LOOPs'24 Fort Lauderdale FLORIDA

A Finite and Computable Spinfoam Model with Cosmological Constant

Qiaoyin Pan

Based on works with Muxin Han and Chen-Hung Hsiao

arXiv: 2310.04537, 2311.08587, 2401.14643 & to appear

Loops '24 conference

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FLORIDA ATLANTIC UNIVERSITY

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Motivation — inclusion of Λ in the 4D spinfoam model

"Spinfoam model is a covariant formalism of quantum gravity"

$$\int \mathcal{D}g_{\mu\nu}\, e^{iS_{\text{EH}}}$$

A *good* formalism is such that

Semi-classically consistent with the Einstein gravity	**Finite**	**Computable**

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Semi-classically consistent with the Einstein gravity	**Finite**	**Computable**
4D: EPRL 3D: Ponzano-Regge Turaev-Viro ($\Lambda \neq 0$)	4D: quantum group deform. of EPRL ($\Lambda \neq 0$?) [Han; Fairbairn, Meusburger] 3D: Turaev-Viro ($\Lambda \neq 0$)	4D: EPRL [*r.f.* Pietro's, Dongxue's & Cong's talk] 3D: Ponzano-Regge Turaev-Viro ($\Lambda \neq 0$)

[Ponzano, Regge, Turaev, Viro, Smolin, Major, Lewandowski, Okołów, Engle, Pereira, Rovelli, Livine, Freidel, Krasnov, Barrett, Dowdall, Fairbairn, Hellmann, Meusburger, Nouri, Roche, Haggard, Han, Kaminski, Riello, Girelli, Dupuis, Dona, Dittrich, Asante, Steinhaus, Qu, Liu, Zhang, *et al.*]

***The focus of this talk*:**

A Lorentzian 4D spinfoam model (SF) with $\Lambda \neq 0$ that has all these features [Han '21]

Plan of this talk

- **Overview of the amplitude construction in the 4D SF with $\Lambda \neq 0$**
- **Concrete testimonies**
  - Finiteness — melonic SF amplitude
  - Consistent with GR — critical point geometry
  - Computable — critical point reconstruction program
- **Future explorations**

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Overview I — Towards 4D SF with Λ

- **Starting point — Plebanski-Holst formulation** of 4D gravity: BF theory + simplicity constraint

$$S_{\mathsf{BF}} = -\frac{1}{2}\int_{M_4} \mathsf{Tr}\left[\left(\star B + \frac{1}{\gamma}B\right)\wedge \mathcal{F}(\mathcal{A})\right] - \frac{|\Lambda|}{12}\int_{M_4}\mathsf{Tr}\left[\left(\star B + \frac{1}{\gamma}B\right)\wedge B\right]$$

$B : \mathfrak{sl}(2,\mathbb{C})$ 2-form; $\mathcal{A} : \mathfrak{sl}(2,\mathbb{C})$ connection; $\gamma \in \mathbb{R}$: Barbero-Immirzi parameter; $\star$: Hodge operator

simplicity constraint: $B = \mathsf{sgn}(\Lambda)\, e \wedge e$

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- **Construct the Lorentzian path integral:** integrating out B field $\xrightarrow{\text{Gaussian integral}} \mathcal{F}[\mathcal{A}] = \frac{|\Lambda|}{3} B$

$$\int \mathrm{d}\mathcal{A} \int \mathrm{d}B\, e^{iS_{\mathsf{BF}}} = \int \mathrm{d}\mathcal{A}\, e^{\frac{3i}{4|\Lambda|}\int_{M_4}\mathrm{Tr}\left[\left(\star + \frac{1}{\gamma}\right)\mathcal{F}\wedge\mathcal{F}\right]}$$

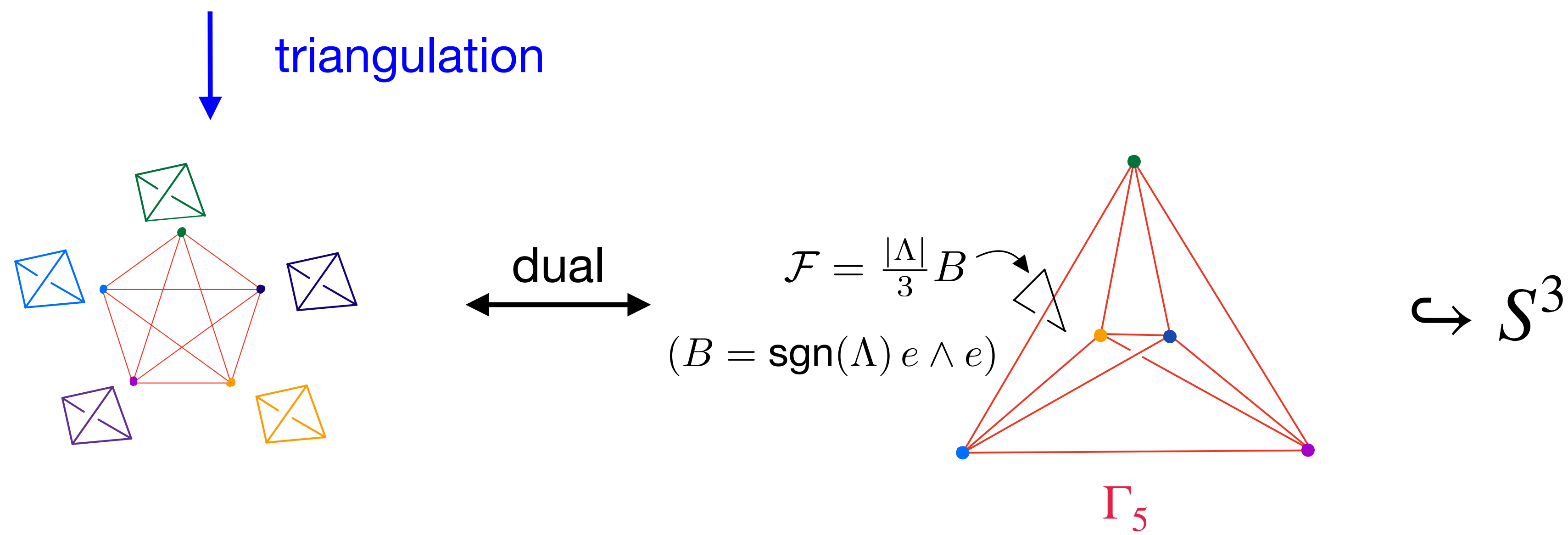
- $\xrightarrow{M_4 \text{ trivial topo.}}$ SL$(2,\mathbb{C})$ **Chern-Simons theory with complex coupling constant on the boundary**

$$\frac{t}{8\pi}\int_{\partial M_4}\mathrm{Tr}\left(A\wedge \mathrm{d}A + \frac{2}{3}A\wedge A\wedge A\right) + \frac{\bar{t}}{8\pi}\int_{\partial M_4}\mathrm{Tr}\left(\overline{A}\wedge \mathrm{d}\overline{A} + \frac{2}{3}\overline{A}\wedge\overline{A}\wedge\overline{A}\right)$$

$t = k(1+i\gamma)$, $\quad k = \frac{12\pi}{|\Lambda|\ell_{\mathrm{p}}^2\gamma} \in \mathbb{Z}_+ \Longrightarrow$ gauge invariant

Overview I — Towards 4D SF with Λ — cont.

- CS theory on ∂M_4: $\frac{t}{8\pi}\int_{\partial M_4}\mathrm{Tr}\left(A\wedge \mathrm{d}A+\frac{2}{3}A\wedge A\wedge A\right)+\frac{\bar{t}}{8\pi}\int_{\partial M_4}\mathrm{Tr}\left(\overline{A}\wedge \mathrm{d}\overline{A}+\frac{2}{3}\overline{A}\wedge \overline{A}\wedge \overline{A}\right)$
- Vertex amplitude: $M_4 = B^4, \partial M_4 = S^3$



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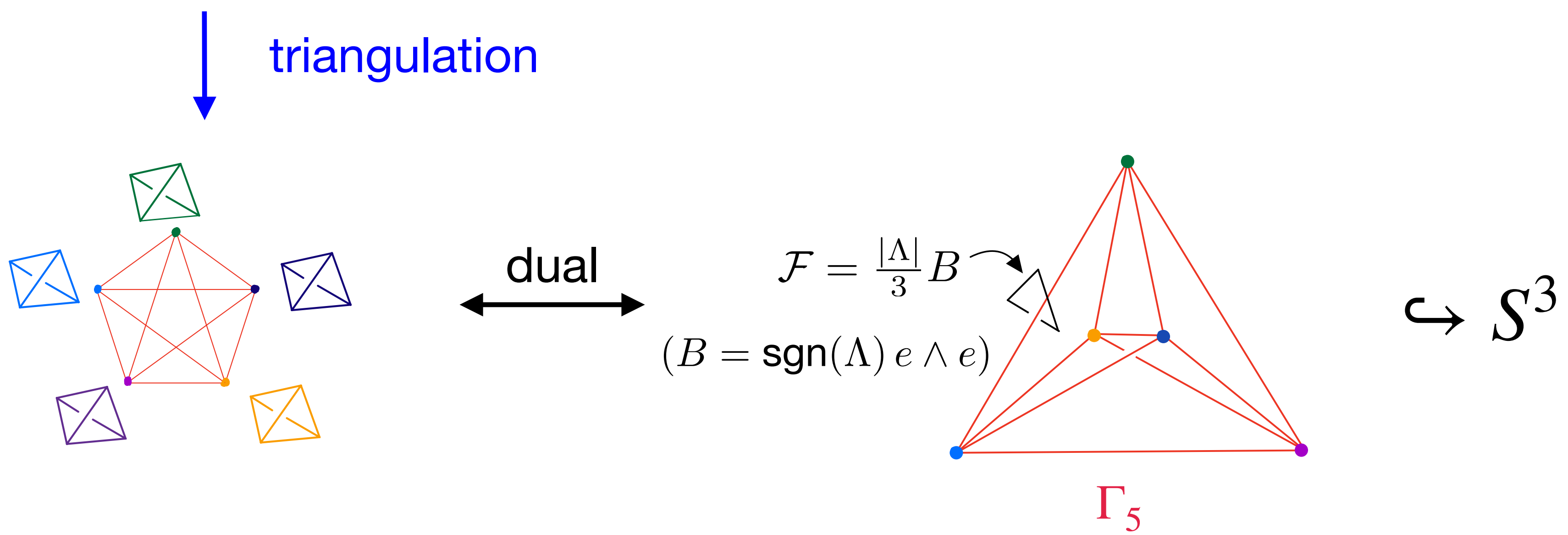
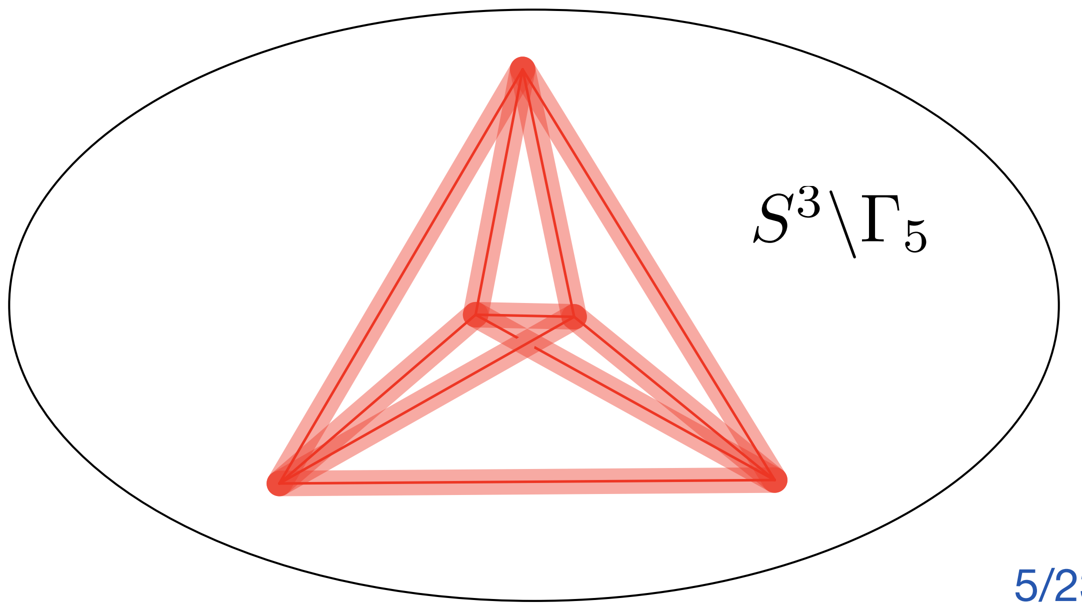


- Curvatures are **line defects** on $S^3 \longrightarrow$ consider CS theory on the **graph complement**

  CS theory on $S^3\backslash\Gamma_5 \longrightarrow$ solution space: moduli space $\mathcal{M}_{\text{flat}}\left(S^3\backslash\Gamma_5, \mathrm{SL}(2,\mathbb{C})\right)$ of $\mathrm{SL}(2,\mathbb{C})$ flat connections



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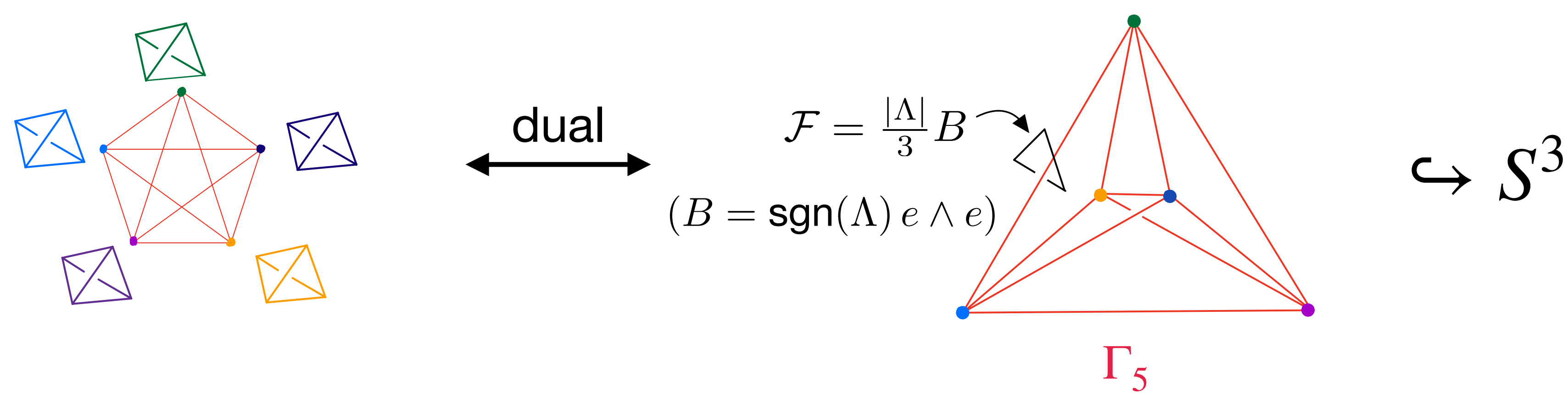
↓ triangulation



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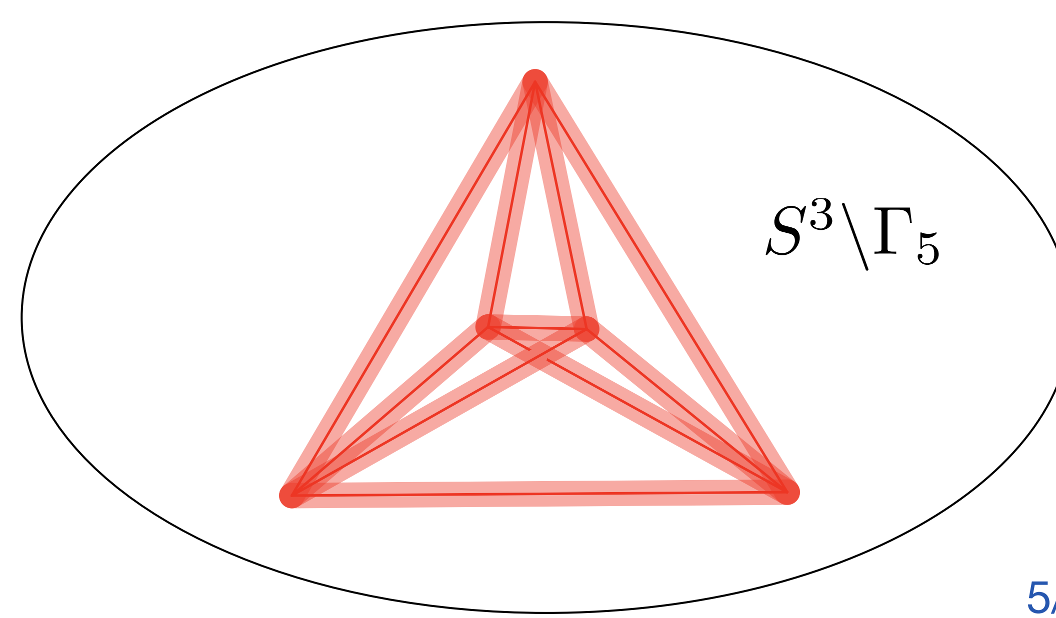
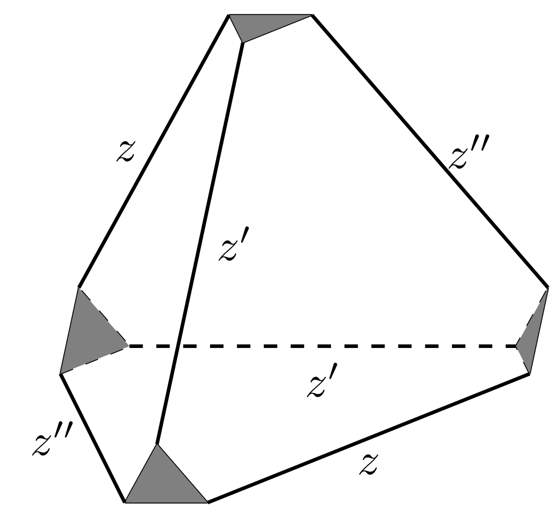
> 4D quantum gravity with $\Lambda \neq 0$
>
> = complex CS theory on boundary\graph
>
> \+ **simplicity constraints** on the graph $\mathcal{F} = \frac{\Lambda}{3}e\wedge e$



Overview II — CS partition function on $S^3\backslash\Gamma_5$

- **Step 1**: CS partition function on $S^3\backslash\Gamma_5$
  - Discretization of $S^3\backslash\Gamma_5$ is composed of 20 ideal tetrahedra $\triangle$'s



ideal tetrahedron
(vertices truncated)

[Faddeev '95, Kashaev '96, Dimofte, Gaiotto, Gukov '14-15]

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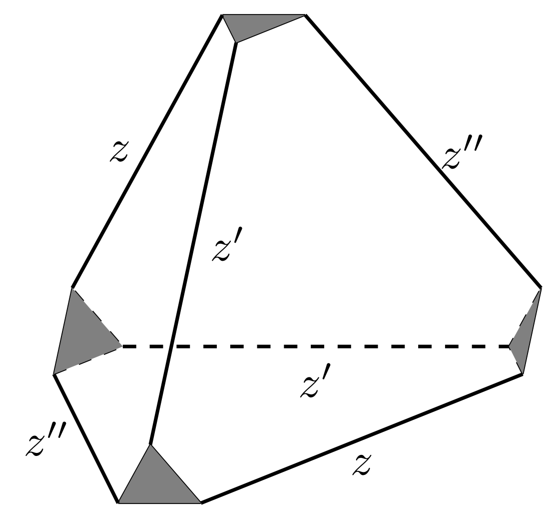
  - CS partition function on one $\triangle$: **quantum dilogarithm function**

$$\Psi_\triangle(z,\tilde{z}) = \prod_{j=0}^{\infty} \frac{1-\tilde{q}^{-j}\tilde{z}^{-1}}{1-q^{-j}z^{-1}} \quad (\text{ meromorphic })$$

$$q = \exp\left(\frac{4\pi i}{t}\right) \qquad \tilde{q} = \exp\left(\frac{4\pi i}{\bar{t}}\right)$$



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CS phase space coordinates on $\triangle$:

$$z(\mu, j) = \exp\left[\frac{2\pi i}{k}(-ib\mu - 2j)\right]$$

$$\tilde{z}(\mu, j) = \exp\left[\frac{2\pi i}{k}\left(-ib^{-1}\mu + 2j\right)\right]$$

$$\left(\ \mu \in \mathbb{R}\ ,\ \text{spin } j \in \mathbb{Z}/2\right)$$

$$b = \sqrt{\frac{1-i\gamma}{1+i\gamma}} \qquad k = \frac{12\pi}{|\Lambda|\ell_{\mathrm{p}}^2\gamma}$$

[Faddeev '95, Kashaev '96, Dimofte, Gaiotto, Gukov '14-15]

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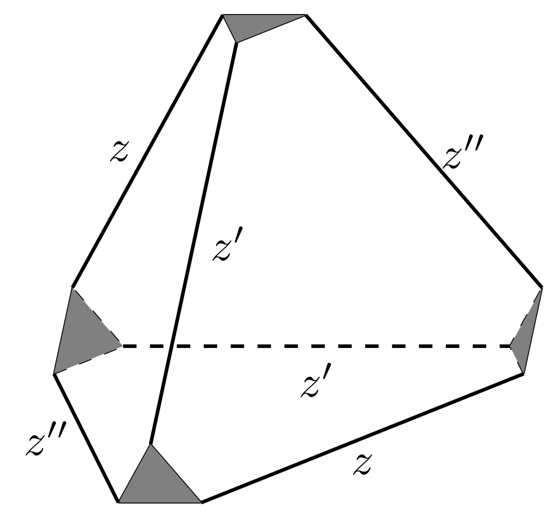

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$$b = \sqrt{\frac{1-i\gamma}{1+i\gamma}} \qquad k = \frac{12\pi}{|\Lambda|\ell_{\rm p}^2\gamma}$$

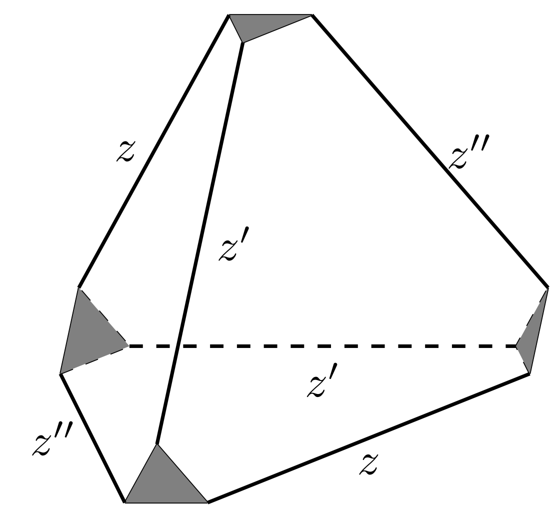
The coordinates are periodic in j: $\quad z(j) = z(j+\mathbb{Z}k/2), \quad \tilde z(j) = \tilde z(j+\mathbb{Z}k/2)$

$$j = 0, \frac{1}{2}, \cdots, \frac{k-1}{2}$$

Truncation in spin **by construction**, implied by Λ

[Faddeev '95, Kashaev '96, Dimofte, Gaiotto, Gukov '14-15]

Overview II — CS partition function on $S^3\backslash\Gamma_5$ — cont.



ideal tetrahedron
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- **Result:** CS partition function $\mathcal{Z}_{S^3\backslash\Gamma_5}$

$$= \text{Unitary transformations} \cdot \text{gluing constraints} \cdot \prod_{a=1}^{20} \Psi_{\triangle}(a)$$

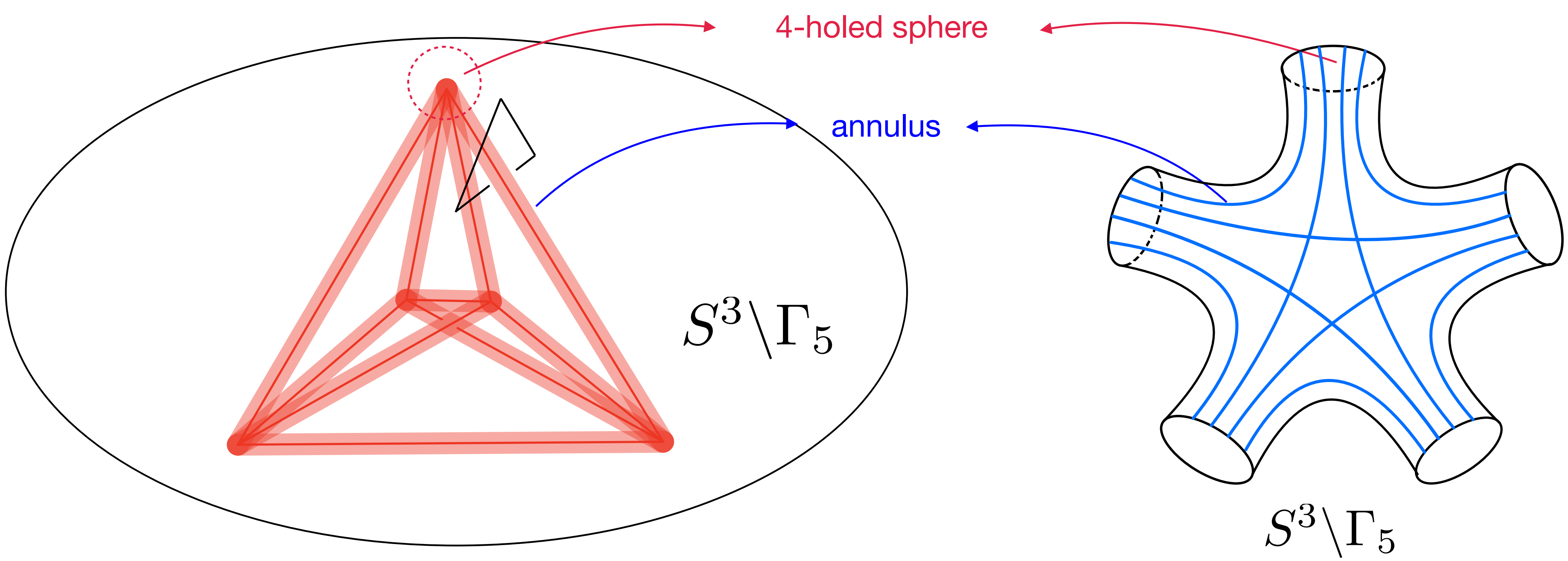
$$= \text{finite sum of convergent state integral}$$

Bounded!

$$\mathcal{Z}_{S^3\backslash\Gamma_5}(\vec{\mu} \mid \vec{j}) = \frac{4i}{k^{15}} \sum_{2\vec{l}\in(\mathbb{Z}/k\mathbb{Z})^{15}} \int_{\mathcal{C}} \mathrm{d}^{15}\vec{\nu}\; e^{S_0} \prod_{i=1}^{20} \Psi_{\triangle}(i)$$

$$S_0 = \frac{\pi i}{k}\left[-2\left(\vec{\mu} - \frac{iQ}{2}\vec{t}\right)\cdot\vec{\nu} + 8\vec{j}\cdot\vec{l} - \vec{\nu}\cdot\mathbf{AB}^\top\cdot\vec{\nu} + 4(k+1)\vec{l}\cdot\mathbf{AB}^\top\cdot\vec{l}\right]$$

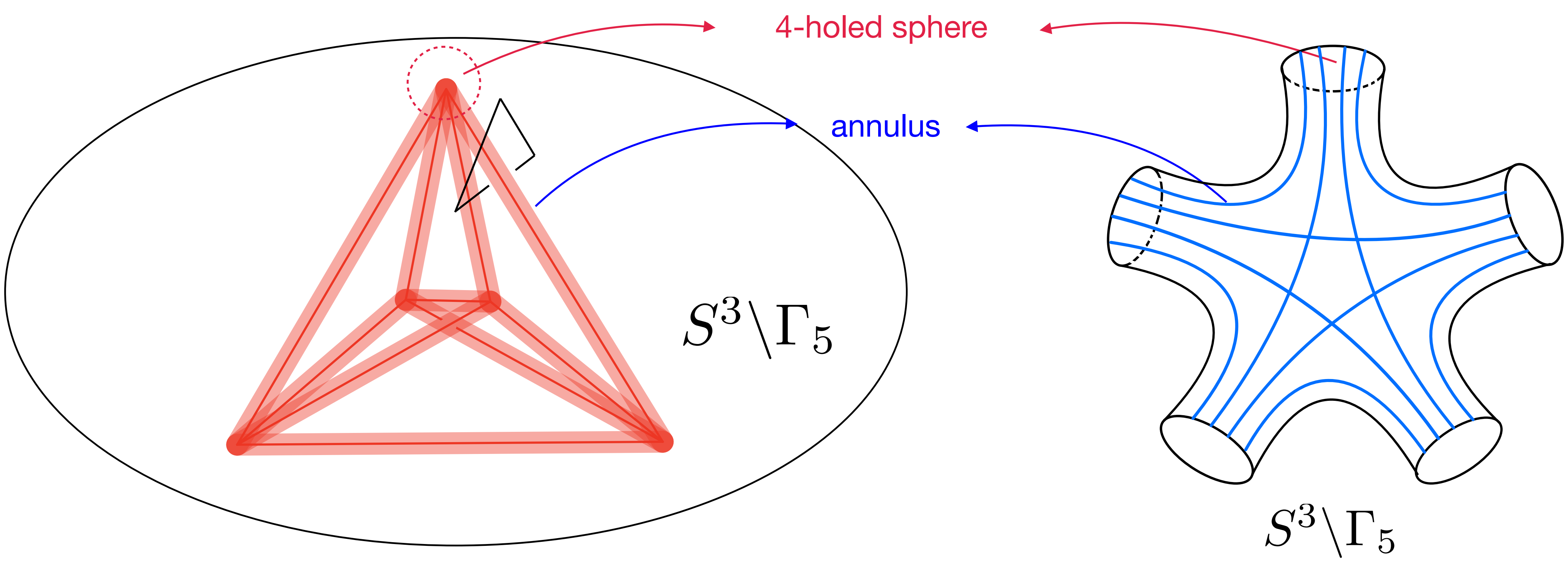
Overview III — simplicity constraints and vertex amplitude



$\mathcal{Z}_{S^3\backslash\Gamma_5}(\vec{\mu} \mid \vec{j}) \Longrightarrow (\vec{\mu}, \vec{j})$ are **localized** coordinates of $\mathcal{M}_{\text{flat}}(\partial(S^3\backslash\Gamma_5), \text{SL}(2,\mathbb{C}))$

[Han '21, Han, QP '23]

Overview III — simplicity constraints and vertex amplitude



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- **Step 2** towards vertex amplitude: impose the **simplicity constraints** on Γ_5

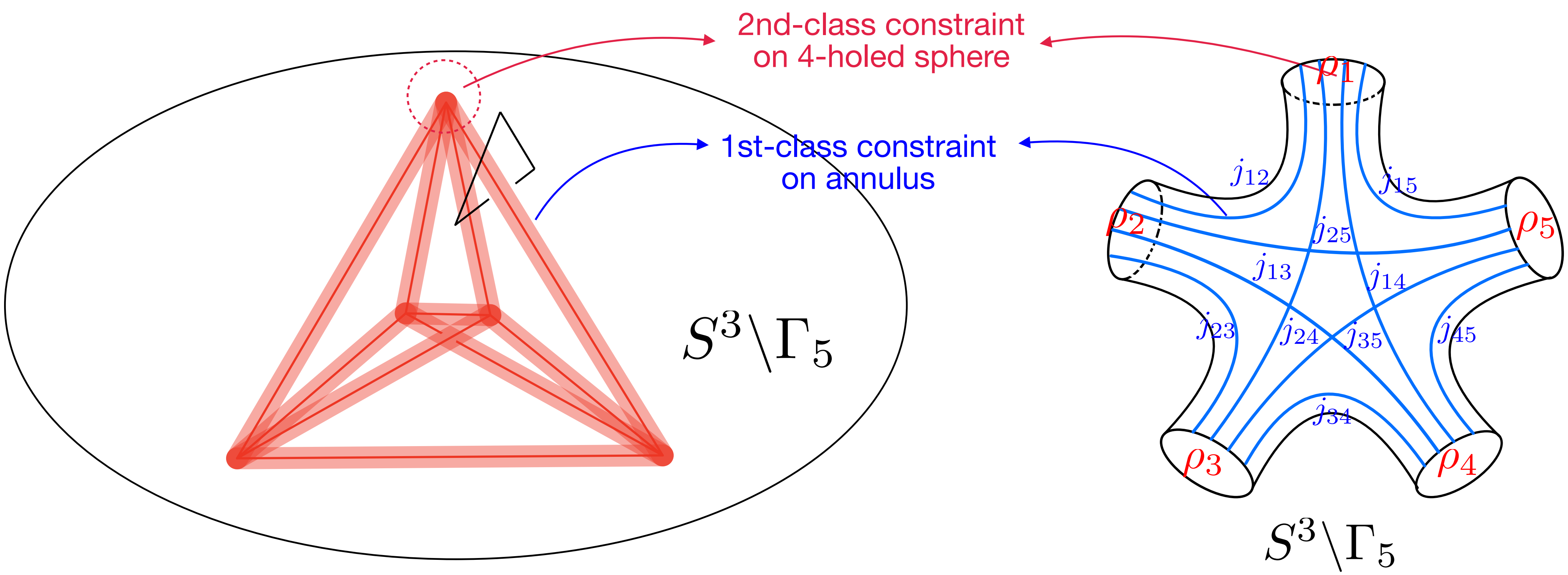
Linear constraint: $\exists\, N^I \in \mathbb{R}^{1,3}$ such that $N^I B_{IJ} = 0\,, \quad \forall$ tetra

$$\xrightarrow{\mathcal{F}=\frac{|\Lambda|}{3}B} \quad N^I \mathcal{F}_{IJ} = 0, \quad \forall \text{ 4-holed sphere } \Sigma_{0,4}$$

$$\mathcal{M}_{\text{flat}}(\Sigma_{0,4}, \text{SL}(2,\mathbb{C})) \xrightarrow{\text{simplicity constraint}} \mathcal{M}_{\text{flat}}(\Sigma_{0,4}, \text{SU}(2))$$

[Han '21, Han, QP '23]

Overview III — simplicity constraints and vertex amplitude



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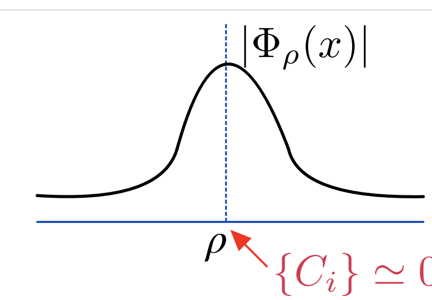
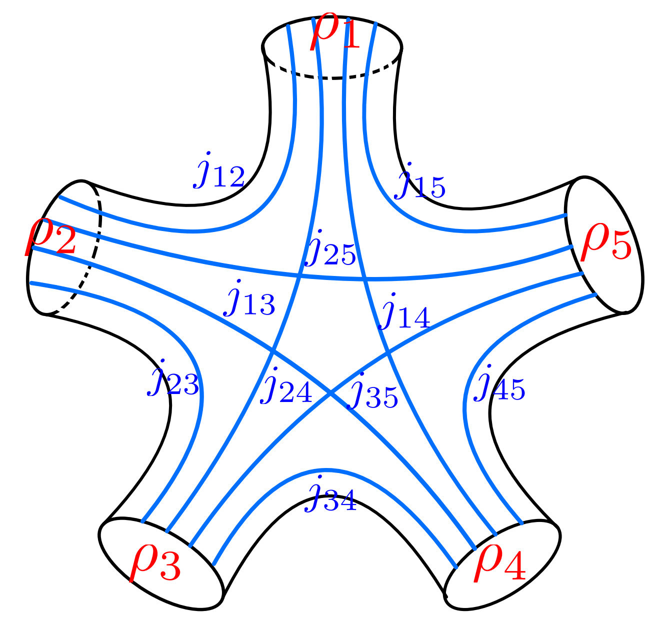
First-class constraints	Second-class constraints
Impose strongly	Impose weakly
$\mathcal{Z}_{S^3\backslash\Gamma_5} = \mathcal{Z}_{S^3\backslash\Gamma_5}\left(\{\lambda_{ab}\}_{a<b};\cdots\right),$ $\lambda_{ab} = \exp\left(\frac{2\pi i}{k} j_{ab}\right) = \exp\left(\frac{i\lvert\Lambda\rvert}{6}\mathfrak{a}_{ab}\right) \in \mathrm{U}(1)$	Couple with coherent state Ψ_ρ 
Fix the triangle **area** by spin	Restrict ρ to label the tetrahedron **shape**

[Han '21, Han, QP '23]

Overview III — simplicity constraints and vertex amplitude — cont.



$S^3\backslash\Gamma_5$

- The **vertex amplitude** is defined by the inner product of the CS partition function with 5 coherent states

$$\mathcal{A}_v(j,\rho) = \langle \Psi_\rho \mid \mathcal{Z}^j_{S^3\backslash\Gamma_5}\rangle$$

  - **Finite** by construction
  - Large k-limit: oscillatory action $\mathcal{A}_v \overset{k\to\infty}{\sim} \int[\mathsf{d}\vec{X}]\, e^{kS(\vec{X})}$ $\quad k = \frac{12\pi}{|\Lambda|\ell_{\mathsf{p}}^2\gamma}$
  - Stationary phase analysis $\Longrightarrow$ reproduce **4D Regge action with constant curvature**

$$\mathcal{A}_v = \left(\mathcal{N}_+ e^{iS_{\text{Regge},\Lambda}} + \mathcal{N}_- e^{-iS_{\text{Regge},\Lambda}}\right)\left[1 + O(1/k)\right]$$

[Haggard, Han, Kaminski, Riello ’14-15; Han ’21]

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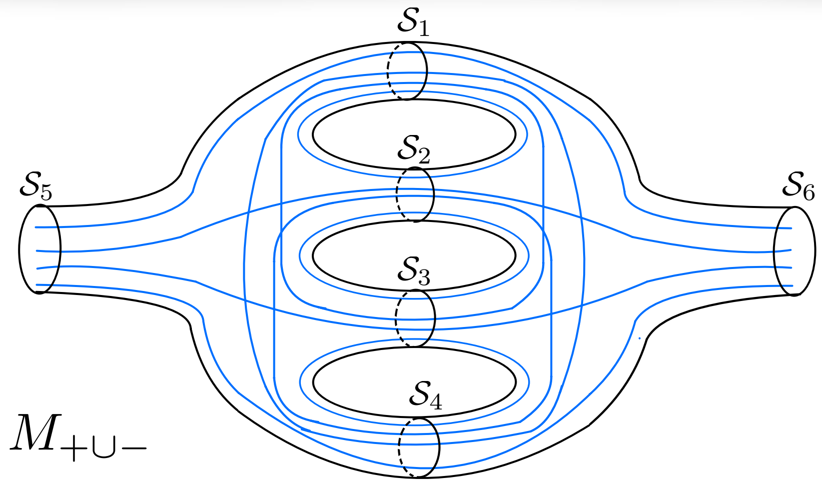
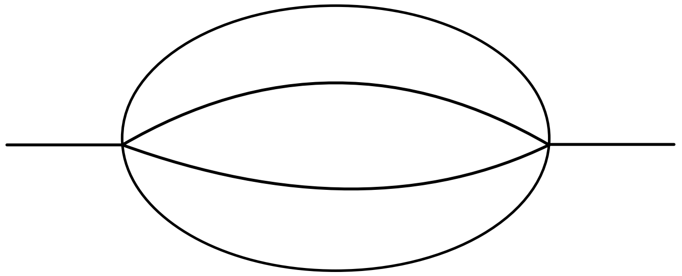
SF amplitude for a melon graph



$\Longleftrightarrow$



melonic spinfoam graph

[Han, QP '23]

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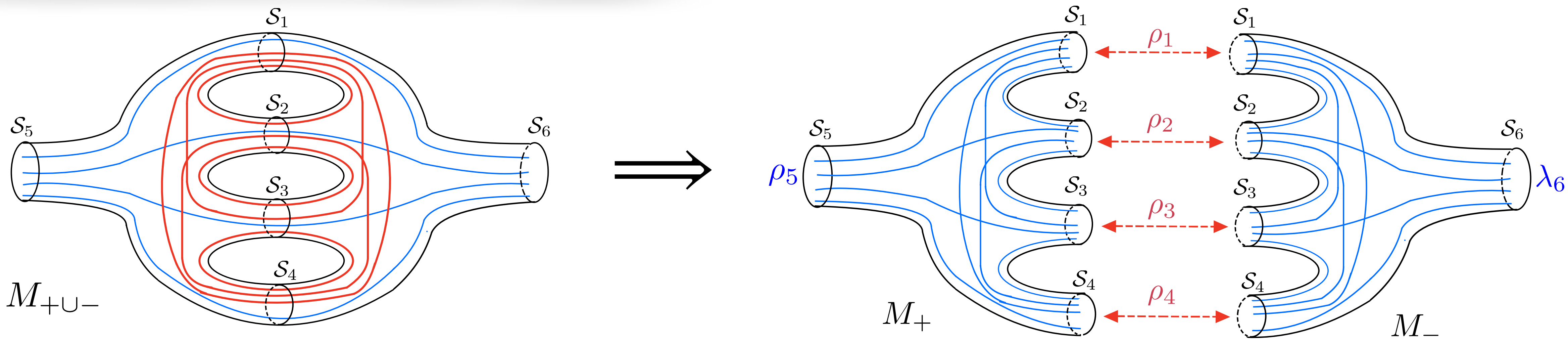


finite sums — Integrals over compact space — bounded amplitudes

$$\mathcal{A}_{\text{melon}}\left(\{j_b\},\rho_5,\lambda_6\right)=\sum_{\{j_f\}=0}^{(k-1)/2}\prod_{f=1}^{6}[2j_f+1]_{\mathfrak{q}}\int[\mathbf{d}\rho_e]^{\times 4}\,\mathcal{A}_{v,+}\left(\{j_f\},\{j_b\},\{\rho_e\},\rho_5\right)\mathcal{A}_{v,-}\left(\{j_f\},\{j_b\},\{\rho_e\},\lambda_6\right)$$

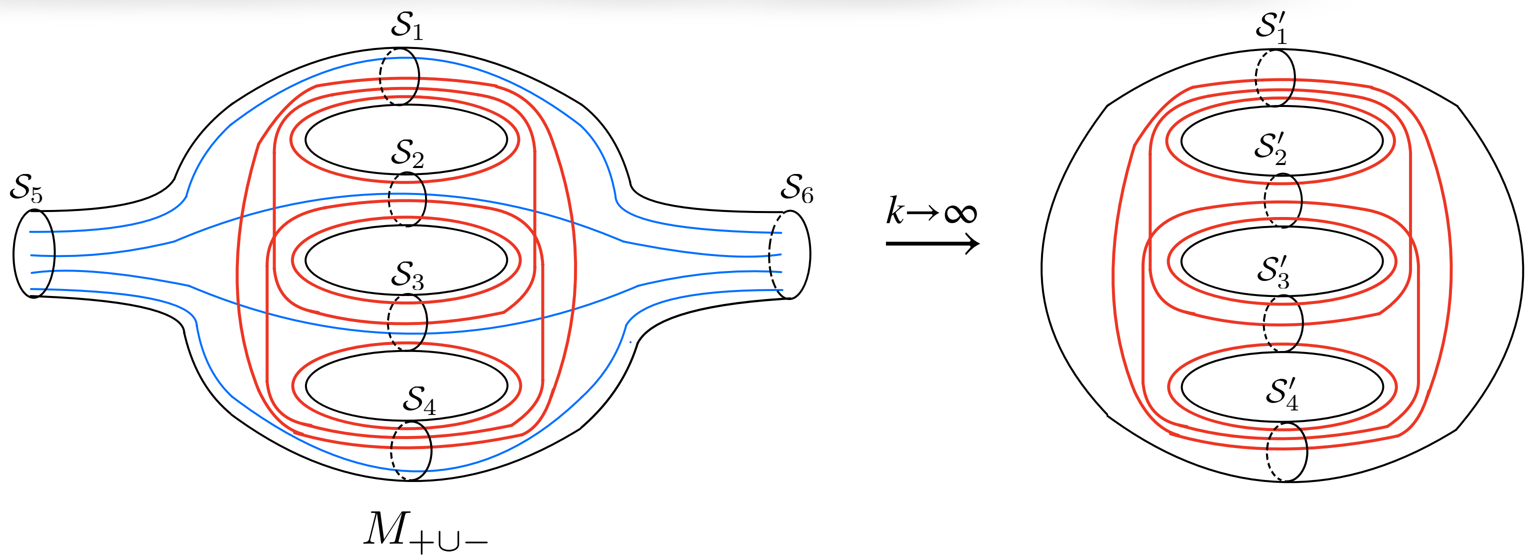
quantum dimension with $\mathfrak{q}=e^{\frac{2\pi i}{k}}$ $\quad k=\frac{12\pi}{|\Lambda|\ell_{\rm p}^2\gamma}$

Finite!

- Face amplitude — consistent with EPRL face amplitude $[2j_f+1]_{\mathfrak{q}}\xrightarrow{\Lambda\to 0}2j_f+1$
- The **finiteness** can be generalized to **any** spinfoam graph using similar mechanism

[Han, QP '23]

SF amplitude for a melon graph — cont.



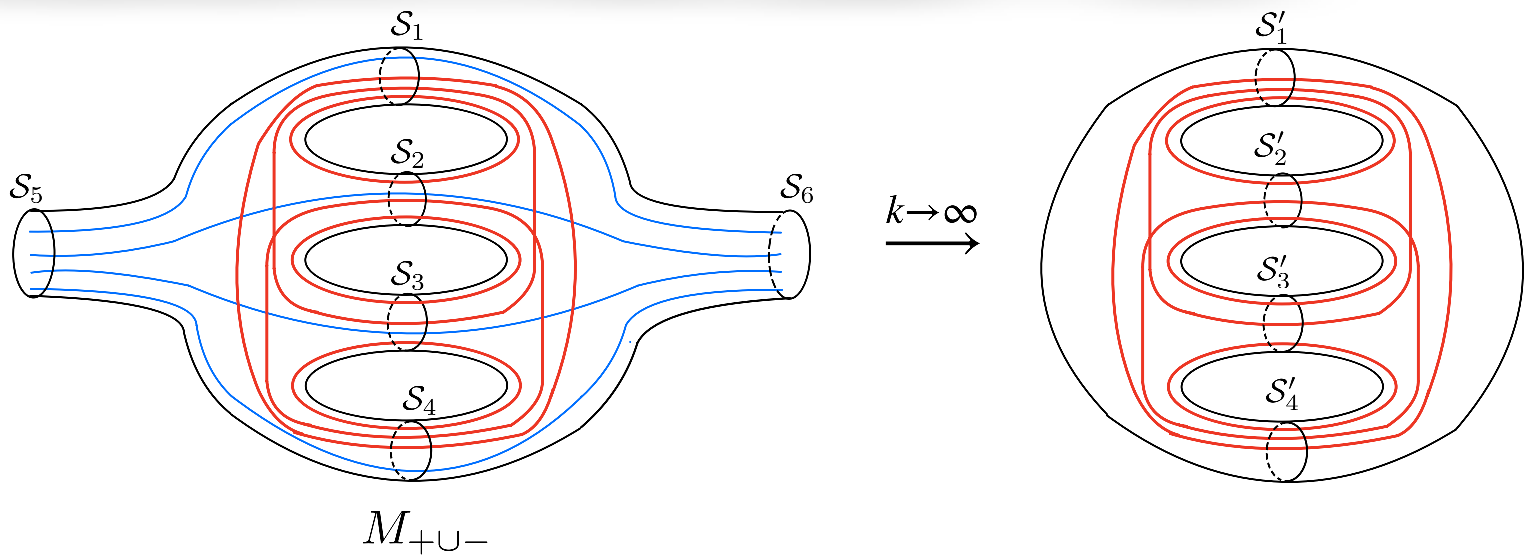
- We fix the boundary data $(\{j_b\}, \rho_5, \lambda_6)$ and consider the $\Lambda \to 0\,(k \to \infty)$ for the melonic amplitude $k = \frac{12\pi}{|\Lambda|\ell_{\mathrm{p}}^2\gamma}$
- Oscillatory action $\mathcal{A}_{\text{melon}} \overset{k\to\infty}{\sim} \int [\mathrm{d}\vec{X}]\, e^{kS(\vec{X})} \implies$ Stationary phase analysis

  $\implies$ Scaling behaviour (lower bound): $\mathcal{A}_{\text{melon}} \sim k^{21}$

Finite!

[Han, QP '23]

SF amplitude for a melon graph — cont.



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  $\implies$ Scaling behaviour (lower bound): $\mathcal{A}_{\text{melon}} \sim k^{21}$ **Finite!**

- Comparison with the melonic radiative correction in the EPRL model
  - Introduce a cut-off for representation label **by hand** $\sum_{j=0}^{\infty} \to \sum_{j=0}^{j_{\text{max}}}$ and consider large j_{max} **Infinite**
  - Divergent behaviour (numerical result): $\mathcal{A}_{\text{melon}} \sim j_{\text{max}}$ [Frisoni, Gozzini, Vidotto '22]

[Han, QP '23]

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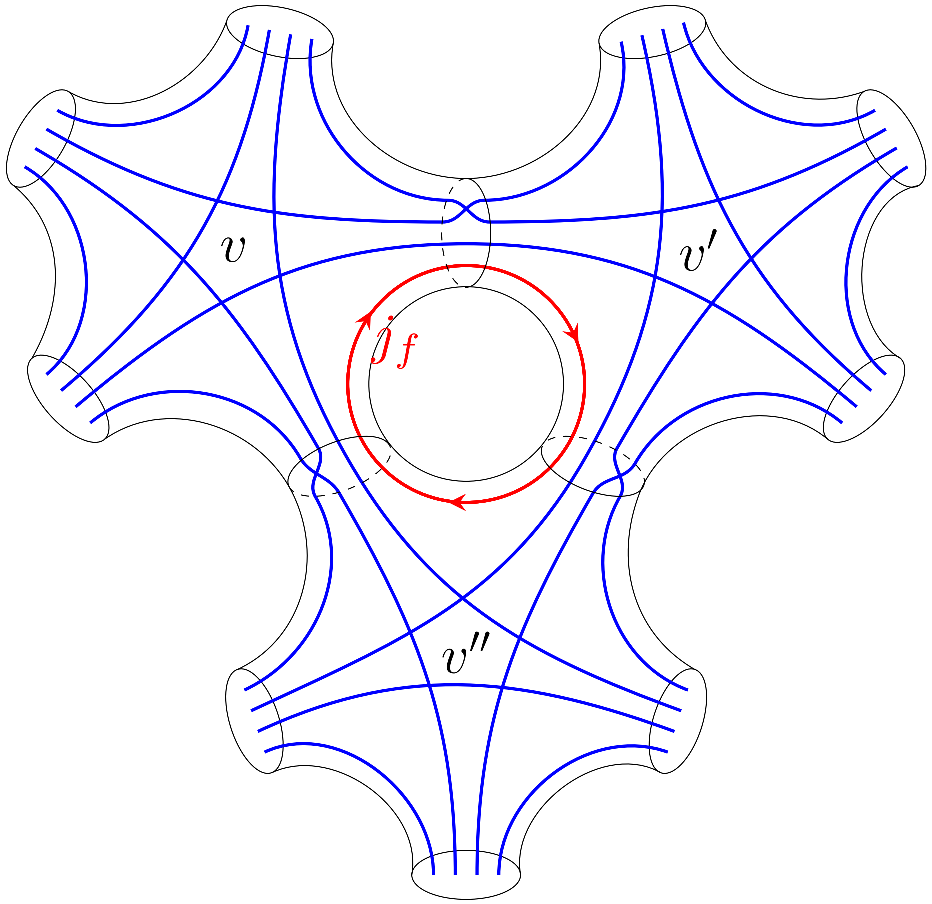
Critical point geometry

- **Critical point geometry:**

Vertex amplitude: **constantly curved 4-simplex** geometry

gluing 4-simplices by identifying boundary **constantly curved tetrahedra**

What if internal triangles form?



Δ_3 three-manifold

[Han, QP '24]

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Stationary phase analysis for internal spin j_f:

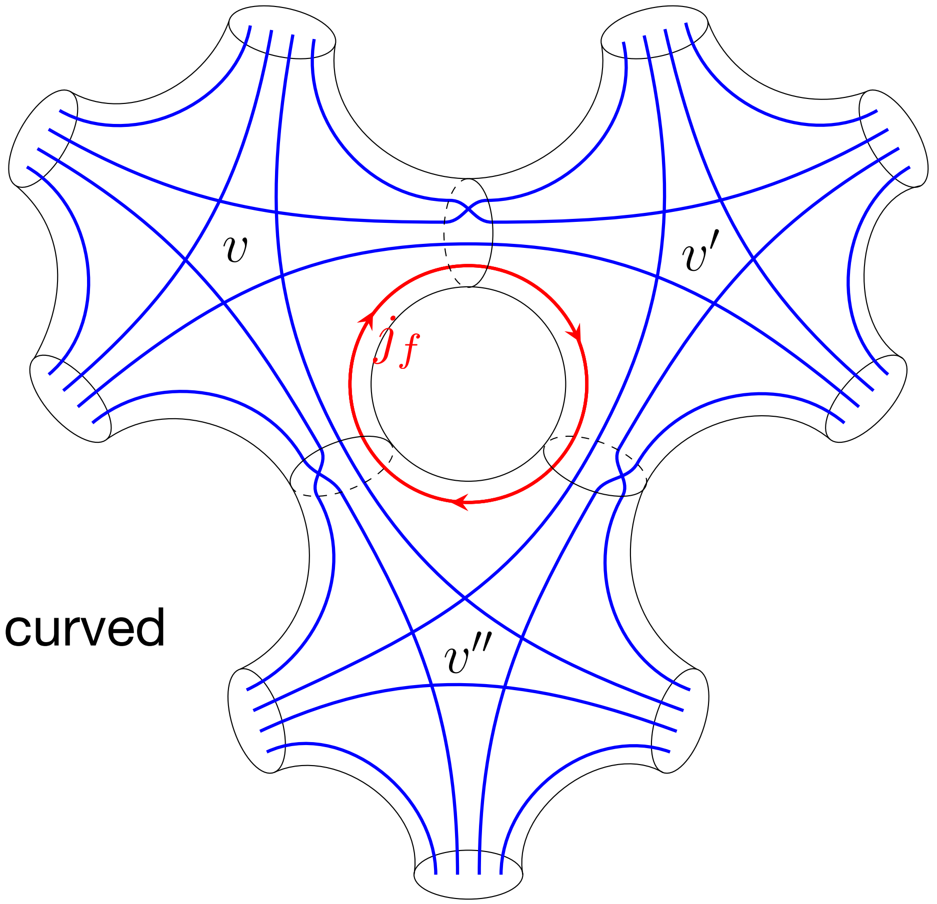
$\Longrightarrow$ critical deficit angle — similar to the flatness in EPRL

[Bonzom '09, Hellmann, Kaminski '12, Han '13, Engle, Kaminski, Oliveira '21]

$$\varepsilon_f \equiv \sum_{v\in f} \Theta_f = 4\pi N_f/\gamma, \quad N_f \in \mathbb{Z} \quad \xrightarrow{N_f=0} \quad 0$$

$\Longrightarrow$ 4D bulk is smoothly dS/AdS $\begin{cases} \text{every 4-simplex is constantly curved} \\ \text{zero deficit angle} \end{cases}$

Valid for any 4-complex with internal triangle(s)



Δ_3 three-manifold

[Han, QP '24]

Critical point geometry — cont.

- **Critical point geometry:**

  - Every **4-simplex** is **constantly curved**
  - 4-simplices are glued by identifying boundary **constantly curved tetrahedra**
  - **Vanishing deficit angle** hinged by each internal triangle $(\text{mod}\, 4\pi\mathbb{Z}/\gamma)$

- in the **semiclassical** regime $\Longrightarrow$ the critical point of the spinfoam amplitude describes a **4D dS spacetime** (when $\Lambda > 0$) or a **4D AdS spacetime** (when $\Lambda < 0$) $-$ **"(A)dS-ness property"**

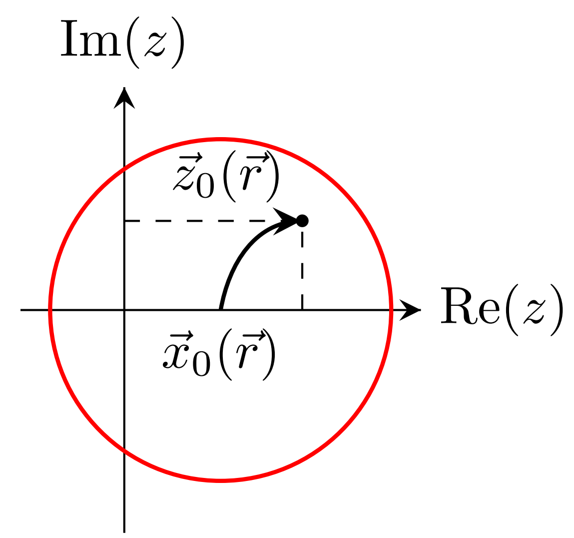
[Han, QP '24]

Critical point geometry — cont.

- **Critical point geometry:** $\longrightarrow$ real critical point
  - Every **4-simplex** is **constantly curved**
  - 4-simplices are glued by identifying boundary **constantly curved tetrahedra**
  - **Vanishing deficit angle** hinged by each internal triangle $(\text{mod } 4\pi\mathbb{Z}/\gamma)$

- in the **semiclassical** regime $\Longrightarrow$ the critical point of the spinfoam amplitude describes a **4D dS spacetime** (when $\Lambda > 0$) or a **4D AdS spacetime** (when $\Lambda < 0$) — **"(A)dS-ness property"**

- **But NOT (A)dS-ness problem!**
- Consider complex critical point, as is done in EPRL *[r.f. Dongxue's talk]*
  - **Non-(A)dS geometries** are captured by the complex critical points



[Han, QP '24]

Plan of this talk

- Overview of the amplitude construction in the 4D SF with $\Lambda \neq 0$
- **Concrete testimonies**
  - Finiteness — melonic SF amplitude
  - Consistent with GR — critical point geometry
  - Computable — stationary phase analysis
- Future explorations

Stationary phase approximation of SF amplitude

1 finite-dimensional integral with oscillatory action

e.g. one $\mathcal{A}_v$: 40-dim integral

$\triangle_3$: 114-dim integral

↓

stationary phase approximation

2 4-simplex geometry

triangle areas & normals

$\{\mathfrak{a}_f, \hat{\mathfrak{n}}_f\}$

↓ [Haggard, Han, Kaminski, Riello '15, Han '15, Haggard, Han, Riello '16]

$\mathcal{M}_{\text{flat}}\left(S^3\backslash\Gamma_5, \mathsf{SL}(2,\mathbb{C})\right)$

coordinates $\{z, \tilde{z}, \cdots\}$

[Han, QP, to appear]

Stationary phase approximation of SF amplitude

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triangle areas & normals
$\{\mathfrak{a}_f, \hat{\mathfrak{n}}_f\}$ ← **Input**

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↓

CS phase space coordinates

$\{z(\{\mathfrak{a}_f, \mathfrak{n}_f\}), \tilde{z}(\{\mathfrak{a}_f, \mathfrak{n}_f\}), \cdots\}$

↓

SF amplitude at (real) critical point → **Output**

$e^{S(w,\tilde{w},\cdots)}\Big|_{w=z,\tilde{w}=\tilde{z},\cdots}$

[Han, QP, to appear]

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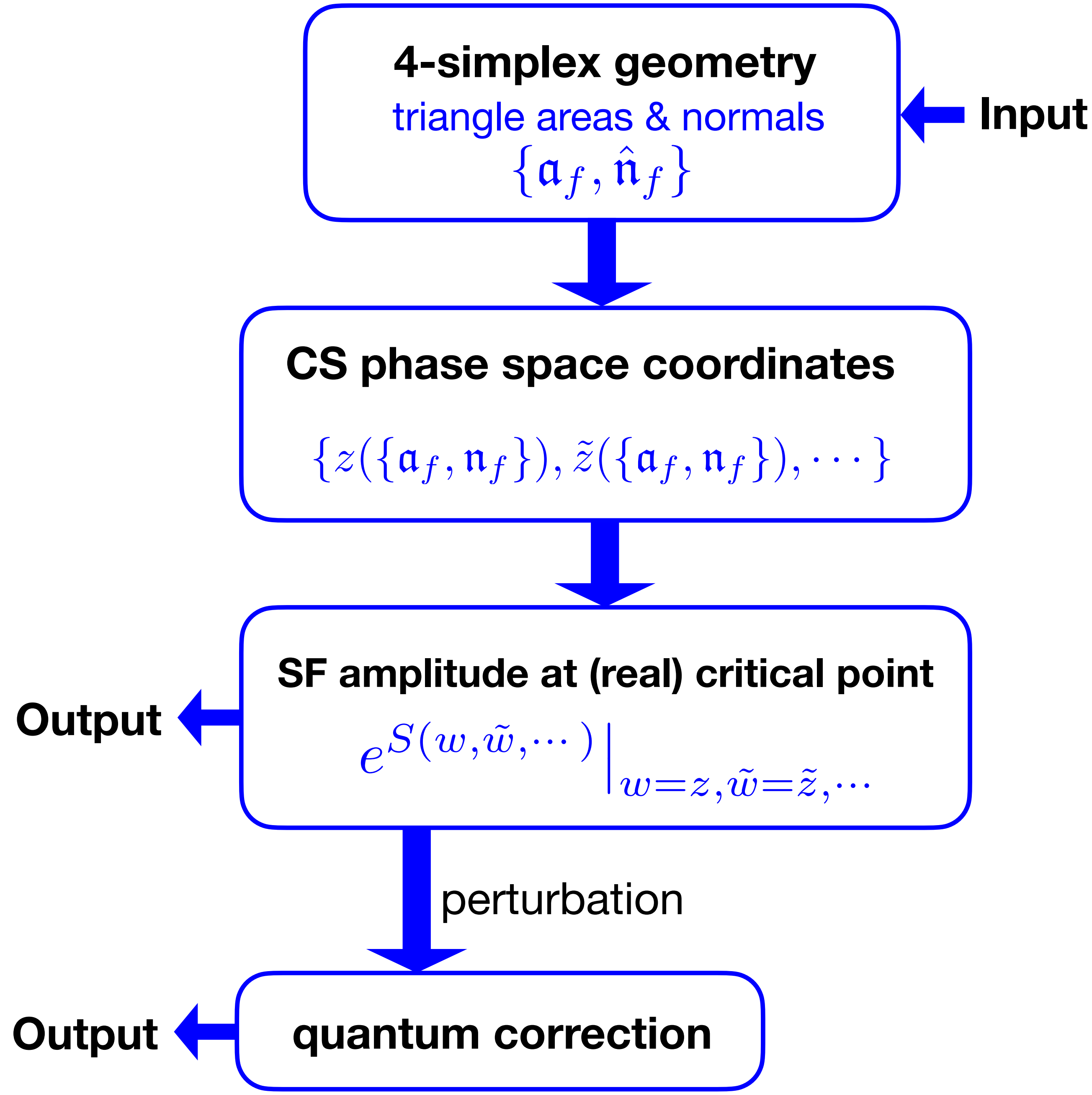

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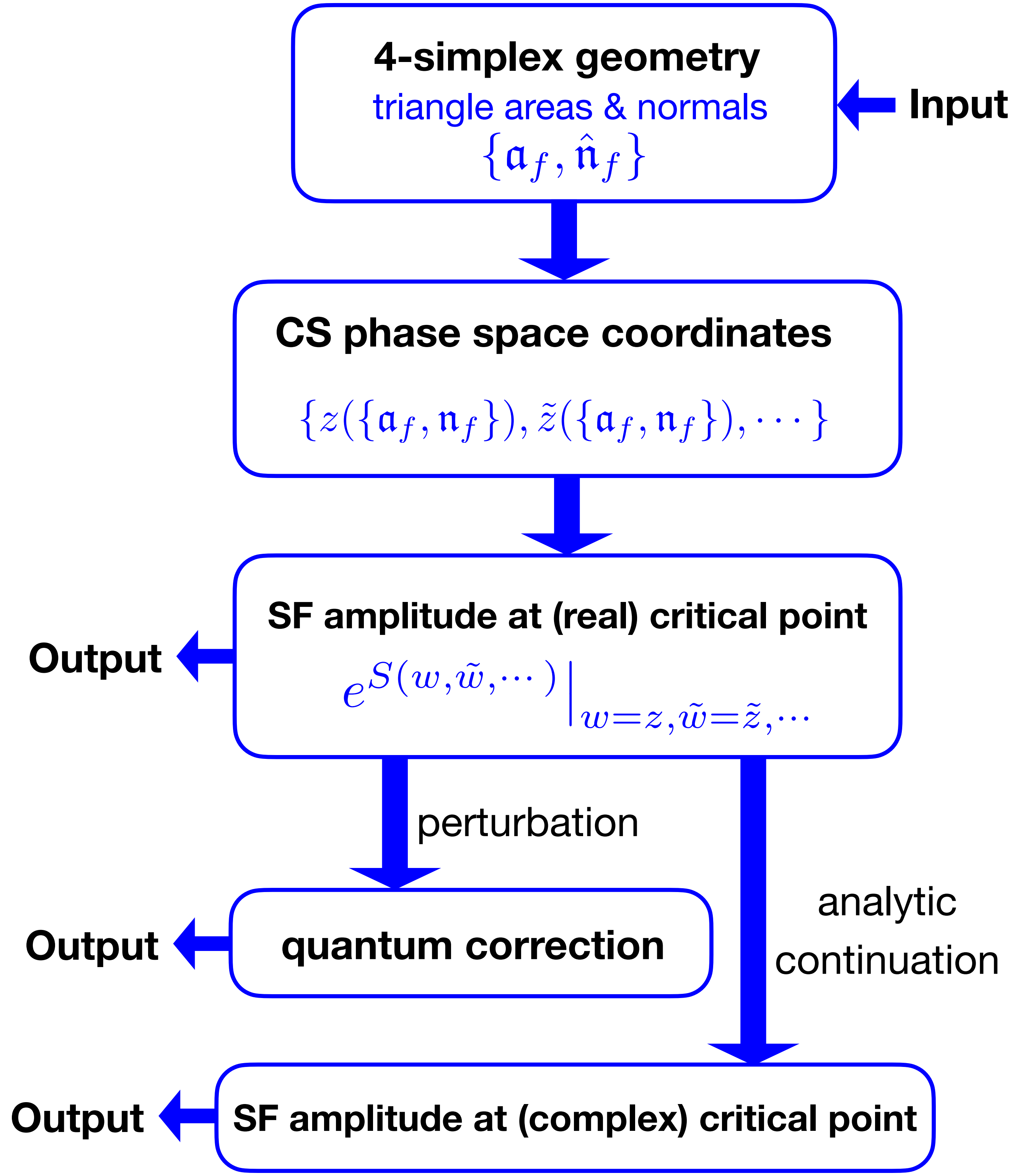
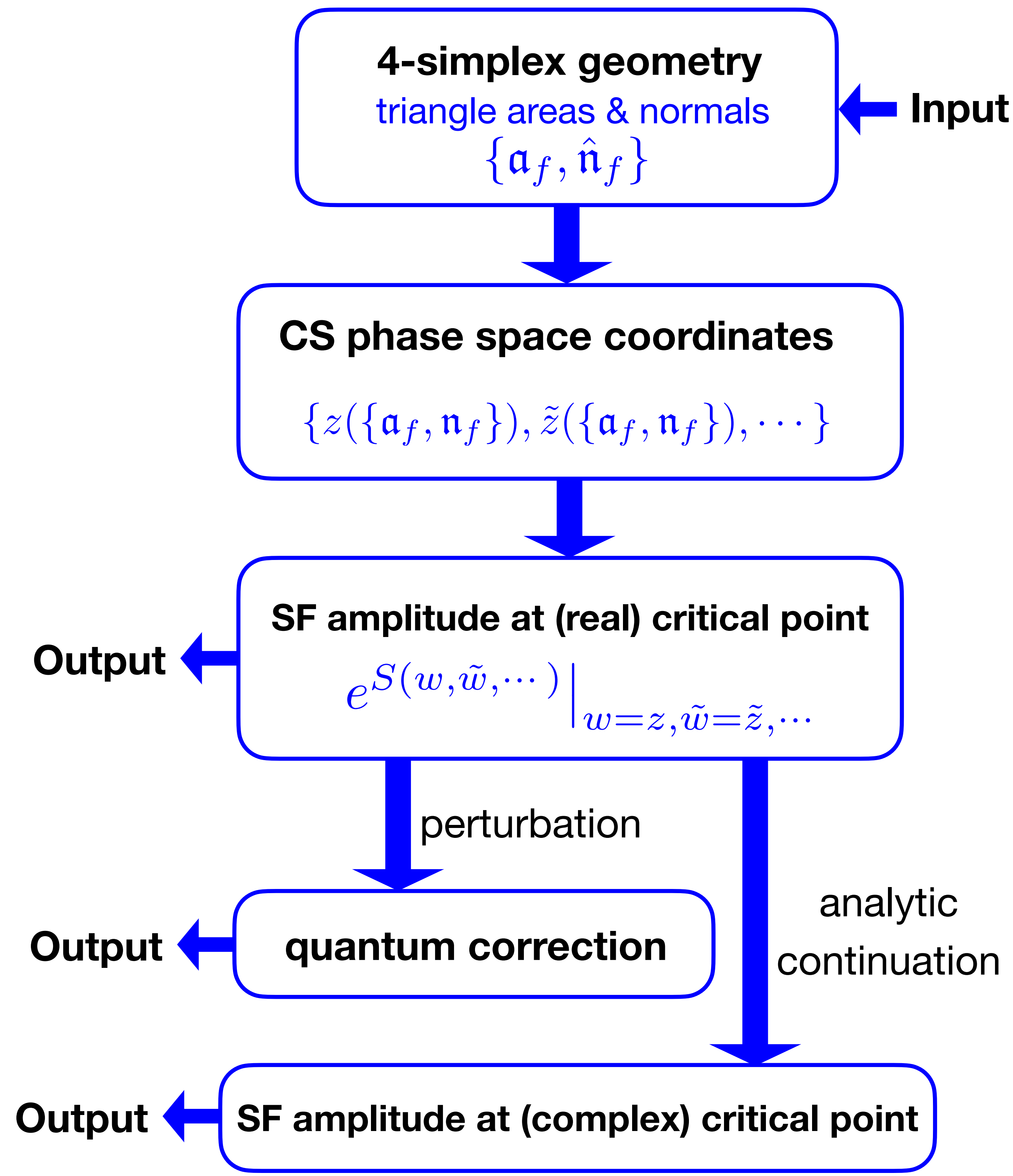


[Han, QP, to appear]

Stationary phase approximation of SF amplitude — cont.

- It is a **concrete, complete and computable** program to calculate the SF amplitude evaluated at the critical point and its quantum perturbation. [r.f. Dongxue's talk for EPRL program]

- Can be easily generalized to **4-complex** geometry and compute for SF amplitude for a general 4-complex.



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Outlook — Can we do better?

- Generalize the model to include **timelike tetrahedra**, as in the Conrady-Hnybida extension of EPRL
  [Conrady, Hnybida ’11, Han, Liu ’18]

- A **quantum group** representation of the SF model?
  - Clue 1: combinatorial quant. of CS theory $\longrightarrow$ quantum group rep. [Alekseev, Grosse, Schomerus ’94-95, Buffenoir, Noui, Roche ’02]
  - Clue 2: Turaev-Viro model
  - Clue 3: quantum state of constantly curved tetrahedron = $\mathfrak{q}$-deformed intertwiner [Han, Hsiao, QP ’23] [*r.f.* Chen-Hung’s talk]

- It is ready to set up the **numerical development** for this SF model, as is done with the EPRL model
  - Realize numerically the real and complex critical points;
  - Consider the higher-order quantum corrections;
  - Application to cosmology;
  - Etc.

Can we do **even** better?

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$$\int \mathcal{D}g_{\mu\nu}\, e^{iS_{\text{EH}}}$$

The SF model with Λ is good 4D QG formalism because
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  - Consider a field $\Psi(j,\iota) : \mathcal{M}_{\text{flat}}(\Sigma_{0,4}, \text{SU}(2)) \to \mathbb{C}$ (j,ι) : configuration of a tetrahedron

[Han, QP, W.I.P.]

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$$S[\Psi] = K[\Psi] + V[\Psi] + c.c.$$

kinetic: $K[\Psi] = \sum_{\{j\}\in(\mathbb{Z}/k\mathbb{Z})^4} \int [\text{d}\iota]\ \Psi(j,\iota^*)\Psi(j,\iota)$

potential: $V[\Psi] = \frac{g}{5!} \sum_{\{j_{ab}\}_{a<b}} \prod_{a=1}^{5} \int [\text{d}\iota_a]\ \mathcal{A}_v(\{j_{ab},\iota_a\}) \prod_{a=1}^{5} \Psi(\{j_{ab}\},\iota_a)$

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    (2) triangulation independent

[Han, QP, W.I.P.]

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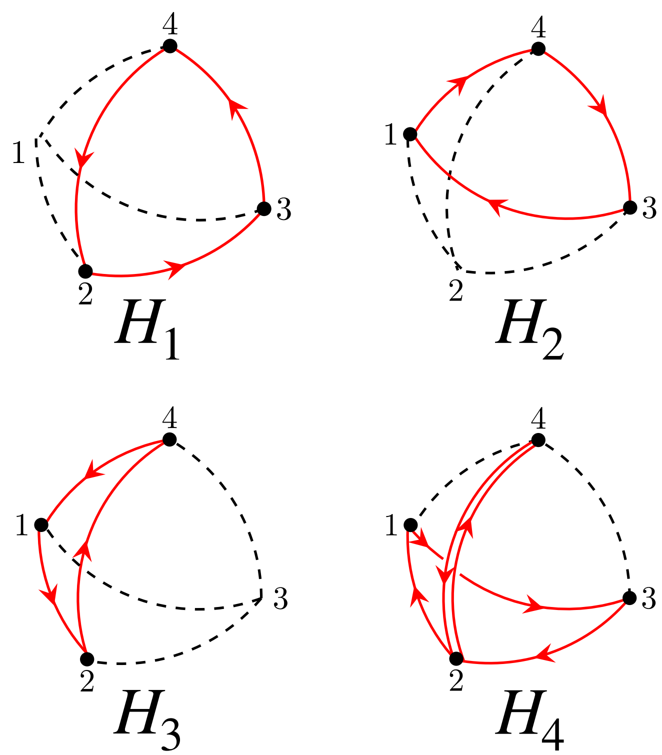
Thank you for your attention!

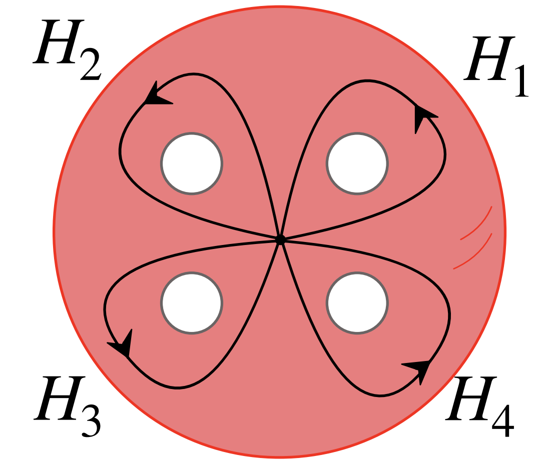
How does a 3D theory describe 4D geometry?



$$\pi_1(\text{sk}_1(\text{ 3-simplex })) \xrightarrow{\text{isomorphism}} \pi_1(\ S^2\backslash 4 \text{ points }\ )$$

$$\omega_{\text{LC}} \searrow \qquad\qquad \swarrow \omega_{\text{flat}}$$

$$\{H_1, H_2, H_3, H_4 \in \text{SU}(2) | H_4 H_3 H_2 H_1 = \mathbb{I}_{\text{SU}(2)}\}/\text{SU}(2)$$



3D geometries are encoded in holonomies on 2D surface: $H_f(\mathfrak{a}_f, \hat{\mathfrak{n}}_f) = e^{\frac{\Lambda}{3}\mathfrak{a}_f \hat{\mathfrak{n}}_f \cdot \vec{\tau}}, \quad \forall f = 1, \cdots, 4$

[Haggard, Han, Riello '16]

One dimension higher:

$$\pi_1(\text{sk}_1(\text{4-simplex})) \xrightarrow{\text{isomorphism}} \pi_1(S^3\backslash\Gamma_5)$$

$$\omega_{\text{LC}} \searrow \qquad\qquad \swarrow \omega_{\text{flat}}$$

$$\{\{\widetilde{H}_{ab}\} \in \text{SL}(2,\mathbb{C}) | \text{closure conditions}\}/\text{SL}(2,\mathbb{C})$$

4D geometries are encoded in holonomies on 3D manifold

[Haggard, Han, Kaminski, Riello '15, Han '15]

Critical deficit angle — compare to EPRL

- A technical advancement in this spinfoam model compared to EPRL SF

Poisson summation

$$\mathcal{Z}_{\text{EPRL}} = \sum_{\{j_f\}\in\mathbb{N}/2} \prod_f \mathcal{A}(j_f) \int \mathrm{d}\mu(X)\, e^{\sum_f j_f F_f(X)} = \sum_{\{u_f\}\in\mathbb{Z}} \int \prod_f \mathcal{A}(j_f)\mathrm{d}(2j_f) \int \mathrm{d}\mu(X)\, e^{\sum_f j_f (F_f(X)+4\pi i u_f)}$$

$$\mathcal{Z}_\Lambda = \int \prod_f \mathcal{A}(j_f)\mathrm{d}(2j_f) \int \mathrm{d}\mu(X)\, e^{S(\{j_f\},X)+\sum_f 4\pi i u_f j_f}\,, \qquad u_f \in \mathbb{Z} \text{ fixed } \forall\, f$$

[Han, QP ’24]