$ ($k^L = k^x - ik^y$).

The covariant vectors are obtained by $x_\mu = g_{\mu\nu}x^\nu$, with the metric tensors $g_{\mu\nu}$ and $g^{\mu\nu}$. The nonzero components of the metric tensors are,

$$g^{+-} = g^{-+} = 2, \qquad g_{+-} = g_{-+} = \frac{1}{2}, \qquad g^{ii} = g_{ii} = -1 \; (i = 1, 2) \; . \tag{A1}$$

Scalar products are

$$a \cdot b = a^\mu b_\mu = a^+ b_+ + a^- b_- + a^1 b_1 + a^2 b_2 = \frac{1}{2}(a^+ b^- + a^- b^+) - \vec{a}_\perp \cdot \vec{b}_\perp \; . \tag{A2}$$

Derivatives are written as

$$\partial_+ = \frac{\partial}{\partial x^+} = \frac{\partial}{2\partial x_-} = \frac{1}{2}\partial^-, \quad \partial_- = \frac{\partial}{\partial x^-} = \frac{\partial}{2\partial x_+} = \frac{1}{2}\partial^+ \; . \tag{A3}$$

We define the integral operators

$$\frac{1}{\partial^+} f(x^-) = \frac{1}{4}\int_{-\infty}^{+\infty} \epsilon(x^- - y^-) f(y^-) \; , \tag{A4}$$

$$\left(\frac{1}{\partial^+}\right)^2 f(x^-) = \frac{1}{8}\int_{-\infty}^{+\infty} |x^- - y^-| f(y^-) \; . \tag{A5}$$

Here, the antisymmetric step function

$$\epsilon(x) = \theta(x) - \theta(-x) \; , \qquad \frac{\partial \epsilon(x)}{\partial x} = 2\delta(x) \; . \tag{A6}$$

with the step function $\theta(x) = 0 (x < 0); 1 (x > 0)$. It follows that $|x| = x\epsilon(x)$.

[Exercise] For the exponential function, check the following relation,

$$\frac{1}{i\partial^+}e^{-ikx} = \frac{1}{k^+}e^{-ikx} \,. \tag{A7}$$

The Levi-Civita tensor is

$$\epsilon^{\mu\nu\rho\sigma} = \frac{1}{\sqrt{|g|}}\begin{cases} +1, & \text{if } \mu,\nu,\rho,\sigma \text{ is an even permutation of } -,+,1,2 \\ -1, & \text{if } \mu,\nu,\rho,\sigma \text{ is an odd permutation of } -,+,1,2 \\ 0, & \text{other cases} \end{cases} \tag{A8}$$

in which $g \equiv \det g_{\mu\nu} = -\frac{1}{2}$.

The full four-dimensional integral is

$$\int \mathrm{d}^4x = \int \mathrm{d}x^0\,\mathrm{d}x^1\,\mathrm{d}x^2\,\mathrm{d}x^3 = \frac{1}{2}\int \mathrm{d}x^+\,\mathrm{d}x^-\,\mathrm{d}^2x_\perp = \int \mathrm{d}^3x\,\mathrm{d}x^+ \,, \tag{A9}$$

where we also define the volume integral as

$$\int \mathrm{d}^3x \equiv \int \mathrm{d}x_+\,\mathrm{d}^2x^\perp = \frac{1}{2}\int \mathrm{d}x^-\,\mathrm{d}^2x^\perp \,. \tag{A10}$$

In the momentum space, the Lorentz invariant integral is,

$$\begin{aligned} \int \frac{\mathrm{d}^4p}{(2\pi)^4}\theta(p^+)(2\pi)\delta(p^+p^- - \vec{p}_\perp^{\,2} - m^2) &= \frac{1}{2}\int \frac{\mathrm{d}p^+\,\mathrm{d}p^-\,\mathrm{d}^2p_\perp}{(2\pi)^4}\theta(p^+)(2\pi)\delta(p^+p^- - \vec{p}_\perp^{\,2} - m^2) \\ &= \int \frac{\mathrm{d}^2p_\perp\,\mathrm{d}p^+}{(2\pi)^3 2p^+}\theta(p^+) \end{aligned} \tag{A11}$$

The Fourier transform of a function $f(\vec{r}_\perp)$ and the inverse transform are defined as

$$f(\vec{r}_\perp) = \int \frac{\mathrm{d}^2p_\perp}{(2\pi)^2}e^{i\vec{p}_\perp\cdot\vec{r}_\perp}\tilde{f}(\vec{p}_\perp), \qquad \tilde{f}(\vec{p}_\perp) = \int \mathrm{d}^2\vec{r}_\perp e^{-i\vec{p}_\perp\cdot\vec{r}_\perp} f(\vec{r}_\perp) \,. \tag{A12}$$

The Dirac deltas read

$$\int \mathrm{d}^2\vec{r}_\perp e^{-i\vec{p}_\perp\cdot\vec{r}_\perp} = (2\pi)^2\delta^2(\vec{p}_\perp), \qquad \int \mathrm{d}^2\vec{p}_\perp e^{i\vec{p}_\perp\cdot\vec{r}_\perp} = (2\pi)^2\delta^2(\vec{r}_\perp) \,. \tag{A13}$$

2. γ matrices

The Dirac matrices are four unitary traceless 4×4 matrices:

$$\gamma^0 = \beta = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \gamma^+ = \begin{pmatrix} 0 & 0 \\ 2i & 0 \end{pmatrix}, \quad \gamma^- = \begin{pmatrix} 0 & -2i \\ 0 & 0 \end{pmatrix}, \quad \gamma^i = \begin{pmatrix} -i\hat{\sigma}^i & 0 \\ 0 & i\hat{\sigma}^i \end{pmatrix} \,. \tag{A14}$$

They are expressed in terms of the 2×2 Pauli matrices,

$$\hat{\sigma}^1 = \sigma^2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \hat{\sigma}^2 = -\sigma^1 = \begin{pmatrix} 0 & -1 \\ -1 & 0 \end{pmatrix} . \tag{A15}$$

Note that $\gamma^3 = \gamma^+ - \gamma^0$. It is also convenient to define $\gamma^R \equiv \gamma^1 + i\gamma^2$ and $\gamma^L \equiv \gamma^1 + i\gamma^2$. The chiral matrix is $\gamma^5 = i\gamma^0\gamma^1\gamma^2\gamma^3$. Some useful relations,

$$\gamma^1\gamma^+\gamma^1 = \gamma^2\gamma^+\gamma^2 = \gamma^+, \quad \gamma^1\gamma^+\gamma^2 = -\gamma^2\gamma^+\gamma^1 = i\gamma^+ \tag{A16}$$

$$\gamma^0\gamma^\mu = \gamma^{\mu\dagger}\gamma^0, \qquad \{\gamma^\mu, \gamma^\nu\} = 2g^{\mu\nu}I \tag{A17}$$

$$\alpha^\kappa = \gamma^0\gamma^\kappa, \quad (\alpha^1)^2 = (\alpha^2)^2 = I, \quad \alpha^1\alpha^2 = -\alpha^1\alpha^2 \tag{A18}$$

Combinations of Dirac matrices as projection operators,

$$\Lambda^\pm = \frac{1}{4}\gamma^\mp\gamma^\pm = \frac{1}{2}\gamma^0\gamma^\pm = \frac{1}{2}(I \pm \alpha^3) . \tag{A19}$$

They have the following properties,

$$\begin{aligned} &\Lambda^+ + \Lambda^- = I\,, \quad (\Lambda^\pm)^2 = \Lambda^\pm\,, \quad \Lambda^\pm\Lambda^\mp = 0\,, \quad (\Lambda^\pm)^\dagger = \Lambda^\pm\,, \\ &\alpha^i\Lambda^\pm = \Lambda^\mp\alpha^i\,, \quad \gamma^0\Lambda^\pm = \Lambda^\mp\gamma^0 \,. \end{aligned} \tag{A20}$$

Appendix B: Spin vectors

We use the following spinor representation, The u, v spinors are defined as,

$$\begin{aligned} u(p,\lambda=\tfrac{1}{2}) &= \frac{1}{\sqrt{p^+}}(p^+, 0, im_q, ip^x - p^y)^\intercal \,, \\ u(p,\lambda=-\tfrac{1}{2}) &= \frac{1}{\sqrt{p^+}}(0, p^+, -ip^x - p^y, im_q)^\intercal \,, \\ \bar{u}(p,\lambda=\tfrac{1}{2}) &= \frac{1}{\sqrt{p^+}}(m_q, p^x - ip^y, -ip^+, 0) \,, \\ \bar{u}(p,\lambda=-\tfrac{1}{2}) &= \frac{1}{\sqrt{p^+}}(-p^x - ip^y, m_q, 0, -ip^+) \,, \end{aligned} \tag{B1}$$

and

$$\begin{aligned} v(p,\lambda=\tfrac{1}{2}) &= \frac{1}{\sqrt{p^+}}(p^+, 0, -im_q, ip^x - p^y)^\intercal \,, \\ v(p,\lambda=-\tfrac{1}{2}) &= \frac{1}{\sqrt{p^+}}(0, p^+, -ip^x - p^y, -im_q)^\intercal \,, \\ \bar{v}(p,\lambda=\tfrac{1}{2}) &= \frac{1}{\sqrt{p^+}}(-m_q, p^x - ip^y, -ip^+, 0) \,, \\ \bar{v}(p,\lambda=-\tfrac{1}{2}) &= \frac{1}{\sqrt{p^+}}(-p^x - ip^y, -m_q, 0, -ip^+) \,. \end{aligned} \tag{B2}$$

Define the spin vector for the massive spin 1 particles with momentum k^μ and spin projection λ:

$$e(k,\lambda=0)=(\frac{k^+}{m},\frac{\vec{k}_\perp^2-m^2}{mk^+},\frac{\vec{k}_\perp}{m}) \tag{B3}$$

$$e(k,\lambda=\pm1)=(0,\frac{2\epsilon_\lambda^\perp\cdot\vec{k}_\perp}{k^+},\epsilon_\lambda^\perp) \tag{B4}$$

where $\epsilon_\pm^\perp=(1,\pm i)/\sqrt{2}$ and m is the mass of the particle.

The polarization vectors for gluon are defined as

$$e(k,\lambda=\pm1)=(0,\frac{2\epsilon_\lambda^\perp\cdot\vec{k}_\perp}{k^+},\epsilon_\lambda^\perp) \tag{B5}$$

where $\epsilon_\pm^\perp=(1,\pm i)/\sqrt{2}$.

Spin vector identities

- Proca equation: $k_\mu e^\mu(k,\lambda)=0$.
- Orthogonality: $e^\mu(k,\lambda)e^*_\mu(k,\lambda')=-\delta_{\lambda,\lambda'}$; .
- Crossing symmetry: $e^*_\mu(k,\lambda)=e_\mu(k,-\lambda)$, $\qquad e^\mu(-k,\lambda)=(-1)^{\lambda+1}e^\mu(k,\lambda)$

1. QCD color space

The specification of the quark state in the color space is by a three-element column vector c,

$$c=\begin{pmatrix}1\\0\\0\end{pmatrix}\text{ for red,}\quad\begin{pmatrix}0\\1\\0\end{pmatrix}\text{ for blue,}\quad\begin{pmatrix}0\\0\\1\end{pmatrix}\text{ for green.} \tag{B6}$$

We use the standard basis for the fundamental representation of SU(3), i.e. the Gell-Mann matrices,

$$\begin{aligned}
T^1&=\frac{1}{2}\begin{pmatrix}0&1&0\\1&0&0\\0&0&0\end{pmatrix},\quad T^2=\frac{1}{2}\begin{pmatrix}0&-i&0\\i&0&0\\0&0&0\end{pmatrix},\quad T^3=\frac{1}{2}\begin{pmatrix}1&0&0\\0&-1&0\\0&0&0\end{pmatrix},\\
T^4&=\frac{1}{2}\begin{pmatrix}0&0&1\\0&0&0\\1&0&0\end{pmatrix},\quad T^5=\frac{1}{2}\begin{pmatrix}0&0&-i\\0&0&0\\i&0&0\end{pmatrix},\quad T^6=\frac{1}{2}\begin{pmatrix}0&0&0\\0&0&1\\0&1&0\end{pmatrix},\\
T^7&=\frac{1}{2}\begin{pmatrix}0&0&0\\0&0&-i\\0&i&0\end{pmatrix},\quad T^8=\frac{1}{2\sqrt{3}}\begin{pmatrix}1&0&0\\0&1&0\\0&0&-2\end{pmatrix}.
\end{aligned} \tag{B7}$$

In the matrix notation, $A^\mu = T^a A^\mu_a$ with the gluon index $a = 1, \ldots, 8$. The color matrix element $A^\mu_{cc'} = T^a_{cc'} A^\mu_a$

$$A^\mu = \frac{1}{2}\begin{pmatrix} \frac{1}{\sqrt{3}}A^\mu_8 + A^\mu_3 & A^\mu_1 - iA^\mu_2 & A^\mu_4 - iA^\mu_5 \\ A^\mu_1 + iA^\mu_2 & \frac{1}{\sqrt{3}}A^\mu_8 - A^\mu_3 & A^\mu_6 - iA^\mu_7 \\ A^\mu_4 + iA^\mu_5 & A^\mu_6 + iA^\mu_7 & -\frac{2}{\sqrt{3}}A^\mu_8 \end{pmatrix} . \tag{B8}$$

2. Discrete symmetries

Consider a particle state with momentum p^μ and parity P,

$$\mathbb{P}\,|\phi(p^\mu, \mathrm{P})\rangle = \mathrm{P}\,|\phi(\mathcal{P}^\mu_\nu p^\nu, \mathrm{P})\rangle \ . \tag{B9}$$

The parity operator is

$$\mathcal{P}^\mu_\nu = (\mathcal{P}^{-1})^\mu_\nu = \begin{pmatrix} +1 & & & \\ & -1 & & \\ & & -1 & \\ & & & -1 \end{pmatrix} . \tag{B10}$$

The current operator under the parity transformation is

$$\mathbb{P}^{-1} J^\mu \mathbb{P} = \mathcal{P}^\mu_\nu J^\nu \ . \tag{B11}$$

For the polarization vector,

$$e^\mu(\mathcal{P}\cdot k, \lambda) = -\mathcal{P}^\mu_\nu e^\nu(k, \lambda) \ . \tag{B12}$$

Consider a particle state with charge conjugation C (if there is one),

$$\mathbb{C}\,|\phi(p^\mu, \mathrm{C})\rangle = \mathrm{C}\,|\phi(p^\mu, \mathrm{C})\rangle \ . \tag{B13}$$

The current operator under the charge conjugation is

$$\mathbb{C}^{-1} J^\mu \mathbb{C} = -J^\mu \ . \tag{B14}$$

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[1] M. Li, *Non-perturbative applications of quantum chromodynamics*, Ph.D. thesis, Iowa State U. (main) (2019).

[2] M. Li, Y. Li, P. Maris, and J. P. Vary, Phys. Rev. **D98**, 034024 (2018), arXiv:1803.11519 [hep-ph].

[3] M. Li, Y. Li, P. Maris, and J. P. Vary, (2019), arXiv:1906.07306 [nucl-th].

[4] L. Adhikari, Y. Li, M. Li, and J. P. Vary, Phys. Rev. **C99**, 035208 (2019), arXiv:1809.06475 [hep-ph].

[5] M. Li, PoS **LC2019**, 057 (2020), arXiv:2001.02108 [nucl-th].

[6] S. Weinberg, *The Quantum theory of fields. Vol. 1: Foundations* (Cambridge University Press, 2005).

[7] Y. Li, P. Maris, and J. P. Vary, Phys. Rev. **D97**, 054034 (2018), arXiv:1712.03467 [hep-ph].

[8] J. Carbonell, B. Desplanques, V. A. Karmanov, and J. F. Mathiot, Phys. Rept. **300**, 215 (1998), arXiv:nucl-th/9804029 [nucl-th].

[9] J. P. B. C. de Melo, J. H. O. Sales, T. Frederico, and P. U. Sauer, *Proceedings, 15th International IUPAP Conference on Few body problems in physics (FB15): Groningen, Netherlands, July 22-26, 1997*, Nucl. Phys. **A631**, 574C (1998), arXiv:hep-ph/9802325 [hep-ph].

[10] S. J. Brodsky and D. S. Hwang, Nucl. Phys. **B543**, 239 (1999), arXiv:hep-ph/9806358 [hep-ph].

[11] D. Melikhov and S. Simula, Phys. Rev. **D65**, 094043 (2002), arXiv:hep-ph/0112044 [hep-ph].

[12] S. Simula, Phys. Rev. **C66**, 035201 (2002), arXiv:nucl-th/0204015 [nucl-th].

[13] A. T. Suzuki, J. H. Sales, and L. A. Soriano, Phys. Rev. **D88**, 025036 (2013), arXiv:1209.4681 [hep-th].

[14] D. Melikhov, Phys. Rev. **D53**, 2460 (1996), arXiv:hep-ph/9509268 [hep-ph].

[15] W. Jaus, Phys. Rev. **D53**, 1349 (1996), [Erratum: Phys. Rev.D54,5904(1996)].

[16] B. L. G. Bakker, H.-M. Choi, and C.-R. Ji, Phys. Rev. **D67**, 113007 (2003), arXiv:hep-ph/0303002 [hep-ph].

[17] N. Isgur and C. H. Llewellyn Smith, Nucl. Phys. **B317**, 526 (1989).

[18] M. Sawicki, Phys. Rev. **D46**, 474 (1992).

[19] M. Burkardt, Nucl. Phys. **A504**, 762 (1989).

[20] J. A. O. Marinho, T. Frederico, and P. U. Sauer, Phys. Rev. **D76**, 096001 (2007).

[21] J. A. O. Marinho, T. Frederico, E. Pace, G. Salme, and P. Sauer, Phys. Rev. **D77**, 116010 (2008), arXiv:0805.0707 [hep-ph].

[22] H.-M. Choi, H.-Y. Ryu, and C.-R. Ji, Phys. Rev. **D99**, 076012 (2019), arXiv:1903.01448 [hep-ph].

[23] S. J. Brodsky, R. Roskies, and R. Suaya, Phys. Rev. **D8**, 4574 (1973).

[24] S. J. Brodsky and S. D. Drell, Phys. Rev. **D22**, 2236 (1980).

[25] M. Tanabashi *et al.* (Particle Data Group), Phys. Rev. **D98**, 030001 (2018).

[26] N. Brambilla, Y. Jia, and A. Vairo, Phys. Rev. **D73**, 054005 (2006), arXiv:hep-ph/0512369 [hep-ph].

[27] G. C. Donald, C. T. H. Davies, R. J. Dowdall, E. Follana, K. Hornbostel, J. Koponen, G. P. Lepage, and C. McNeile, Phys. Rev. **D86**, 094501 (2012), arXiv:1208.2855 [hep-lat].

[28] D. Ebert, R. N. Faustov, and V. O. Galkin, Phys. Rev. **D67**, 014027 (2003), arXiv:hep-ph/0210381 [hep-ph].

[29] T. Barnes, S. Godfrey, and E. S. Swanson, Phys. Rev. **D72**, 054026 (2005), arXiv:hep-ph/0505002 [hep-ph].

[30] A. Pineda and J. Segovia, Phys. Rev. **D87**, 074024 (2013), arXiv:1302.3528 [hep-ph].

[31] J. J. Dudek, R. G. Edwards, and D. G. Richards, Phys. Rev. **D73**, 074507 (2006), arXiv:hep-ph/0601137 [hep-ph].