Towards Covariant LQG 2.0

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In collaboration with H. Chen, M. Dupuis, O. Hrytseniak, T. Oliveira Ferreira, C. Pollack, A. Riello, P. Tsimiklis.

Towards covariant LQG 2.0

Under the influence (of topological models),
making the case for using higher structures in LQG

What are they? Definition through examples:
Revisiting Stokes theorem
4d BF as 2-gauge theory
Some elements of representation theory

Comments and a Road map

Making the case for using higher structures in LQG

Topological 4d BF theory $\mathscr{A} \in \mathfrak{g} \otimes \Lambda^1 M$

$$\int \mathscr{B} \wedge \mathscr{F}(\mathscr{A}) - \frac{1}{2}\mathscr{B} \wedge t(\mathscr{B})$$

t-map is "boundary map"
Lie algebra homo
$t : \mathfrak{g}^* \to \mathfrak{g}$

2-form is related to a connection data $\mathfrak{A}$
"Boundary data"

$$t(\mathscr{B}) = \mathscr{F}$$

$$d_{\mathscr{A}}\mathscr{B} = 0$$

Well known fact: 4d gravity and BF thy share structural features.

Making the case for using higher structures in LQG

Einstein-Cartan(-Holst) gravity formulation

$$\int (\star e \wedge e) \wedge (\mathscr{F} + \frac{\lambda}{4} e \wedge e) + \gamma e \wedge e \wedge \mathscr{F}$$

$$\star\, e \wedge (\mathscr{F} + \lambda e \wedge e) = 0 \qquad \text{Einstein eq}$$

many other formulations rely on extending the number of variables, and use a 2 form B

Making the case for using higher structures in LQG

Plebanski $\mathscr{A} \in \mathfrak{so}(3,1) \otimes \Lambda^1 M$ $\mathcal{S}_{Pl} = \int \mathscr{B} \wedge \mathscr{F}(\mathscr{A}) - \frac{1}{2}\lambda \mathscr{B} \wedge \mathscr{B} - \phi(\mathscr{B}) \wedge \mathscr{B}$

$$T_{Pl}(\mathscr{B}) = (\lambda \mathrm{id} + \phi)\mathscr{B} = \mathscr{F}$$

Cf review by Freidel-Speziale

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Gives Immirzi

$\alpha \propto G\lambda \sim 10^{-120}$

Freidel-Starodubtsev
(MacDowell-Mansouri) $\mathscr{A} = (\omega, e) \in \mathfrak{so}(4,1) \otimes \Lambda^1 M$

$$\int \mathscr{B} \wedge \mathscr{F}(\mathscr{A}) - \frac{1}{2}\beta \mathscr{B} \wedge \mathscr{B} - \frac{\alpha}{4}\epsilon^{4IJKL}\mathscr{B}_{IJ} \wedge \mathscr{B}_{KL}$$

$$T_{MM}(\mathscr{B}) = (\beta \mathrm{id} + \frac{\alpha}{2}\epsilon)\mathscr{B} = \mathscr{F}$$

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Mikovic-Vojinovic $\mathscr{A} = (\omega, e) \in \mathfrak{iso}(3,1) \otimes \Lambda^1 M$

$$\mathscr{S}_{MV} = \int \mathscr{B} \wedge \mathscr{F}(\mathscr{A}) + \tilde{\phi}(\mathscr{B}) \wedge \mathscr{B}$$

$$T_{MV}(\mathscr{B}) = \tilde{\phi}\mathscr{B} = \mathscr{F}$$

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Herffray-Krasnov $\mathscr{A} \in \mathfrak{so}(3,\mathbb{C}) \otimes \Lambda^1 M$

$$S_{HK} = \int \mathscr{B} \wedge \mathscr{F} - \frac{\lambda}{2}\mathscr{B} \wedge \mathscr{B} + \frac{\alpha}{2}\left(Tr(\sqrt{\mathscr{B}^I \wedge \mathscr{B}^J})\right)^2$$

$$T^I_{HK}(\mathscr{B}) = \left(\lambda\delta^{IJ} - \alpha Tr(\sqrt{X})\,(X^{-1})^{IJ}\right)\mathscr{B}_J = \mathscr{F}^I, \quad X^{IJ} = \mathscr{B}^I \wedge \mathscr{B}^J$$

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2-form is related to a connection data $\mathfrak{A}$
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$$t(\mathscr{B}) = \mathscr{F}$$

$$\downarrow$$

$$T_{grav}(\mathscr{B}) = \mathscr{F}$$

$$d_{\mathscr{A}}\mathscr{B} = 0$$

$\downarrow$

Gravity

Well known fact: 4d gravity and BF thy share structural features.

Spinfoam model: consider the topological theory and tweak its state-sum to reproduce gravity.

Making the case for using higher structures in LQG

It is only a conjecture that the Crane-Yetter state sum (q-deformed 15j symbol) corresponds to the partition function of BF + BB action.

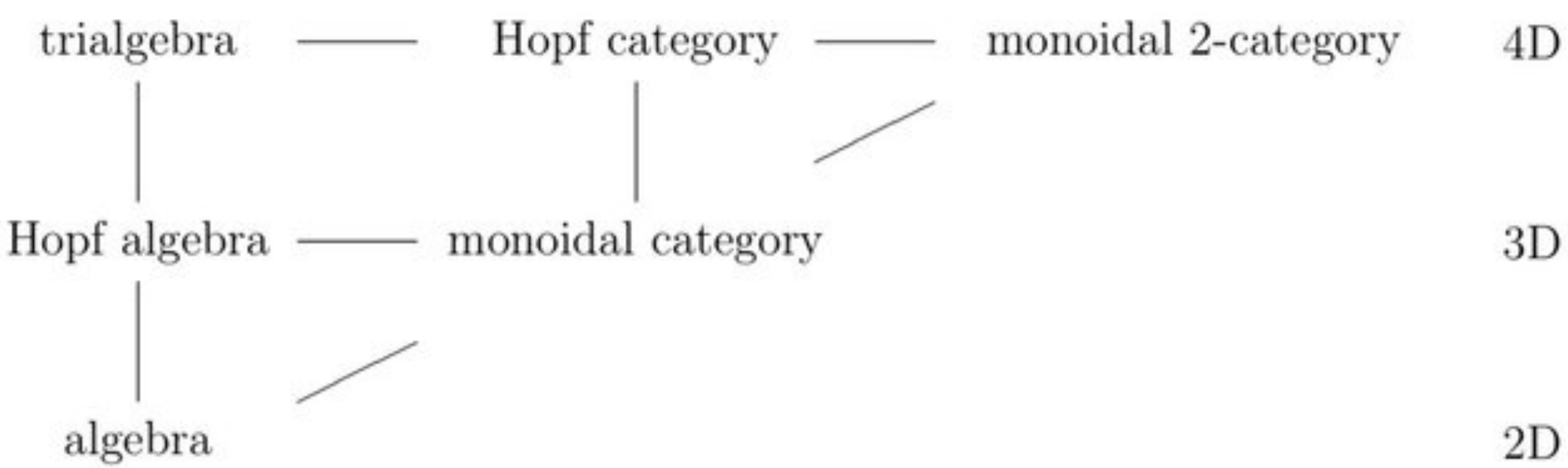
There is some evidence [4] that BF theory with nonzero cosmological constant can be quantized to obtain the so-called Crane–Yetter model [35, 37], which is a spin foam model based on the category of representations of the quantum group associated to G. Indeed, in some circles this is taken almost as an article of faith. But a rigorous argument, or even a fully convincing argument, seems to be missing. So, this issue deserves more study.

An invitation to higher gauge theory (2010)
Baez, Huerta

Making the case for using higher structures in LQG

Spinfoam model: consider the topological theory and tweak its state-sum to reproduce gravity.

Crane-Frenkel categorical/dimensional ladder proposal to characterize **topological models**



Pic from Hank Chen's thesis

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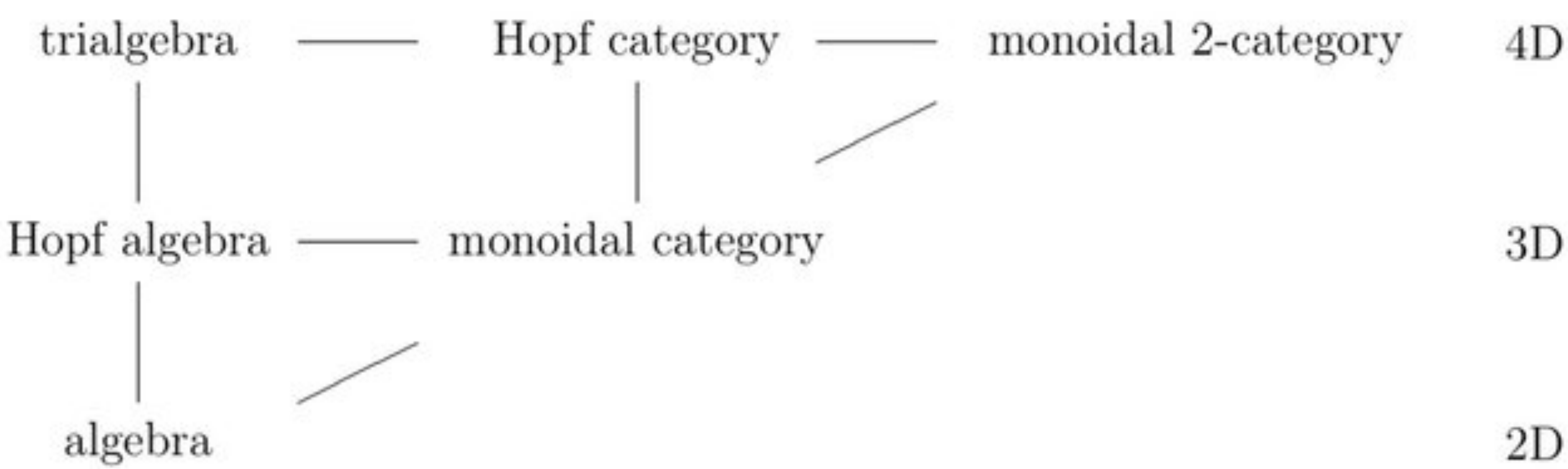

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State-sum model for monoidal 2-category has been constructed but for **finite** 2-groups.

(Baratin-Freidel-Korepanov 2-state-sum for 2-Poincare gp)

Cui
Douglas Reutter

In 4d, higher symmetries are symmetries of the topological theory.

From a spinfoam perspective, we should be using 2-symmetries/2-state sum

Making the case for using higher structures in LQG

Consider **matter fields Φ** with no spin on a 4d spacetime, which geometry is given by **frame field** e and **spin connection** A.

- We assume **no curvature** and **no torsion**: $F(A) = 0, \quad T(e, A) = d_A e = 0$

$$\mathscr{L}(e, A, \Phi) \approx \mathscr{L}(e, A, \Phi) + \mathfrak{B} \wedge \mathfrak{F}$$

$$\mathfrak{F} = (F(A), T(e, A)) \in \mathfrak{iso}(3,1) \otimes \Lambda^2 M, \quad \mathfrak{B} = (B, \Sigma) \in \mathfrak{iso}^*(3,1) \otimes \Lambda^2 M$$

4d topological theory is naturally present in 4d, hence 2-symmetries according to the categorical/ dimensional ladder.

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Conversely, given Feynman diagrams (scalar field in Minkowski), one can recover 2-symmetries!!

Feynman diagrams (scalar field in Minkowski) = particle excitations in a 2-state-sum

Baratin-Freidel

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$$\mathscr{L}(e,A,\Phi)\approx\mathscr{L}(e,A,\Phi)+\mathfrak{B}\wedge\mathfrak{F}$$

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4d topological theory is naturally present in 4d, hence 2-symmetries according to the categorical/ dimensional ladder.

We assume **constant curvature** and **no torsion**: $F(A)=\frac{\lambda}{2}e\wedge e, \quad T(e,A)=d_A e=0$

$$\mathfrak{F}=(F(A)-\frac{\lambda}{2}e\wedge e,T(e,A))\in\mathfrak{so}(4,1)\otimes\Lambda^2, \quad \mathfrak{B}=(B,\Sigma)\in\mathfrak{so}^*(4,1)\otimes\Lambda^2$$

In order to add gravity, add a symmetry breaking term, to recover the MacDowell-Mansouri action

$$\mathcal{S}_{MM}=\int\mathfrak{B}\wedge\mathfrak{F}-\frac{1}{2}\beta\mathfrak{B}\wedge\mathfrak{B}-\frac{\alpha}{4}\epsilon^{4IJKL}\mathfrak{B}_{IJ}\wedge\mathfrak{B}_{KL}$$

Under the influence (of topological models), Making the case for using higher structures in LQG

recap

Formulations of gravity in the terms of B field have some “holographic features”

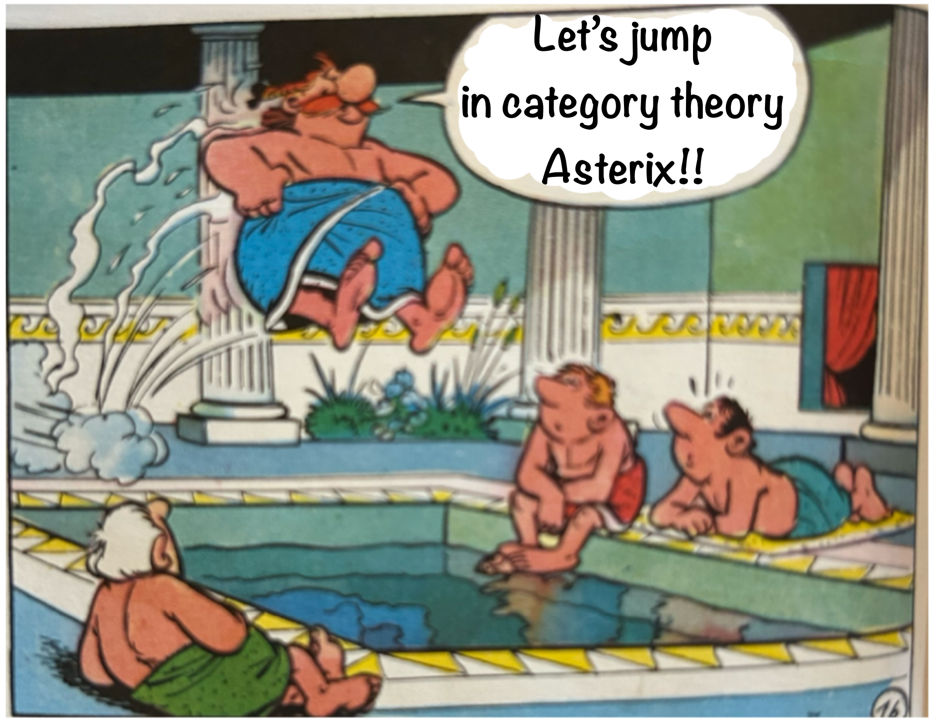
$$T(\mathscr{B}) = \mathscr{F}$$

(Some involve the frame field as part of the connection data (similar to 3d))

2-symmetries are hidden in Feynman diagrams.

2-structures should be present in spinfoams due to the nature of topological models, **but are not fully leveraged**

What are those 2-structures?



Asterix gladiator

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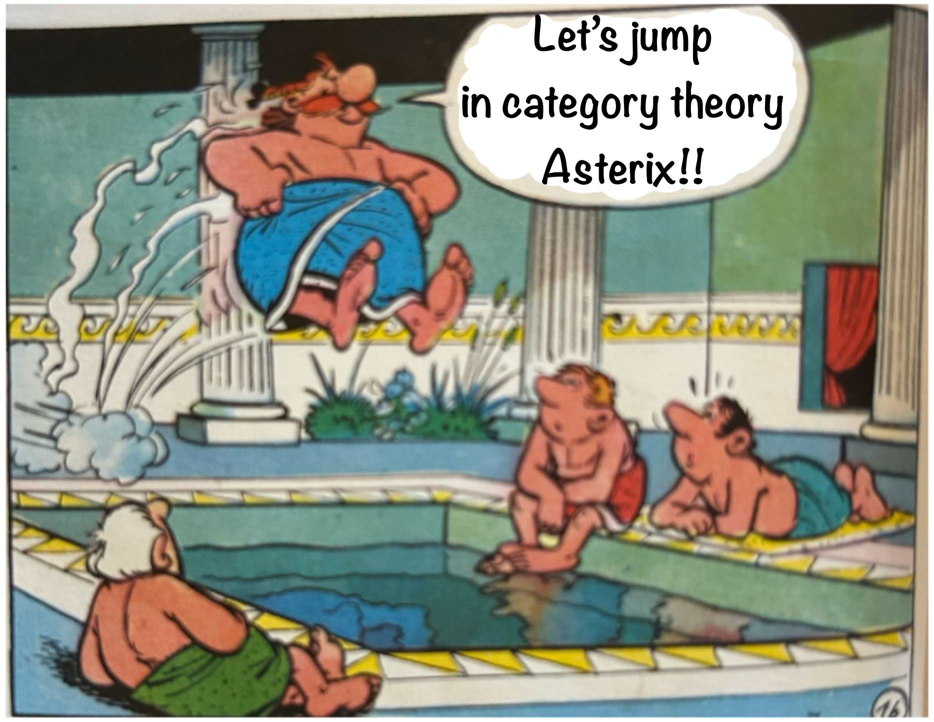

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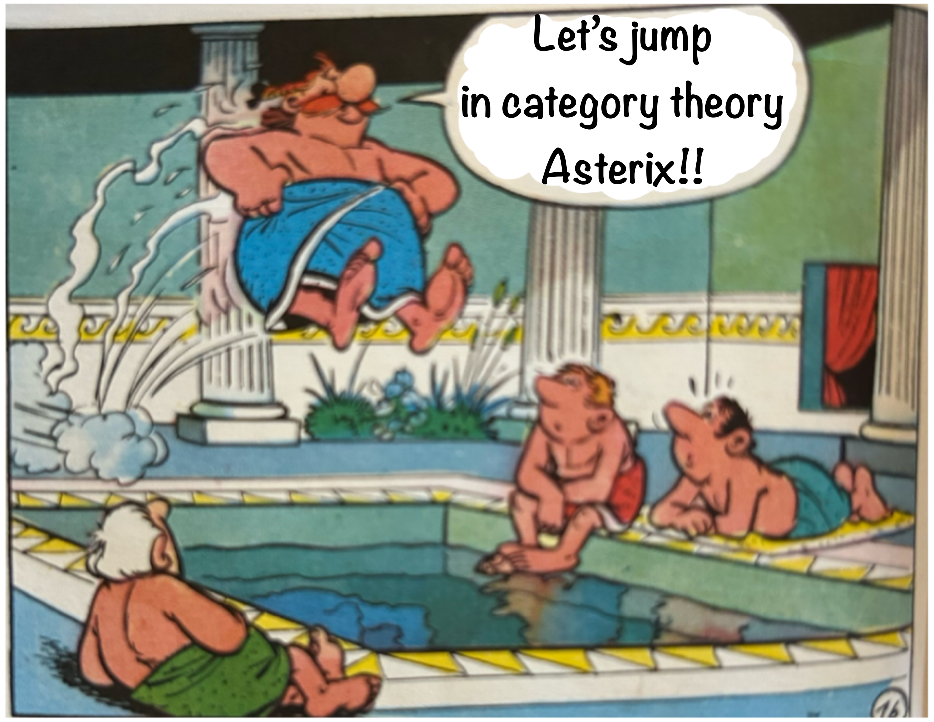

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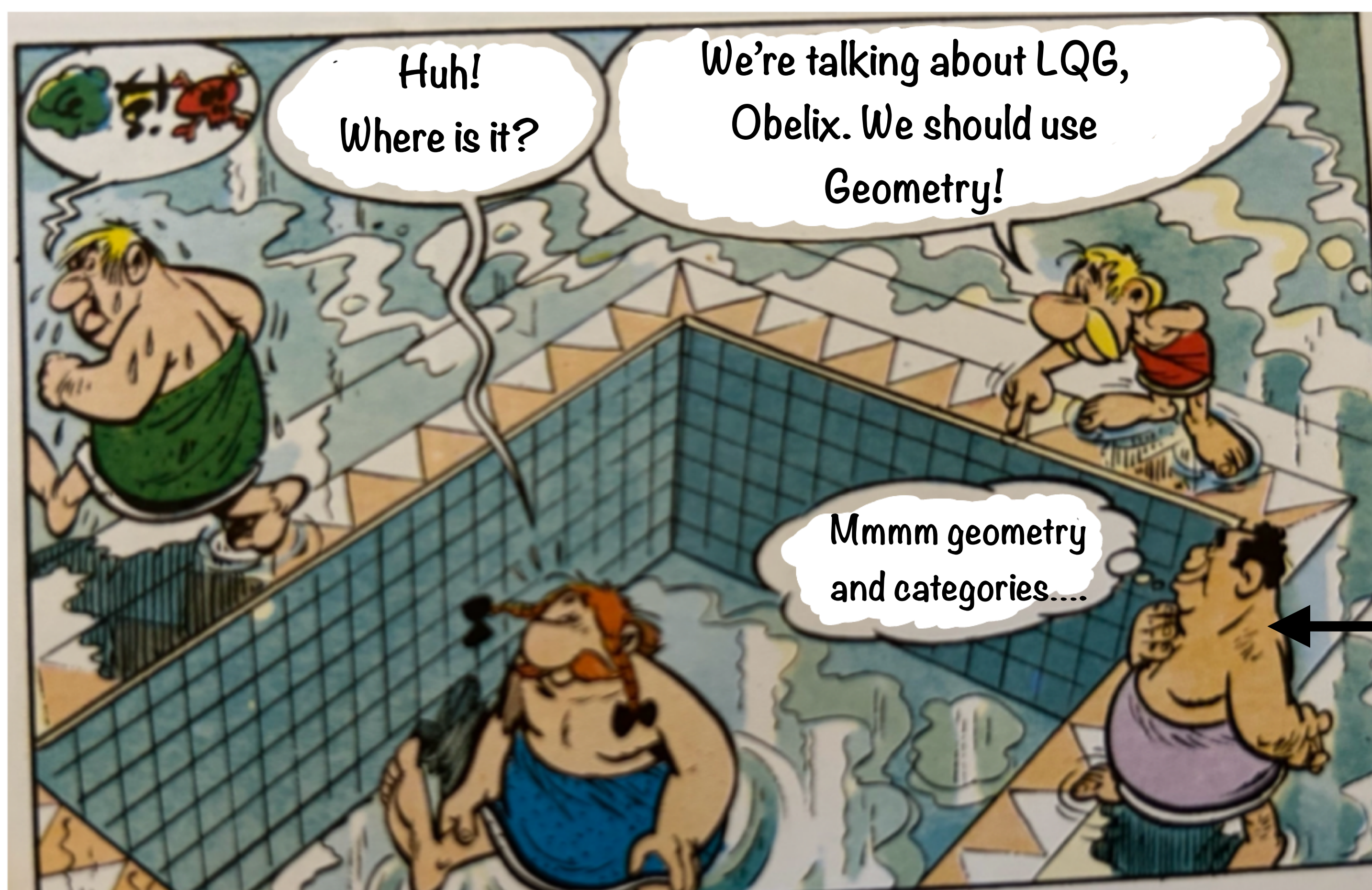
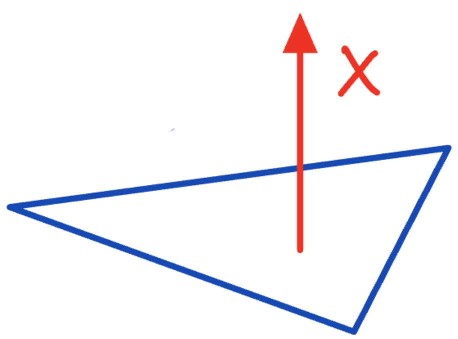


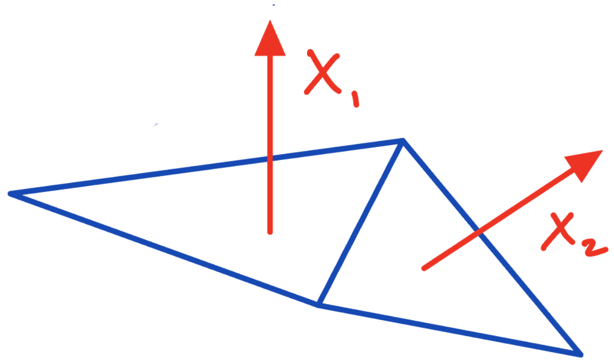
Johnus Baezus Categoricus

2-vector space, Lie 2-algebras and Lie 2-groups

Revisiting Stokes theorem

Consider the discretized flux. $X \in \mathbb{R}^3$ encoding the normal of the face S.



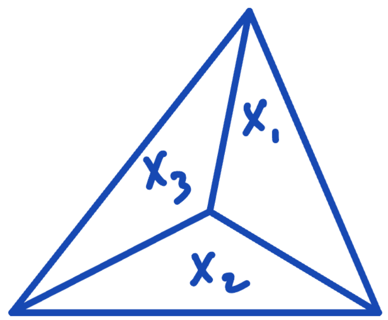


We deal with a decorated surface by a vector. Composing/combining surface means adding the normals.

We need to be able to compose decorated surfaces: Gauss law.

$$X_1 + X_2 + X_3 + X_4 = 0$$

Constrained spinning tops



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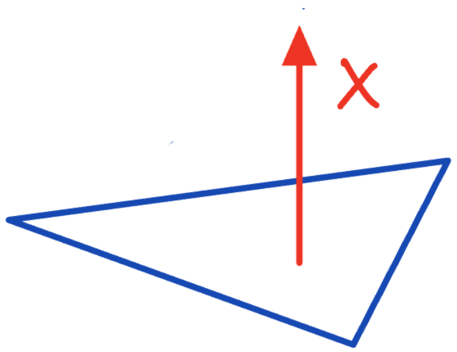

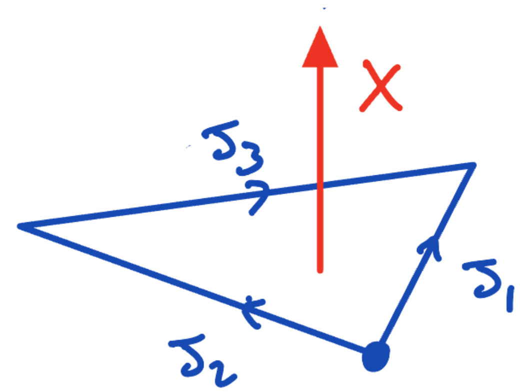

Flux, as a simple bivector, can be discretized

$$X = \int_S dz \times dz = \int_S d(z \times dz) = \int_{\partial S} (z \times dz) = \sum_e \int_{e \in \partial S} (z \times dz) = \sum J_e$$

This appeared both in *Fairbain-Perez* to discuss string like defects

and

in *Freidel-Ziprick* to discuss twisted/spinning geometries

Can replace the **face information** by the **edge information**

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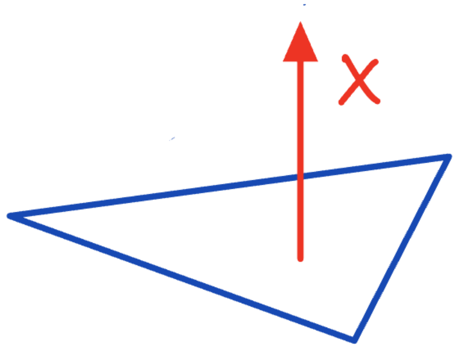

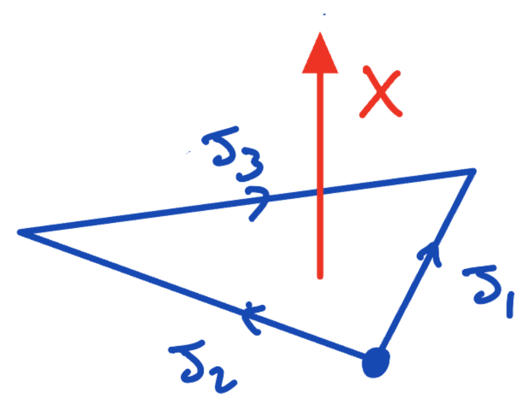

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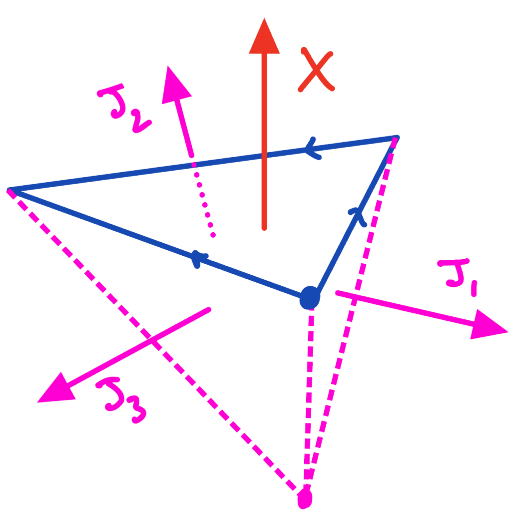
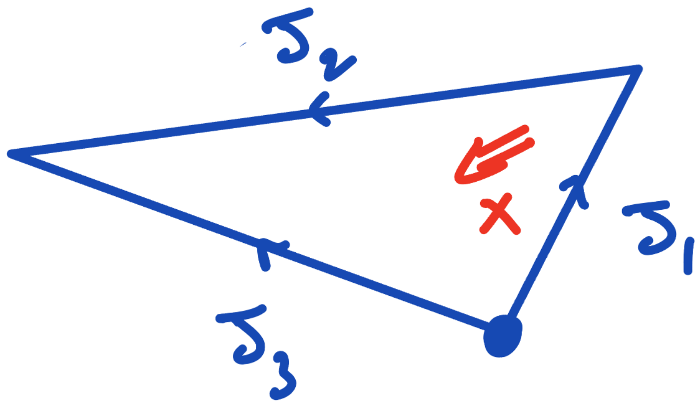
Note that J_e is associated to edge ***e*** but is **not** the edge vector.
It can be seen as the normal of a face in a ghost tetrahedron.
Cf **Haggard-Han-Riello**



Revisiting Stokes theorem



$$X + J_1 + J_2 = J_3$$

Reinterpret this equation: face info relates some “edge” info

$tX + J_1 + J_2 = J_3$, ***t* is “boundary map”** for the face information

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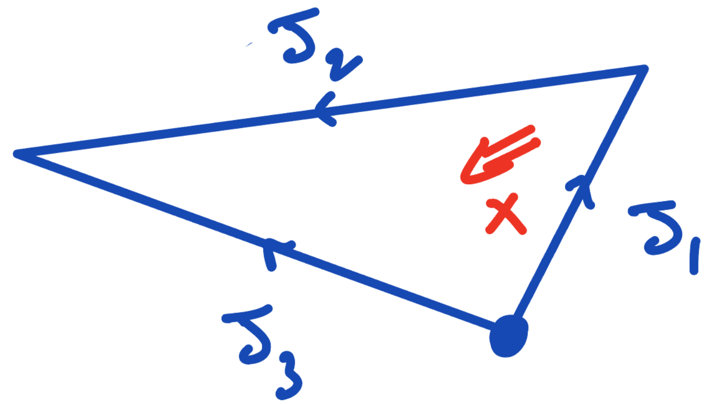

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This provides an example of **2-vector space (of the Baez Crans type)**.

Such 2-vector space is given in terms of a pair of vector spaces, with a linear map $t : W \to V$

W and t encode the transformations on V $\quad v \to v + tw$

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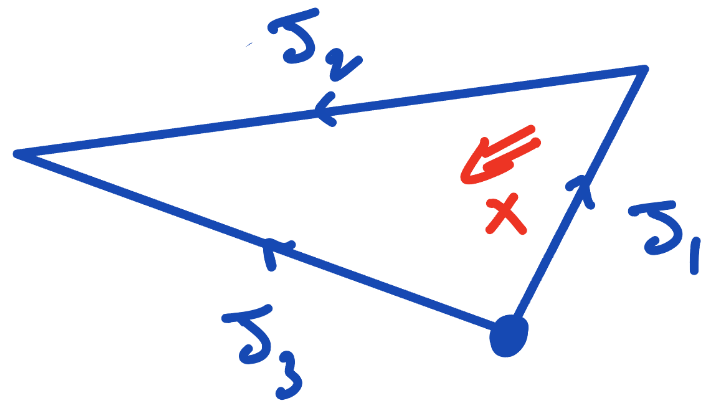

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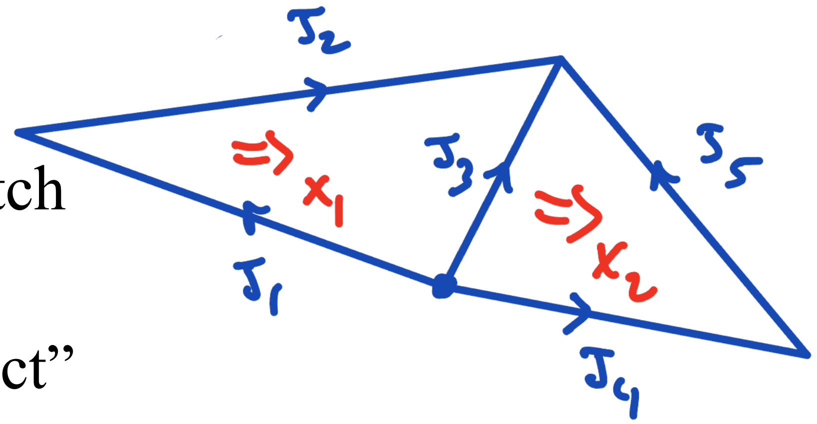
Other way to formulate, if ***v* is the source** then $v' = v + tw$ **is the target** of the transformation w

$$V \oplus W \rightrightarrows V$$
$$s(J_1 + J_2, X) = J_1 + J_2$$
$$\tau(J_1 + J_2, X) = tX + J_1 + J_2$$
$$\tau - s = tX$$

can compose/add 2-vectors,
provide the source of second match
the target of first

"groupoid product"



2-vector space cane be made a Lie 2-algebra, with a 2-bracket.

Replace the vector spaces by Lie algebras, with an action of $\mathfrak{g}$ on $\mathfrak{h}$

$$V \rightsquigarrow \mathfrak{g}, \quad W \rightsquigarrow \mathfrak{h}, \quad V \oplus W \rightsquigarrow \mathfrak{g} \ltimes \mathfrak{h}$$

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Lie 2-algebra structure $\quad \mathfrak{g} \ltimes \mathfrak{h} \rightrightarrows \mathfrak{g}$

- Lie algebras $\mathfrak{g}$, $\mathfrak{h}$
- Source map: $s(J, X) = J$
- Target map: $\tau(J, X) = tX + J$
- Boundary map: $t(X) = \tau(J, X) - s(J, X)$

Lie 2-bracket:
(Satisfies 2-Jacobi)

$$\begin{aligned}[(J_1, X_1); (J_2, X_2)]_2 &= ([J_1; J_2]\,, \tau(J_1, X_1) \triangleright X_2 - s(J_2, X_2) \triangleright X_1) \\ &= ([J_1; J_2]\,, [X_1; X_2] + J_1 \triangleright X_2 - J_2 \triangleright X_1)\end{aligned}$$

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Source and target maps are compatible with brackets

$$s[(J_1, X_1); (J_2, X_2)]_2 = [s(J_1, X_1); s(J_2, X_2)]$$

$$\tau[(J_1, X_1); (J_2, X_2)]_2 = [\tau(J_1, X_1); \tau(J_2, X_2)]$$

Spinning geometry example: $\quad \mathfrak{su}(2) \ltimes \mathfrak{su}(2) \rightrightarrows \mathfrak{su}(2)$

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Lie 2-algebras space cane be integrated: Lie 2-groups

Convenient to see 2-holonomies (=decorated surfaces) as maps between 1-holonomies (decorated paths).

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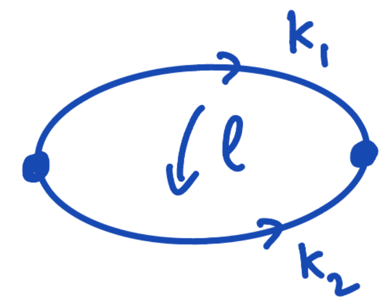
Lie 2-group structure $G \ltimes H \rightrightarrows G$



- Lie group $G \ni k_i$ decorates edges,
- Lie group $H \ni \ell$ decorates faces.
- Source map: $s(k, \ell) = k$
- Target map: $\tau(k, \ell) = t(\ell)k$
- Boundary map: $t(\ell) = \tau(k, \ell)s^{-1}(k, \ell)$

t-map is group homomorphism

$t(k \triangleright \ell) = k\, t(\ell)\, k^{-1}, \quad (t(\ell)) \triangleright \ell' = \ell\, \ell'\, \ell^{-1}$

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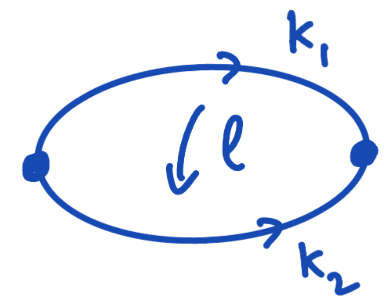


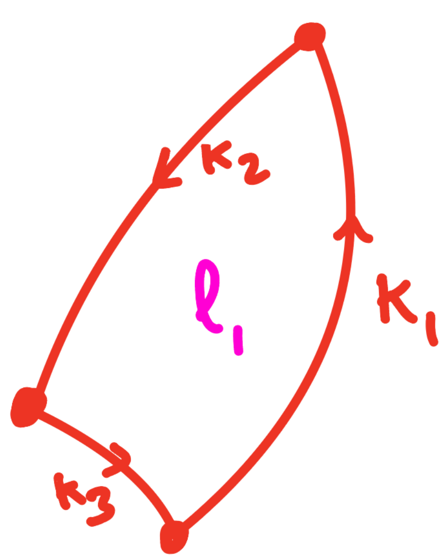
- Lie group $G \ni k_i$ decorates edges,
- Lie group $H \ni \ell$ decorates faces.
- Source map: $s(k, \ell) = k$
- Target map: $\tau(k, \ell) = t(\ell)k$
- Boundary map: $t(\ell) = \tau(k, \ell)s^{-1}(k, \ell)$

t-map is group homomorphism

$$t(k \triangleright \ell) = k\, t(\ell)\, k^{-1}, \quad (t(\ell)) \triangleright \ell' = \ell\, \ell'\, \ell^{-1}$$

$\ell \in H, \quad k_i \in G$

$$t(\ell_1) = k_2 k_3 k_1^{-1}$$

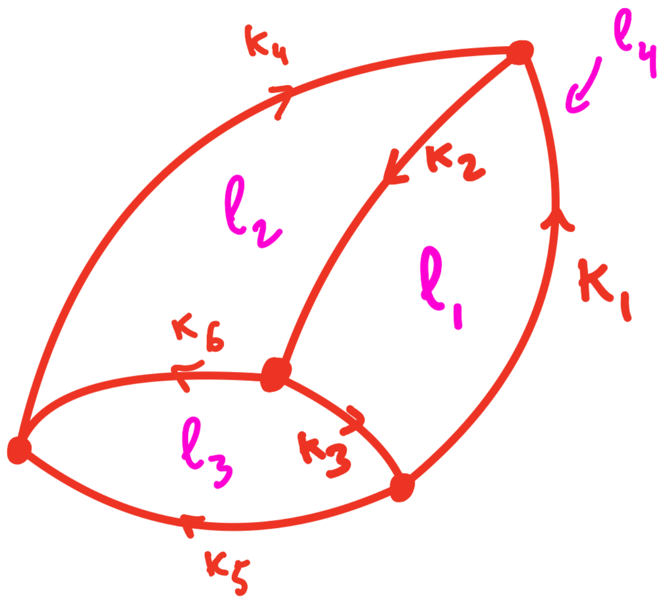


Take $t = id$, it's like implementing Stokes thm!

$\ell \in SU(2), \quad k_i \in SU(2)$ $\qquad \ell = k_3 k_1^{-1} k_2^{-1}$ $\qquad$ **"Fake-flatness"**

$\ell = Pexp^{\oint A} = k$

What are those 2-structures?



Curved tetrahedra

Take $t = id$, it's like implementing Stokes thm!

$\ell \in SU(2), \quad k_i \in SU(2)$ $\quad \ell_1 = k_2 k_3 k_1^{-1}$ **"Fake-flatness"**

Gauss constraint = Bianchi id

$$\ell_1 \ell_2 \ell_3 (k_3 \triangleright \ell_4) = 1$$

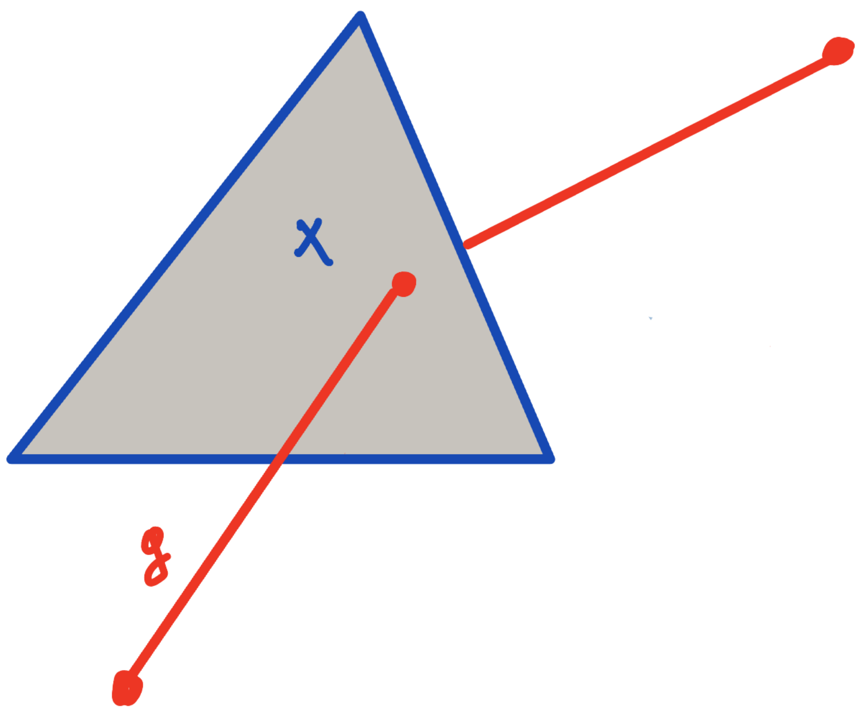
Can replace the face information by the edge information by reducing the fake flatness condition.

Underlying motivation/approach to different works to define curved tetrahedron

Haggard-Han-Riello
Charles-Livine
Han-Hsiao-Pan

What are those 2-structures?

Take $t = id$, it's like implementing Stokes thm!



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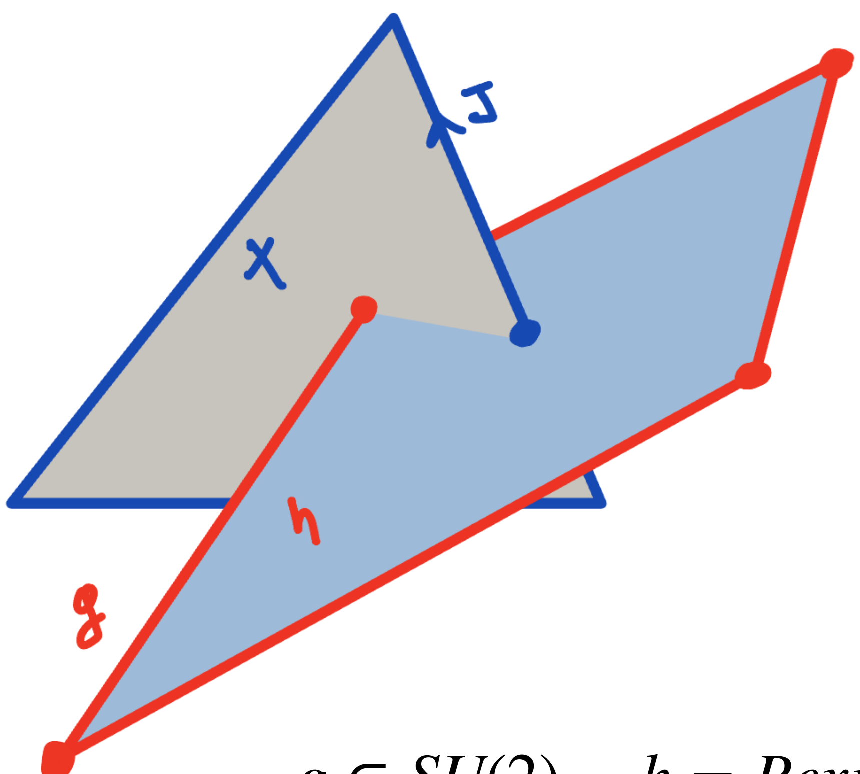


$$X \in \mathbb{R}^3, \quad J_i \in \mathbb{R}^3 \qquad X_1 = J_3 - J_2 - J_1$$

$$X = \int_S dz \times dz = \int_S d(z \times dz) = \int_{\partial S} (z \times dz) = \sum_e \int_{e \in \partial S} (z \times dz) = \sum J_e$$

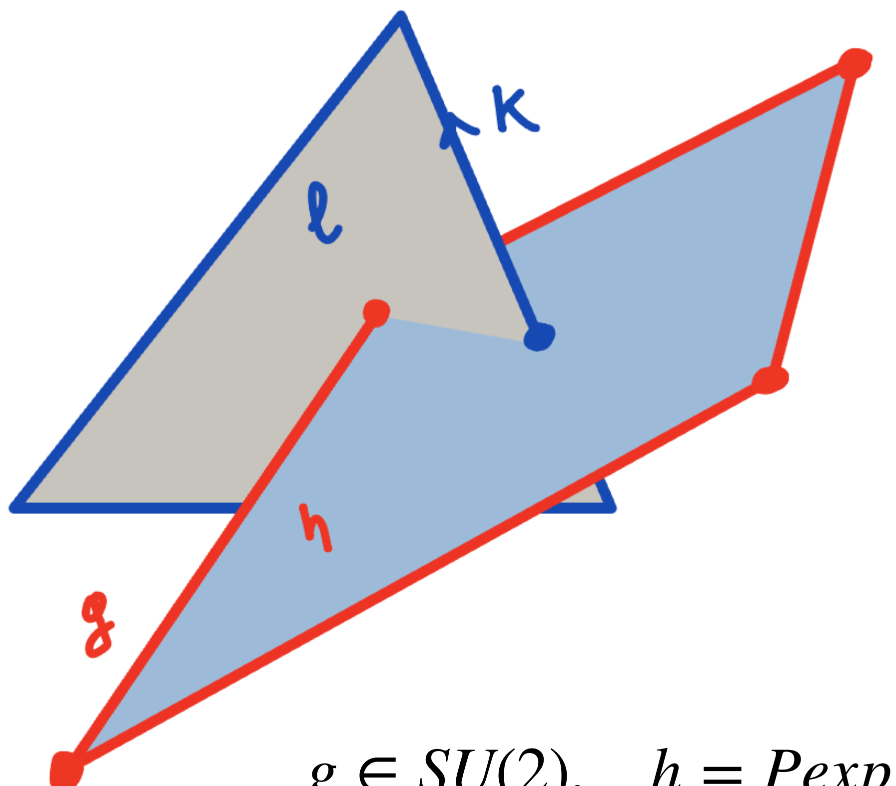
$$g \in SU(2), \quad h = Pexp^{\oint A} \in SU(2)$$

What is $T^*\mathbb{G}$? **WIP Girelli-Oliveira-Riello**

Ready to generalize to the curved case

What are those 2-structures?

Take $t = id$, it's like implementing Stokes thm!



$\ell \in AN_2, \quad k \in AN_2$

$\ell \in SU(2), \quad k \in SU(2)$

$g \in SU(2), \quad h = Pexp^{\oint A} \in SU(2)$

2-gauge theory and BF theories

Symmetries of 4d BF theory, aka 4d Chern-Simons thy

Higher gauge thy for Lie 2-algebra $\mathfrak{g} \ltimes \mathfrak{h} \rightrightarrows \mathfrak{g}$

Jurco, Raspollini, Saemann, Wolf

Connection $\mathscr{A} = (A, \Sigma) \in (\mathfrak{g} \otimes \Lambda^1 M) \oplus (\mathfrak{h} \otimes \Lambda^2 M)$

Curvature data, $\mathscr{F} = (F - t\Sigma, \quad d_A\Sigma = d\Sigma + A \triangleright \Sigma)\,.$

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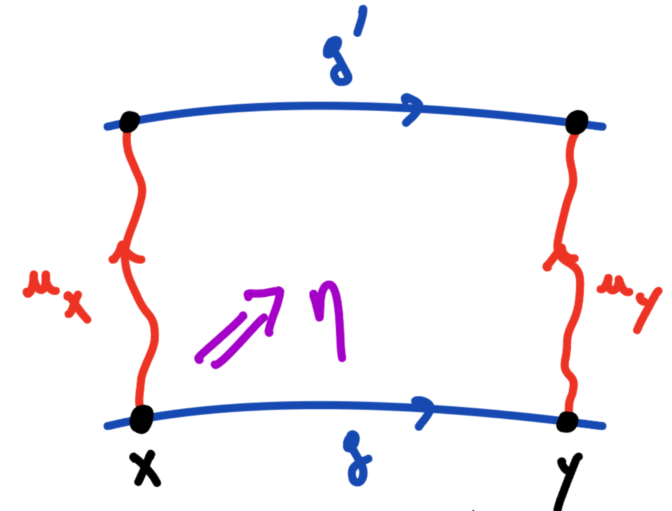

$g' = u_x^{-1}\, t(\eta)\, g\, u_y$

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1- and 2-gauge transf parameterized by
$(\alpha, \phi) \in \mathfrak{g} \otimes \Lambda^0 M \oplus \mathfrak{h} \otimes \Lambda^1 M$,

$$\delta_{(\alpha,\phi)}\mathscr{A} = d(\alpha,\phi) + [\mathscr{A}, (\alpha,\phi)]_2 = (d\alpha + [A,\alpha], \alpha \triangleright \Sigma + d_A\phi)$$

$$\delta_{(\alpha,\phi)}\mathscr{F} = [\mathscr{F}, (\alpha,\phi)]_2 = ([F - t\Sigma, \alpha], \alpha \triangleright d_A\Sigma + (F - t\Sigma) \triangleright \phi]$$

2-co-adjoint action!

Symmetries of 4d BF theory, aka 4d Chern-Simons thy

Construct a 4d action to implement flat 2-curvature $\mathscr{F} = 0$, a “2-BF” theory

This is called *4d Chern-Simons thy* by some.

$$\langle \mathscr{B} \wedge \mathscr{F} \rangle = \langle B \wedge (F - t\Sigma) \rangle + \langle C \wedge d_A \Sigma \rangle \, .$$

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Lagrange multipliers
$\mathscr{B} = (C, B) \in (\mathfrak{h}^* \otimes \Lambda^1 M) \oplus (\mathfrak{g}^* \otimes \Lambda^2 M)$,
which also transform under the 2-adjoint action,

Use an invariant bilinear form $\langle , \rangle$
invariant under the 2-(co)adjoint action.

$$\langle [\chi_1, \chi_2], \chi_3 \rangle = - \langle \chi_2, [\chi_1, \chi_3] \rangle, \quad \chi_{1,3} \in Lie\mathbb{G}, \chi_2 \in Lie\mathbb{G}^*$$

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Canonical variables

$\theta = B \wedge \delta A + C \wedge \delta \Sigma$ $\qquad (B, C) \leftrightarrow (A, \Sigma)$ $\qquad$ 1-2-connections dual to each other.

There is a dual *2-symmetry*, for the $\mathscr{B}$ sector, due to *2-Bianchi identity*.

Natural symmetry given by a matched pair of Lie 2-algebras, $Lie\mathbb{G}^* \bowtie Lie\mathbb{G}$

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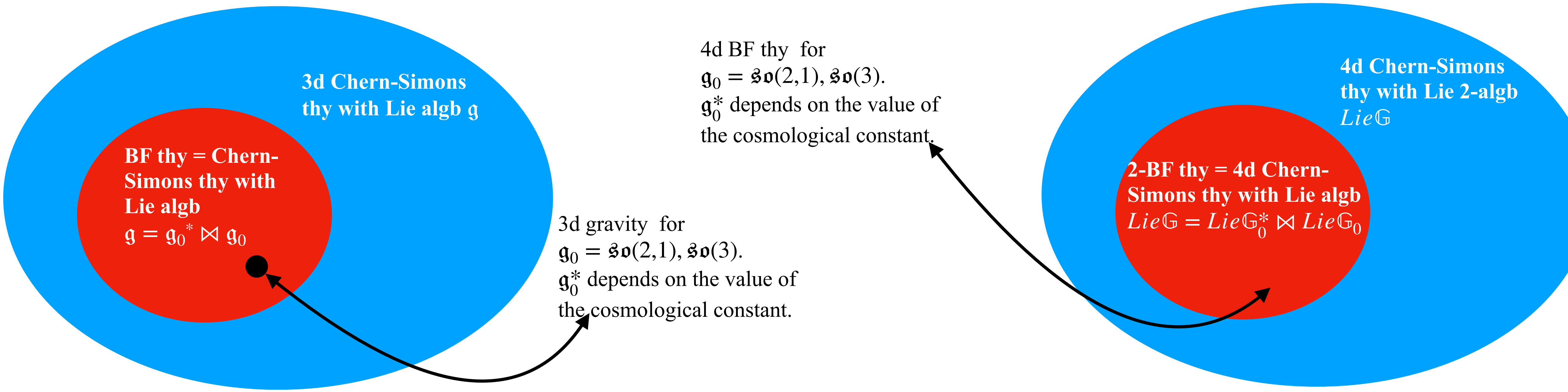
$$\langle \mathscr{B} \wedge \mathscr{F} \rangle + d\langle C \wedge \Sigma \rangle = \langle B \wedge (F - t\Sigma) \rangle + \langle \Sigma \wedge d_A C \rangle = \langle \mathfrak{B} \wedge \mathfrak{F} \rangle .$$

Canonical variables

$$\theta' = B \wedge \delta A + \Sigma \wedge \delta C = \mathfrak{B} \wedge \delta \mathfrak{A} \qquad \mathfrak{B} = (B, \Sigma) \leftrightarrow (A, C) = \mathfrak{A}$$

Symmetries of 4d BF theory, aka 4d Chern-Simons thy

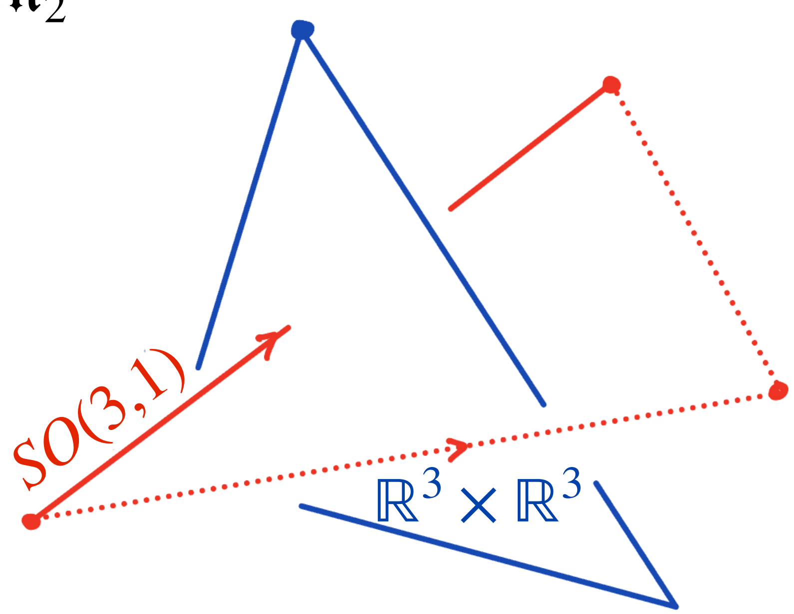
Not all BF theories are double of 2-symmetries. This is the analogue of Chern-Simons vs BF theory in 3d

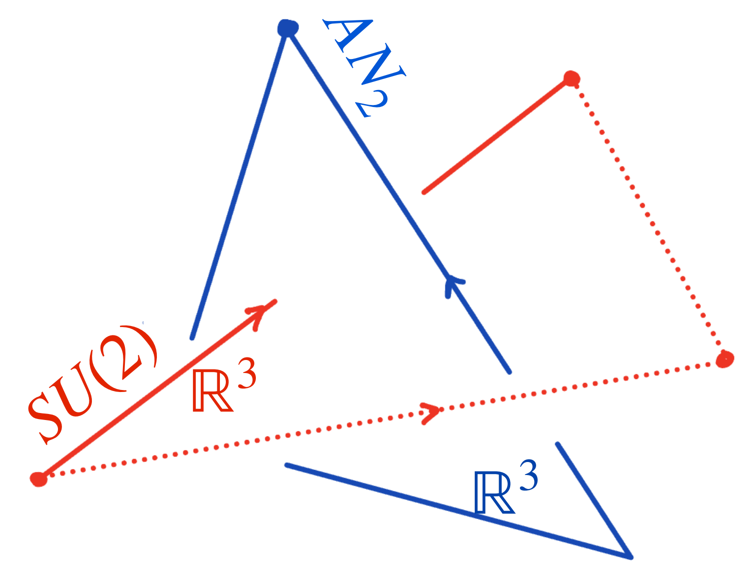


Symmetries of 4d BF theory, aka 4d Chern-Simons thy

Standard $\mathfrak{so}(3,1)$ BF theory (no BB interaction) can be viewed either as a double of trivial 2-symmetries, or as a double of (deformed) 2-symmetries. This is the same as the change of polarization from before.

$\mathfrak{so}(3,1) \approx \mathfrak{su}(2) \bowtie \mathfrak{an}_2$
$\mathfrak{so}(3,1)^* \approx \mathbb{R}^3 \times \mathbb{R}^3$





Partition function as state sum in terms of 1-category of $\mathfrak{so}(3,1)$ representations.

Girelli-Tsimiklis
Girelli-Laudonio-Tsimiklis

Partition function as state sum in terms of 2-category of Euclidian 2-group $\mathfrak{su}(2) \ltimes \mathbb{R}^3$ 2-representations.

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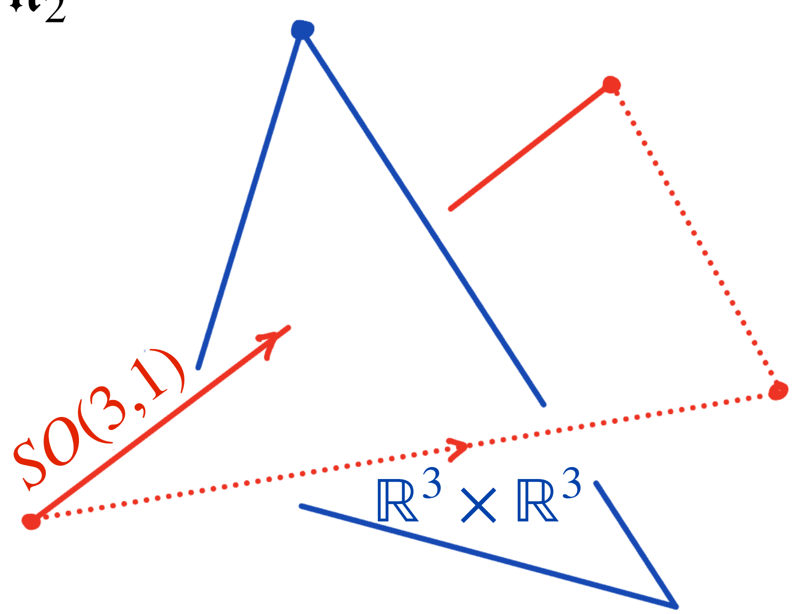

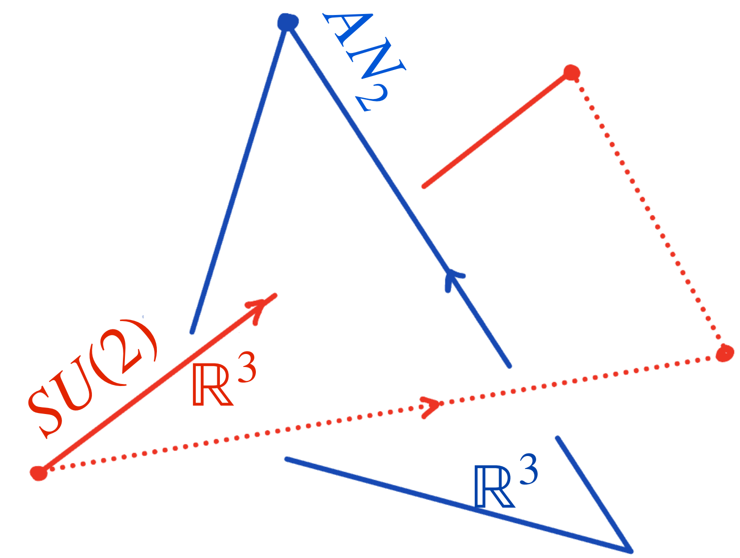

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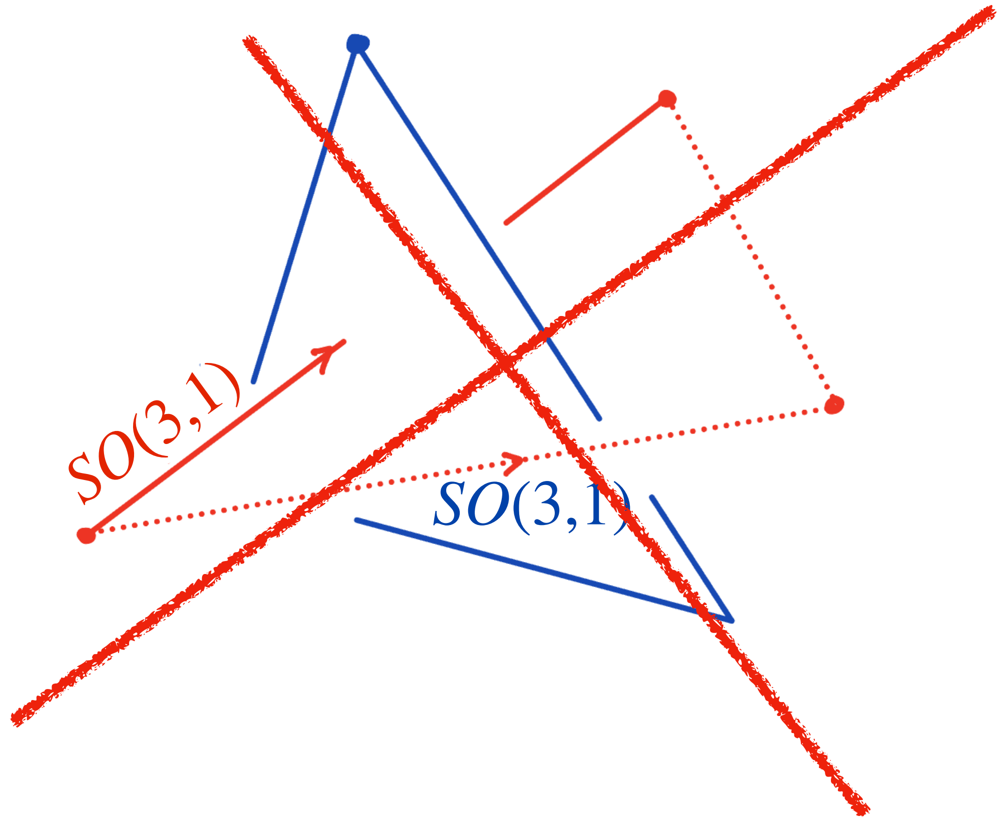
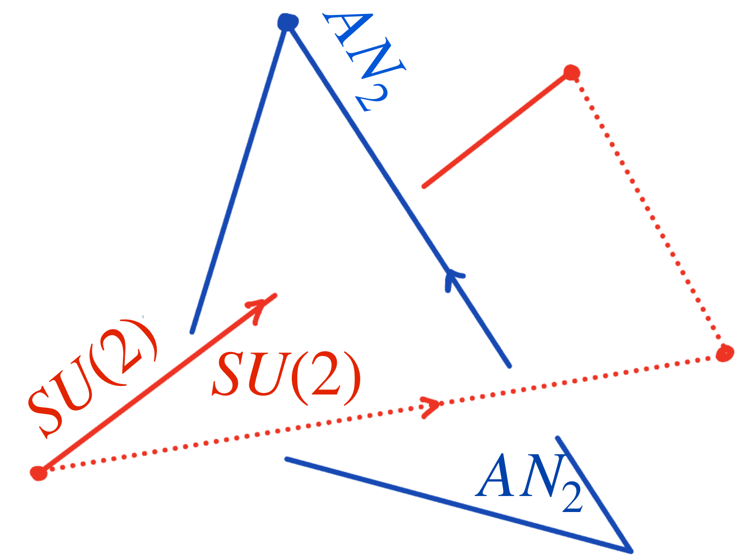
Partition function as state sum in terms of 2-category of Euclidian 2-group $\mathfrak{su}(2) \ltimes \mathbb{R}^3$ 2-representations.

This would imply some relations between symbols based on standard representations and 2-representations.

Baratin Freidel for the 2-Poincare group

Symmetries of 4d BF theory, aka 4d Chern-Simons thy





Standard Lorentz BF theory **with BB interaction** cannot be viewed as a *standard gauge theory. This is why quantum 2-group structure should arise.*

$$Lie\mathbb{G} = \left(\mathfrak{so}(3,1) \ltimes \mathfrak{so}(3,1) \rightrightarrows \mathfrak{so}(3,1)\right) \approx Lie\mathbb{G}^*$$

$$\mathfrak{so}(3,1)^* \approx \mathfrak{so}(3,1)$$

$$\mathfrak{so}(3,1) \approx \mathfrak{su}(2) \bowtie \mathfrak{an}_2$$

Elements of 2-representations

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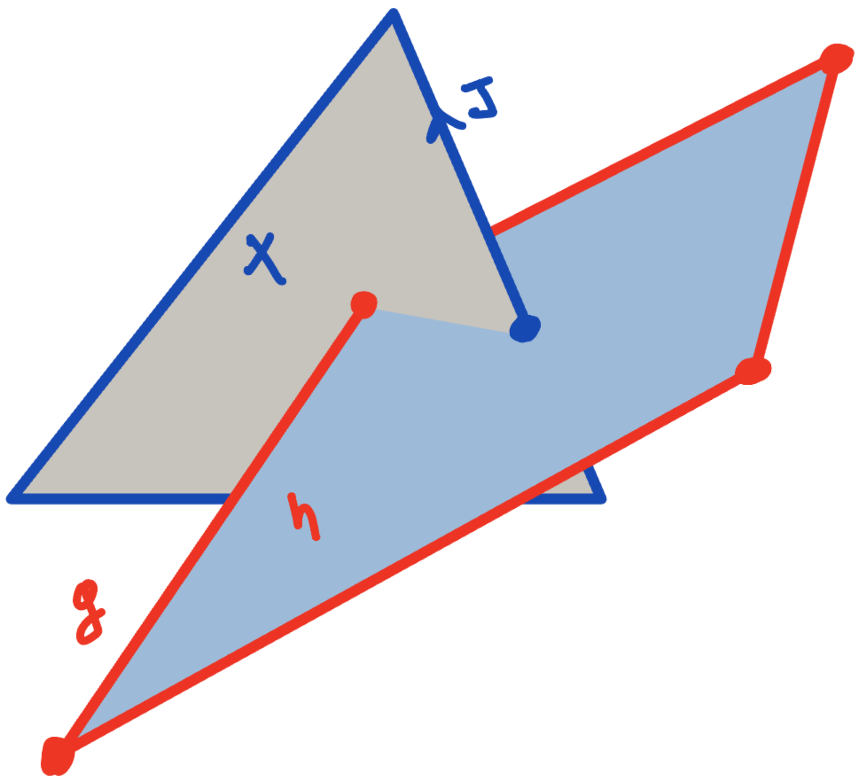
We can define 2-spin networks, functions invariant under 1- and 2-gauge transformations, what is the representation picture?

We lack of a Peter-Weyl theorem to recover it.



Cf Maite's talk this afternoon

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For some choices of Lie 2-group, we can discretise the partition function

$$\int [dA][dC][d\Sigma][dB] e^{i\int \langle \mathscr{B} \wedge \mathscr{F} \rangle} \sim \int [dg][dh] \prod_{face} \delta(t(h) \prod_{links} g)\, \delta(\prod_{polyhedra} h)$$

We can construct a GFT to generate such amplitudes as Feynman diagrams **Girelli-Laudonio-Tanasa-Tsimiklis**

We lack of a Peter-Weyl theorem to recover a 2-state-sum.
What are the 2-characters??

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We lack of a Peter-Weyl theorem to recover a 2-state-sum.

But, for the Poincare 2-group, ie BF thy for Poincare, using geometry we could recover the Baratin-Freidel-Korepanov 2-state-sum model.

Asante-Dittrich-Girelli-Riello-Tsimiklis

Nice mathematical problem! Head start with respect to mathematicians: we know 1-2-gauge invariance should be equivalent to constraints in the triangulation picture.

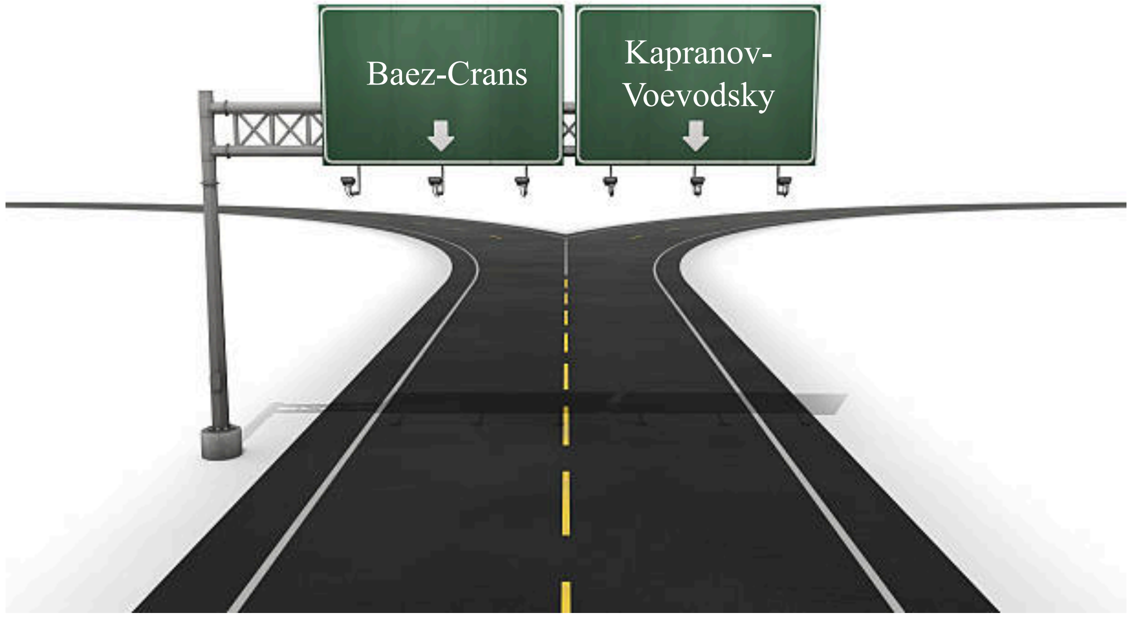
WIP
Dupuis-Girelli-Hrytseniak

Elements of 2-representations

__Attention__: there are 2 different notions of 2-vector spaces.

- Angulo's representation theory
- 2-Hopf algebras (Chen-Girelli)
- 2-coadjoint orbits (Chen-Girelli)

"Natural" from an algebra perspective 2-adj action!



- Condensed matter models (topological order): mainly inspired by finite groups and category theory
- Douglas Reuter state sum.
- Baez-Baratin-Freidel-Wise 2-representation theory (building up on Crane,Yetter…..)

"Natural" from a category perspective

Elements of 2-representations

Consider a vector space V, a representation of a Lie algebra/Lie group on V is given in terms of a **matrix**

What is a "matrix" on a Baez-Crans 2-vector space $\mathbb{V} = (W \xrightarrow{\partial} V)$?

Sheng-Zhu
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What is a "matrix" on a Baez-Crans 2-vector space $\mathbb{V} = (W \xrightarrow{\partial} V)$?

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It is given in terms of a pair of matrices (M, m) and a map A

$(M, m) \in GL(W) \oplus GL(V) = GL_0(\mathbb{V})$: objects $\quad \partial M = m\partial$

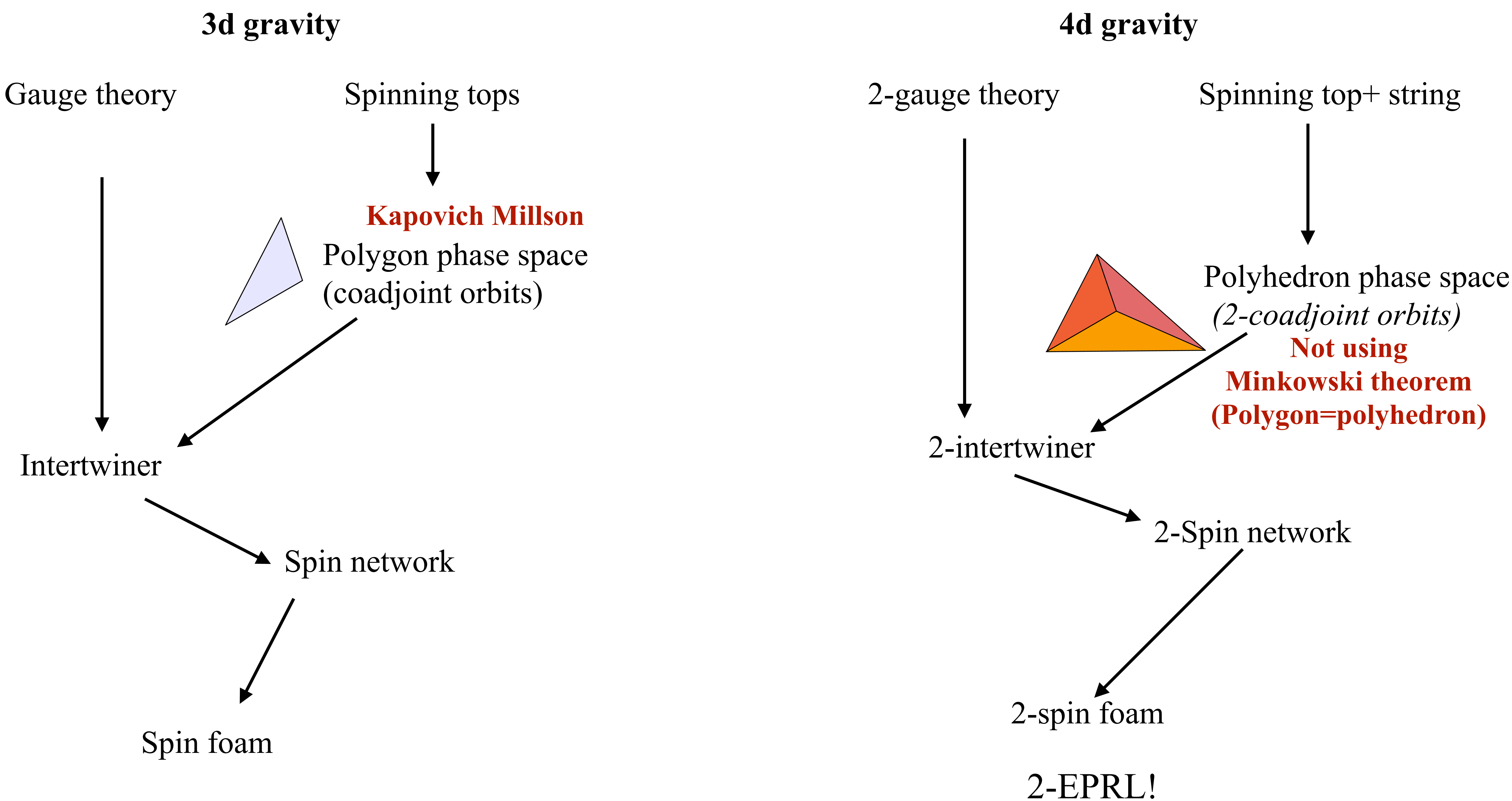
$A \in Hom(V, W) = GL_1(\mathbb{V})$: maps between objects $\quad \Delta A = (id + A\partial, id + \partial A)$

$GL(\mathbb{V})$ is itself a 2-group.

2-(co-)Adjoint action is a 2-representation of this kind!

Road map and comments

Road map to LQG 2.0
Work in progress



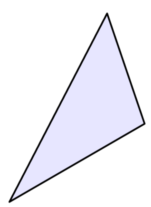
Road map to LQG 2.0
Work in progress

3d gravity

Gauge theory

Spinning tops

Kapovich Millson



Polygon phase space
(coadjoint orbits)

Intertwiner

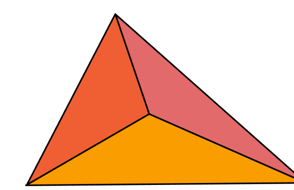
Spin network

Spin foam

4d gravity

2-gauge theory

Spinning top+ string



Polyhedron phase space
(2-coadjoint orbits)

Not using
Minkowski theorem
(Polygon=polyhedron)

2-intertwiner

2-Spin network

2-spin foam

For example:

$$\mathcal{S}_{MM} = \int \mathfrak{B} \wedge \mathfrak{F} - \frac{1}{2}\beta \mathfrak{B} \wedge \mathfrak{B} - \frac{\alpha}{4}\epsilon^{4IJKL}\mathfrak{B}_{IJ} \wedge \mathfrak{B}_{KL}$$

$$\int d\mathfrak{B} d\mathfrak{A} e^{iS_{MM}} \sim \int d\mathfrak{B} d\mathfrak{A} \left(1 + \alpha\epsilon\mathfrak{B}\mathfrak{B} + \dots\right) e^{iS_{BF}}$$

2-state-sum

Outlook

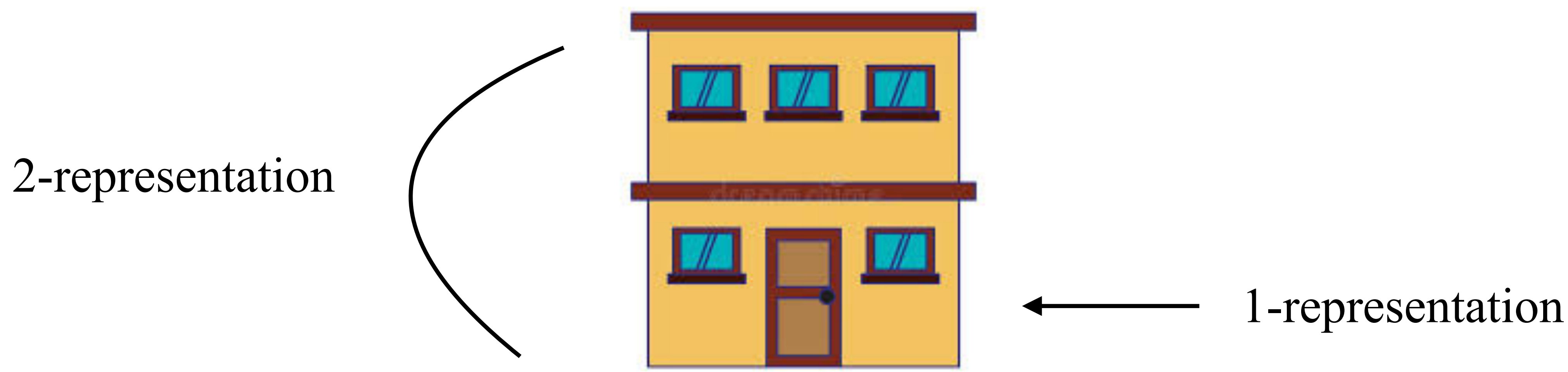
2-symmetries could provide a better understanding of the QG regime

Topological models are based on 2-symmetries/representations, should matter for QG spinfoam construction!

The presence of a cosmological constant reveals, at the discrete level, that 2-group holonomies are the natural tool to describe discretized quantities.

2-symmetries appear in Feynman diagrams

2-symmetries are naturally present in 4d, but have not been leveraged



Don’t stop at 2, use also 3-groups! Matter content could be related to a 3-group structure (see Vojinovic’talk)

Higher structures are already present

Symmetries of 4d BF theory, aka 4d Chern-Simons thy

Standard gauge theory for Lie algebra $\mathfrak{g}$

Data:
Gauge connection $A \in \mathfrak{g} \otimes \Lambda^1$,
Curvature $F = dA + \frac{1}{2}[A \wedge A] \in \mathfrak{g} \otimes \Lambda^2$

Gauge transformations, $\alpha \in \mathfrak{g} \otimes \Lambda^0$

$$\delta_\alpha A = d_A \alpha,$$

$$\delta_\alpha F = [F, \alpha]$$

Co-adjoint action!

Maurer Cartan form/equation, $A \in \mathfrak{g} \otimes \Lambda^1$

$$dA = \pm \frac{1}{2}[A \wedge A]$$

solution: $A = g^{-1}dg, or A = dgg^{-1}$

"Pure gauge"

Higher gauge thy for Lie 2-algebra $\mathfrak{g} \ltimes \mathfrak{h} \Rightarrow \mathfrak{g}$

Data:
Connection $\mathscr{A} = (A, \Sigma) \in (\mathfrak{g} \otimes \Lambda^1 M) \oplus (\mathfrak{h} \otimes \Lambda^2 M)$
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$$\delta_{(\alpha,\phi)}\mathscr{F} = [\mathscr{F}, (\alpha, \phi)]_2 = ([F - t\Sigma, \alpha], \alpha \rhd d_A \Sigma + (F - t\Sigma) \rhd \phi]$$

2-co-adjoint action!

Generalized MC forms: $\mathscr{A} = (A, \Sigma) \in (\mathfrak{g} \otimes \Lambda^1 M) \oplus (\mathfrak{h} \otimes \Lambda^2 M)$

$$dA + \frac{1}{2}[A \wedge A] = t\Sigma, \quad d_A \Sigma = 0$$

solution: $A = g^{-1}dg + t\phi \qquad B = d_A \phi + \frac{1}{2}[\phi \wedge \phi]$

"Pure 2-gauge"

Proper surface holonomy when $F - t\Sigma = 0$

From spinning tops to strings

Recall that *spinning top* action for group G

$$\int (\pi \cdot g^{-1}\frac{dg}{dt} - \frac{1}{2}\pi \cdot \pi)dt$$

Balachandran et al.

Co-adjoint orbits
momentum $\pi \in \mathfrak{g}^*$ transforming under the co-adjoint action and characterized by Casimir $\pi \cdot \pi$

$$\pi_0 \in \mathfrak{g}^*, \quad \pi = g^{-1}\pi_0 g, \quad \pi \cdot \pi = s^2$$

Maurer-Cartan form
transforms under coadjoint action for **global transformations** $u^{-1}du = 0$

$$u^{-1}g^{-1}dgu + \cancel{u^{-1}du},$$

From spinning tops to strings

Recall that *spinning top* action for group G

$$\int (\pi \cdot g^{-1}\frac{dg}{dt} - \frac{1}{2}\pi \cdot \pi)dt$$

Balachandran et al.

Co-adjoint orbits
momentum $\pi \in \mathfrak{g}^*$ transforming under the co-adjoint action and characterized by Casimir $\pi \cdot \pi$

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Maurer-Cartan form
transforms under coadjoint action
for **global transformations** $u^{-1}du = 0$

$$u^{-1}g^{-1}dgu + \cancel{u^{-1}du},$$

2-co-adjoint orbits
momentum $(\pi, p) \in \mathfrak{g}^* \otimes \Lambda \oplus \mathfrak{h}^* \otimes \Lambda$ transforming under the 2-co-adjoint action and characterized by Casimir

$$(\pi, p) \sim (\pi_0, p_0) + [(\pi_0; p_0), (\alpha; \phi)]_2$$

Generalized Maurer-Cartan form

$$(g^{-1}dg + t(q); d_{g^{-1}dg}q + \frac{1}{2}[q \wedge q]) \in (\mathfrak{g} \otimes \Lambda^1) \oplus (\mathfrak{h} \otimes \Lambda^2)$$

transforms under 2-coadjoint action
for **global transformations** $d(\alpha, \phi) = (0,0)$

$$[(g^{-1}dg + t(q); d_{g^{-1}dg}q), (\alpha; \phi)]_2 + \cancel{d(\alpha, \phi)}$$

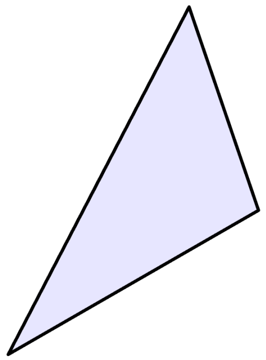
$$\int (\pi; p) \cdot \left(g^{-1}dg + t(q); d_{g^{-1}dg}q\right) - \pi \cdot p = \int \pi \cdot \left(g^{-1}dg + t(q)\right) + p \cdot d_{g^{-1}dg}q - \pi \cdot p$$

String coupled to a spinning top

WIP
Girelli-Pollack-Riello-Tsimiklis

From polygon to polyhedron

Polygon phase space (with fixed edge length) **Kapovich-Millson**



$$C = \sum_a X^a = 0$$

$$\mathscr{P}_{\triangle} = \left(S^2_\ell \times S^2_\ell \times S^2_\ell\right) // C$$

$SU(2)$ co-adjoint orbit

$$X \in S^2_\ell \subset \mathfrak{su}^*(2) \sim \mathbb{R}^3$$

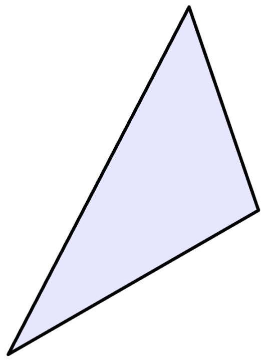
$$X^2 = \ell^2$$

$+$ Kirillov symplectic form

$$\{X_i, X_j\} = \epsilon_{ij}^k X_k$$

From polygon to polyhedron

Polygon phase space (with fixed edge length) Kapovich-Millson



$$\mathscr{P}_{\triangle} = \left(S^2_\ell \times S^2_\ell \times S^2_\ell\right) // C \longleftarrow C = \sum_a X^a = 0$$

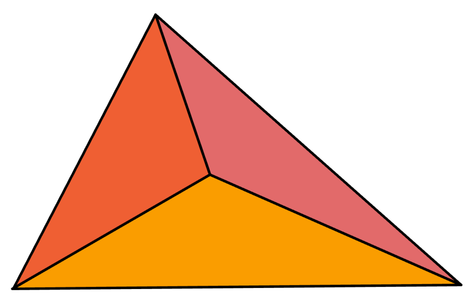
$SU(2)$ co-adjoint orbit

$$\left.\begin{array}{l} X \in S^2_\ell \subset \mathfrak{su}^*(2) \sim \mathbb{R}^3 \\ X^2 = \ell^2 \end{array}\right) + \text{Kirillov symplectic form}$$

$$\{X_i, X_j\} = \epsilon_{ij}^{k} X_k$$

Polyhedron phase space?

WIP
Girelli-Oliveira-Riello



$$C_i = X_i - \sum_j J_j = 0$$
$$C = \sum X_i = 0$$

$$\mathbb{G} = (SU(2) \ltimes SU(2) \rightrightarrows SU(2)) \text{ 2-coadjoint orbits}$$

$$\left.\begin{array}{l} (X, J) \in \mathfrak{su}^*(2) \oplus \mathfrak{su}^*(2) \\ X \cdot J = s, \quad J^2 = \ell^2 \end{array}\right) + \text{2-Kirillov symplectic form}$$

$$\{X_i, X_j\} = \epsilon_{ij}^{k} X_k$$
$$\{J_i, X_j\} = \epsilon_{ij}^{k} X_k$$
$$\{J_i, J_j\} = \epsilon_{ij}^{k} J_k$$

2-representation theory

Representation theory:

consider a vector space V, a representation of a Lie algebra/Lie group is a map

$$\rho : \mathfrak{g} \to GL(V) \qquad\qquad \rho : G \to GL(V)$$

$$(X, Y) \in \mathfrak{g} \qquad \rho([X, Y]) = [\rho(X), \rho(Y)] \qquad \rho(g_1 g_2) = \rho(g_1)\rho(g_2) \qquad (g_1, g_2) \in G$$

2-Representation theory for Baez-Crans 2-vector space:

consider a 2-vector space $\mathbb{V} = (W \to V)$, a representation of a 2-Lie algebra/Lie 2-group is a map

$$\rho : \mathfrak{g} \to GL(\mathbb{V}) \qquad\qquad \rho : \mathbb{G} \to GL(\mathbb{V})$$

$$(X, Y) \in \mathfrak{g} \qquad \rho([X, Y]_2) = [\rho(X), \rho(Y)]_2 \qquad \rho(g_1 \,.\, g_2) = \rho(g_1) \,.\, \rho(g_2) \qquad (g_1, g_2) \in \mathbb{G}$$

what is $GL(\mathbb{V})$?

$GL(\mathbb{V})$ is itself a 2-group, $GL(\mathbb{V}) = (GL_1(\mathbb{V}) \to GL_0(\mathbb{V}), \rhd)$.

$GL_0(\mathbb{V}) = GL(W) \oplus GL(V) \ni (M, m)$: objects $\qquad \partial M = m\partial$

$GL_1(\mathbb{V}) = Hom(V, W) \ni A$: maps between objects $\qquad \Delta A = (A\partial, \partial A)$

$$s((M, m), A) = (M, m), \quad \tau((M, m), A) = \Delta A(M, m) = (A\partial M, \partial A m)$$

2-representation theory: 2-adjoint action example

consider the Lie 2-algebra $Lie\mathbb{G} = (\mathfrak{h} \to \mathfrak{g}, \rhd), \quad \alpha \in \mathfrak{g}, \quad \phi \in \mathfrak{h},$
the 2-adjoint representation consists in representing it on itself, so on $W = \mathfrak{h} \ni X, V = \mathfrak{g} \ni J$

$$\rho : Lie\mathbb{G} \to GL(Lie\mathbb{G})$$
$$\rho = ((\rho_0^1, \rho_0^0), \rho^1) = ((M, m), A)$$
$$\rho_0^1(\alpha)X = \alpha \rhd X, \quad \rho_0^0(\alpha)(J) = [\alpha, J], \quad \rho^1(\phi)(J) = -J \rhd \phi$$

2-adjoint representation:

$$^2ad_{(\alpha,\phi)}(J, X) \equiv (\rho_0^0(\alpha)J, \rho_0^1(\tau(\alpha, \phi))X + \rho^1(\phi)J) = [(\alpha, \phi), (J, X)]$$

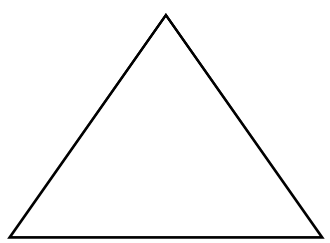
The 2-bracket encodes the 2-adjoint action

From polygon to polyhedron

Let us consider the co-adjoint orbit of $G = SU(2)$ with Lie algebra $\mathfrak{su}(2)$.

$$\left.\begin{array}{l} X \in S^2_\ell \subset \mathfrak{su}^*(2) \sim \mathbb{R}^3 \\ X^2 = \ell^2 \end{array}\right) + \text{Kirillov symplectic form} \qquad \{X_i, X_j\} = \epsilon_{ij}^k X_k$$

Polygon phase space (with fixed edge length): Each edge is an element in the phase space S^2_ℓ, and we reduce with respect to the constraint $C = \sum_a X^a = 0$.



$$\mathscr{P}_\triangle = \left(S^2_\ell \times S^2_\ell \times S^2_\ell\right)//C$$

Can be generalized to positive or negative curvature.

$$AN_2 \sim H_3$$

$$\mathfrak{su}^*(2) \sim \mathbb{R}^3$$

$$SU(2) \sim S^3$$

$$\mathbb{R}^3 \rtimes SU(2) \sim T^*SU(2)$$

To make the edge length dynamical, extend phase space. $\mathfrak{su}^*(2) \sim \mathbb{R}^3$

$$\mathbb{C}^2 \sim \mathbb{R}^4$$

Let us consider the specific case, with 2-group $\mathbb{G} = (SU(2) \to SU(2), \rhd\ )$ with 2-Lie algebra $Lie\mathbb{G} = (\mathfrak{su}(2) \to \mathfrak{su}(2), \rhd\ )$.

4d Actions with Lie 2-symmetries

Let us generalize this to a 2-group $\mathbb{G}$ with 2-Lie algebra $Lie\mathbb{G} = (\mathfrak{h} \to \mathfrak{g}, \rhd)$.

$$(g^{-1}dg + t(q); d_{g^{-1}dg}q + \frac{1}{2}[q \wedge q]) \equiv (A, B) \in (\mathfrak{g} \otimes \Lambda^1) \oplus (\mathfrak{h} \otimes \Lambda^2)$$

- **Generalized Maurer-Cartan form** $F(A) = t(B), \quad d_A B = 0$
    - Fake flatness and no 2-curvature,

ϕ is an exact 1-form

-
    - transforms under adjoint action for global transformations $d(\alpha, \phi) = (0,0)$

$${}^2ad_{(\alpha,\phi)}(g^{-1}dg + t(q); d_{g^{-1}dg}q) + \cancel{d(\alpha, \phi)} = ([g^{-1}dg + t(q), \alpha], \alpha \rhd d_{g^{-1}dg}q + (g^{-1}dg + t(q)) \rhd \phi)$$

$p \cdot \pi = s$

- **2-co-adjoint orbits** — momentum $(\pi, p) \in \mathfrak{g}^* \otimes \Lambda \oplus \mathfrak{h}^* \otimes \Lambda$ transforming under the co-adjoint action and characterized by Casimir $(\pi, p) = (\pi_0, p_0) + {}^2ad_{(\alpha,\phi)}(\pi_0; p_0) = ([\pi_0, \alpha], \alpha \rhd p_0 + \pi_0 \rhd \phi)$

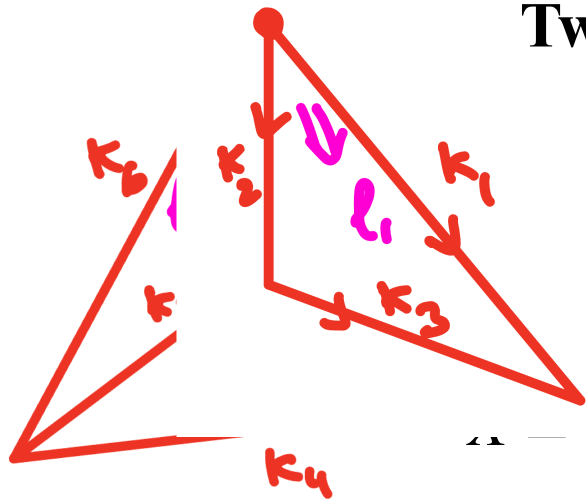
$$\int \left(g^{-1}dg + t(q); d_{g^{-1}dg}q\right) \cdot (\pi, p) - \pi \cdot p = \int p \cdot \left(g^{-1}dg + t(q)\right) + \pi \cdot d_{g^{-1}dg}q - \pi \cdot p$$

$$\begin{aligned}
\delta\pi : &\quad d_{g^{-1}dg}q - p = 0 \\
\delta g : &\quad d(g^{-1}\pi g) = [] \\
\delta p : &\quad g^{-1}dg + t(q) = \pi \\
\delta q : &\quad t(p) = d_{g^{-1}dg}\pi
\end{aligned}$$

$$\theta = \pi \cdot \delta g g^{-1} \longrightarrow \qquad T^*\mathbb{G}?$$

Higher symmetries are already present in many different places.

Twisted geometry=spinning geometry=2-geometry



der the discretized flux. $X \in \mathbb{R}^3$ encoding the normal of the face S.

$$X = \int_S dz \times dz = \int_S d(z \times dz) = \int_{\partial S} (z \times dz) = \sum_e \int_{e \in \partial S} (z \times dz) = \sum J_e$$

Could replace the face information by the edge information

by reducing the fake flatness.

Note that J_e is associated to edge but is not edge vector.
It can be seen as the normal of a face in a ghost tetrahedron.

Gauge fields and their transformations

Usual gauge theory with Lie group G on manifold M:

- connection $A \in \mathfrak{g} \otimes \Lambda^1 M$ and its curvature $F = dA + \frac{1}{2}[A \wedge A] \in \mathfrak{g} \otimes \Lambda^2 M$
- gauge transf parameterized by $\alpha \in \mathfrak{g} \otimes \Lambda^0 M$, $\delta_\alpha A = d\alpha + [A, \alpha]$, $\quad \delta_\alpha F = [F, \alpha]$
- Maurer Cartan form ("pure gauge"): $A = g^{-1}dg$ or $A = dgg^{-1}$ and $F = 0$

- Connection $\mathscr{A} = (A, \Sigma) \in \mathfrak{g} \otimes \Lambda^1 M \oplus \mathfrak{h} \otimes \Lambda^2 M$
- Curvature data, $\mathscr{F} = (F - t\Sigma, \quad d_A\Sigma = d\Sigma + A \triangleright \Sigma)$.
- 1- and 2-gauge transf parameterized by $(\alpha, \phi) \in \mathfrak{g} \otimes \Lambda^0 M \oplus \mathfrak{h} \otimes \Lambda^1 M$,
  $\delta_{(\alpha,\phi)}\mathscr{A} = d(\alpha, \phi) + [\mathscr{A}, (\alpha, \phi)] = d(\alpha, \phi) + ([A, \alpha], \alpha \triangleright \Sigma + A \triangleright \phi] = (d_A\alpha, \alpha \triangleright \Sigma + d_A\phi)$

  $\delta_{(\alpha,\phi)}\mathscr{F} = [\mathscr{F}, (\alpha, \phi)] = ([F - t\Sigma, \alpha], \alpha \triangleright d_A\Sigma + (F - t\Sigma) \triangleright \phi]$
- Generalized Maurer Cartan forms: $A = g^{-1}dg + t\phi$, $B = d_A\phi$ and $F = t\Sigma, \quad d_A\Sigma = 0$
- Proper surface holonomy when $F - t\Sigma = 0$

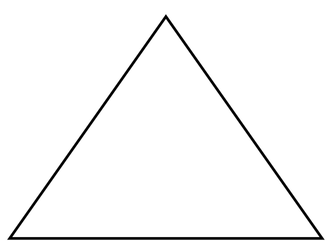
Curvature is 2-gauge!

Geometry from co-adjoint orbits

Let us consider the co-adjoint orbit of $G = SU(2)$ with Lie algebra $\mathfrak{su}(2)$.

$$\left.\begin{array}{l} X \in S^2_\ell \subset \mathfrak{su}^*(2) \sim \mathbb{R}^3 \\ X^2 = \ell^2 \end{array}\right) + \text{Kirillov symplectic form} \qquad \{X_i, X_j\} = \epsilon_{ij}^{k} X_k$$

Polygon phase space (with fixed edge length): Each edge is an element in the phase space S^2_ℓ, and we reduce with respect to the constraint $C = \sum_a X^a = 0$.



$$\mathscr{P}_{\triangle} = \left(S^2_\ell \times S^2_\ell \times S^2_\ell\right) // C$$

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$$SU(2) \sim S^3$$

$$\mathbb{R}^3 \rtimes SU(2) \sim T^*SU(2)$$

To make the edge length dynamical, extend phase space. $\quad \mathfrak{su}^*(2) \sim \mathbb{R}^3$

$$\mathbb{C}^2 \sim \mathbb{R}^4$$

Let us consider the specific case, with 2-group $\mathbb{G} = (SU(2) \to SU(2), \triangleright\,)$ with 2-Lie algebra $Lie\mathbb{G} = (\mathfrak{su}(2) \to \mathfrak{su}(2), \triangleright\,)$.

4d Actions with Lie 2-symmetries

Construct an action to implement flat 2-curvature $\mathscr{F} = 0$:

- Use Lagrange multipliers $\mathscr{B} = (C, B) \in \mathfrak{h}^* \otimes \Lambda^1 M \oplus \mathfrak{g}^* \otimes \Lambda^2 M$, which also transform under the 2-adjoint action,

- Use an invariant bilinear form $\langle , \rangle$ on under the 2-adjoint action.

$$\langle [\chi_1, \chi_2], \chi_3 \rangle = -\langle \chi_2, [\chi_1, \chi_3] \rangle, \quad \chi_{1,3} \in Lie\mathbb{G}, \chi_2 \in Lie\mathbb{G}^*$$

$$\langle \mathscr{B} \wedge \mathscr{F} \rangle = \langle B \wedge (F - t\Sigma) \rangle + \langle C \wedge d_A \Sigma \rangle \, .$$

This is called *4d Chern-Simons thy* by some.

$\theta = B \wedge \delta A + C \wedge \delta\Sigma$ $\qquad$ $(B, C) \leftrightarrow (A, \Sigma)$ $\qquad$ 1-2-connections dual to each other.

There is a dual 2-symmetry, from the $\mathscr{B}$ sector.

Natural symmetry given by a matched pair of Lie 2-algebras, $Lie\mathbb{G}^* \bowtie Lie\mathbb{G}$

It can be repackaged as a big BF theory with symmetry $(\mathfrak{g}^* \ltimes \mathfrak{h} \to \mathfrak{h}^* \ltimes \mathfrak{g}, \triangleright)$ by adding a boundary term

$$\langle B \wedge (F - t\Sigma) \rangle + \langle \Sigma \wedge d_A C \rangle + d\langle C \wedge \Sigma \rangle = \langle B \wedge (F - t\Sigma) \rangle + \langle \Sigma \wedge d_A C \rangle + d\langle C \wedge \Sigma \rangle = \langle \mathfrak{B} \wedge \mathfrak{F} \rangle + d\langle C \wedge \Sigma \rangle$$

$\theta' = B \wedge \delta A + \Sigma \wedge \delta C = \mathfrak{B} \wedge \delta\mathfrak{A}$ $\qquad$ $\mathfrak{B} = (B, \Sigma) \leftrightarrow (A, C) = \mathfrak{A}$

Recall that *spinning top* action for group G

$$\int (\pi \cdot g^{-1}\frac{dg}{dt} - \frac{1}{2}\pi \cdot \pi)dt$$

$$\delta\pi : \quad g^{-1}dg - \pi = 0$$
$$\delta g : \quad d(g\pi g^{-1}) = 0$$
$$\delta\lambda : \quad \pi^2 = s^2$$

$F(A) = 0$

$$\cancel{\theta = \pi \cdot \delta g g^{-1}} \longrightarrow T^*G$$

$$g^{-1}dg \to u^{-1}g^{-1}dgu + u^{-1}du, \qquad u^{-1}du = 0$$

$$g^{-1}dg \to g^{-1}dg + [g^{-1}dg, \alpha] \quad u \sim 1 + \alpha, \quad \alpha \in$$

$$\pi_0 \in \mathfrak{g}^*, \quad \pi_1 = g^{-1}\pi_0 g \sim \pi_0 + [\pi_0, \alpha], \quad \pi \cdot \pi = s^2$$

- **Maurer-Cartan form** $g^{-1}dg \equiv A \in \mathfrak{g} \otimes \Lambda^1$
  - no curvature,
  - transforms under adjoint action for global transformations
- **Co-adjoint orbits** — momentum π transforming under the co-adjoint action and charac Casimir