Matter coupled to 3d Quantum Gravity: One-loop Unitarity

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Introduction

3D Quantum Gravity: No **local** freedom degree but only **global** (topological).

Path integral formulation (Ponzano Regge) of matter coupled to 3D QG

$$\mathrm{Z} = \int \mathcal{D}g\mathcal{D}\Psi \; e^{iS[\Psi,g]+iS_{\mathrm{GR}}[g]} \longrightarrow \int \mathcal{D}\Psi \; e^{iS^{\kappa}_{\mathrm{eff}}[\Psi]}$$

The resulting QFT is **noncommutative** ($\mathbb{R}^3_\kappa$), **effective, and invariant under deformed κ-Poincaré symmetries.** [Freidel,Livine 2006]
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Question: Is this theory unitary ?

Unitarity: Motivations and Tools

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Cutting rule

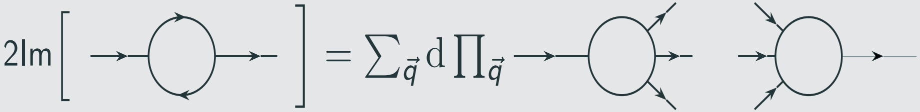
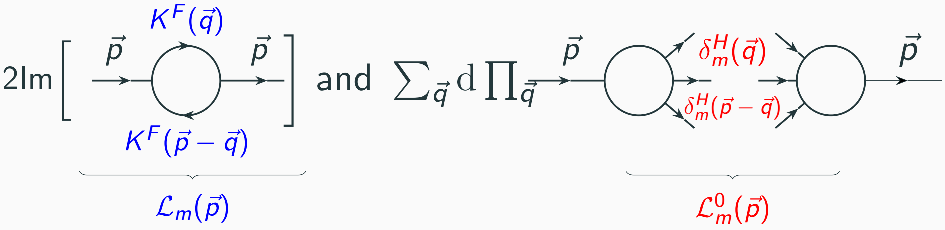


Figure 1: *On the left side the imaginary part of the amplitude (off-shell) and on the right the amplitude decomposed over all possible intermediate states (on-shell).*



Cutting rule in Euclidean theory	Cutting rule in Lorentzian theory
$i\,\mathrm{Im}(\mathcal{L}_m(\vec{p})) + \mathcal{L}_m^0(\vec{p}) = \frac{i}{8\lvert\vec{p}\rvert}$	$i\mathrm{Im}(\mathcal{L}_m(\vec{p})) = \mathcal{L}_m^0(\vec{p}) = \begin{cases} 0 & \text{if} \quad \lvert p\rvert < 2m \\ \frac{i}{8\lvert p\rvert} & \text{if} \quad \lvert p\rvert > 2m \end{cases}$

The 3D quantum gravity theory as a group field

$$S_{\text{eff}}[g] = \int_G \frac{dg}{2}(P^2(g) - \frac{1}{2}M^2)\tilde{\Psi}(g)\tilde{\Psi}(g^{-1}) + \frac{\lambda}{3!}\int_G dgdhdk\delta(ghk)\tilde{\Psi}(g)\tilde{\Psi}(h)\tilde{\Psi}(k)$$

- Euclidian case $g \in$ **SO(3)** or **SU(2)** $\to \vec{P}(g) \in \mathbb{R}^3_\kappa$
- Lorentzian case $g \in$ **SO(2,1)** or **SU(1,1)** $\to \vec{P}(g) \in \mathbb{R}^3_\kappa$

Caution: the map between g and $\vec{P}(g)$ is only bijective for SO(3) and SO(2,1).

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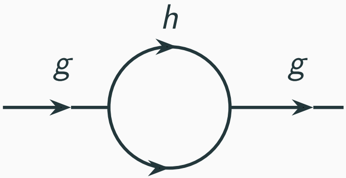
Mass, momentum parametrizations for **Euclidean** and **timelike Lorentzian** elements

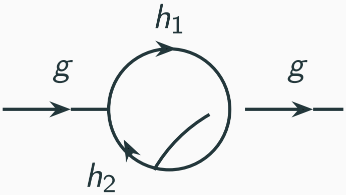
$$\vec{P}(g) = \kappa \sin\theta\, \hat{u} \quad M = \kappa \sin(\varphi)$$

with $\theta \in [0, \pi]$, $\hat{u} \in \mathbb{R}^3$, $\hat{u}^2 = 1$, $\varphi \in [0, \pi/2]$

Feynman diagrams at one loop for Ψ^3 interaction



Planar one loop



Non planar one loop

[Sasai,Sasakura 2009]

Euclidean case: SO(3) Propagators:

Feynman propagator (for off-shell one loop):

$$K_\varphi(g(\theta, \hat{u})) = \frac{1}{\kappa^2(\sin^2(\theta) - \sin^2(\varphi) + i\epsilon)} \quad \xrightarrow[\kappa\to\infty]{} \quad \frac{1}{p^2 - m^2 + i\epsilon}$$

Hadamard propagator (for on-shell one loop) :

$$\delta_\varphi(g(\theta, \hat{u})) \propto \int_{\mathrm{SO}(3)} \mathrm{d}h \, \delta(gh\, g_{(\varphi,\hat{u})}\, h^{-1}) \quad \text{fix the class angle of } g \text{ to } \varphi \ \sim \delta(p^2 - m^2)$$

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Basis of SO(3): $\chi_j(g(\theta,\hat{u})) = \sin(d_j\theta)/\sin\theta \quad d_j = 2j+1,\ j \in \mathbb{N}$

SO(3) Feynman and Hadamard propagators

$$K_\varphi(g) = \frac{2}{\kappa}\sum_{j\in\mathbb{N}} \frac{e^{-id_j(\varphi - i\epsilon)}}{\cos(\varphi)}\chi_j(\theta) \qquad \delta_\varphi(g) = \frac{4\tan(\varphi)}{\kappa^2}\sum_{j\in\mathbb{N}}\chi_j(\varphi)\chi_j(\theta)$$

Euclidean case: SU(2) Propagators:

No bijection between SU(2) and $\mathbb{R}^3_\kappa$: Ambuiguity to define SU(2) Feynman propagator, some solutions have been proposed. [Freidel,Livine 2005],[Dupuis,Girelli,Livine 2011],[Freidel,Majid 2005]

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$$K_\varphi(g) = \frac{2}{\kappa^2} \sum_{j \in \mathbb{N}/2} e^{-i d_j (\varphi - i\epsilon)} \chi_j(\theta) + 1/\kappa^2 \sin\theta$$

$$= \frac{1}{2\kappa^2(\sin^2(\theta/2) - \sin^2(\varphi/2) + i\epsilon)} + \frac{1}{\kappa^2 \sin\theta} \xRightarrow[\kappa\to\infty]{} \frac{1}{p^2 - m^2 + i\epsilon}$$

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$$= \frac{1}{2\kappa^2(\sin^2(\theta/2)-\sin^2(\varphi/2)+i\epsilon)} + \frac{1}{\kappa^2\sin\theta} \quad \underset{\kappa\to\infty}{\Longrightarrow} \quad \frac{1}{p^2-m^2+i\epsilon}$$

Interpretation of the extra term as SU(2) zero mode :

- Eigenvectors of the Casimir with eigenvalues $-1/4$.
- Correspond to negative spin $j=-1/2$ with $d_j = 0$.
- $\chi_{-1/2}(\theta) := 2/\sin\theta \Longrightarrow K_\varphi(g) = \frac{2}{\kappa}\sum_{d_j\in\mathbb{N}} e^{-id_j(\varphi-i\epsilon)}\chi_j(\theta)$

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$$\delta_\varphi(g(\theta,\hat{u})) = \int_{\mathrm{SU}(2)} \mathrm{d}h\, \delta(gh\, g_{(\varphi,\hat{u})}\, h^{-1})$$

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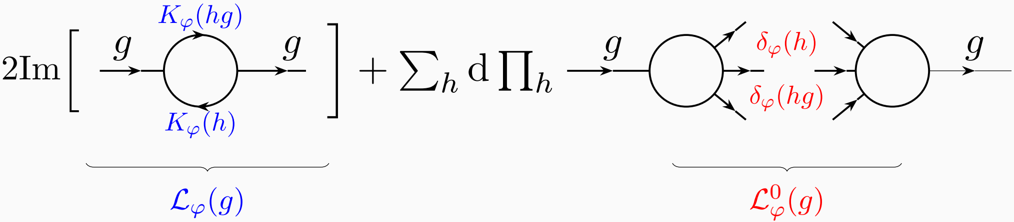
Remark: In both SO(3) and SU(2) Feynman and Hadamard propagators are related, similarly to standard QFT, by the following identity:

$$iK_\varphi(g) - iK_{-\varphi}(g) = \delta_\varphi(g)$$

Euclidean case: Cutting rule



.	Cutting rule in SO(3) for $\theta \in [0, \pi/2]$	Cutting rule in SU(2) for $\theta \in [0, \pi]$
$\varphi \in [0, \frac{\pi}{4}]$	$i\mathrm{Im}\,(\mathcal{L}_\varphi(g)) + \mathcal{L}^0_\varphi(g) = \dfrac{i}{8\kappa \cos^2\varphi \sin\theta}$,	$i\,\mathrm{Im}\,(\mathcal{L}_\varphi(g)) + \mathcal{L}^0_\varphi(g) = \dfrac{i}{4\kappa \sin\theta}$
$\varphi \in [\frac{\pi}{4}, \frac{\pi}{2}]$	Failure of Euclidean cutting rule	$i\,\mathrm{Im}\,(\mathcal{L}_\varphi(g)) + \mathcal{L}^0_\varphi(g) = \dfrac{i}{4\kappa \sin\theta}$;

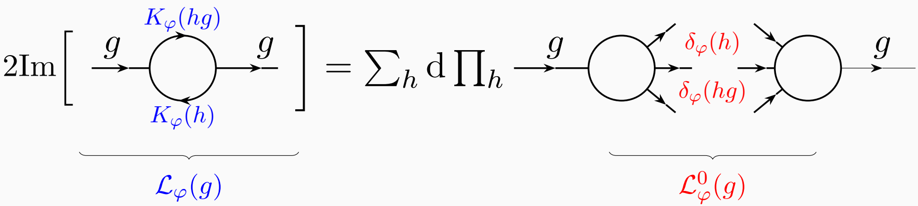
Lorentzian case

Continuous and discret characters of SU(1,1)

$$\chi_s(\theta) = 0\,, \qquad \chi_s(t,\pm) = \frac{\cos st}{|\sinh t|}\,, \quad \chi_j(\theta) = -\frac{e^{i\,d_j\theta}}{2i\sin\theta}\,, \quad \chi_j(t,\pm) = (\pm)^{2j}\frac{e^{-d_j|t|}}{2|\sinh t|}$$

$s \in \mathbb{R}, j \in \mathbb{N}/2$

$$= \begin{cases} \text{Feynman SO(2,1)} & K_\varphi(g) = \frac{-4i\tan\varphi}{\kappa^2}[\sum_{j\in\mathbb{N}} \chi_j^-(\varphi - i\epsilon)\chi_j(g) + \int s\, F_s(\varphi)\chi_s(g)] \\ \text{Hadamard SO(2,1)} & \delta_\varphi(g) = \sum_{j\in\mathbb{N}}[\chi_j^+(\varphi)\chi_j^-(g) + \chi_j^-(\varphi)\chi_j^+(g)] + \int \mathrm{d}s F_s(\varphi)\chi_s(g) \\ \text{Feynman SU(1,1)} & K_\varphi(g) = [\frac{-4i\sin(\varphi)}{\kappa^2}\sum_{d_j\in\mathbb{N}} \chi_j^-(\varphi - i\epsilon)\chi_j(g) + \int s\, f_s(\varphi)\chi_s(g)] \\ \text{Hadamard SU(1,1)} & \delta_\varphi(g) = \sum_{j\in\frac{1}{2}\mathbb{N}}[\chi_j^+(\varphi)\chi_j^-(g) + \chi_j^-(\varphi)\chi_j^+(g)] + \int \mathrm{d}s f_s(\varphi)\chi_s(g) \end{cases}$$



.	Cutting rule in SO(2,1) for $\theta \in [0, \pi/2]$	Cutting rule in SU(1,1) for $\theta \in [0, \pi]$
$\varphi \in [0, \frac{\pi}{4}]$	$i\mathrm{Im}\left(\mathcal{L}_\varphi(g)\right) = \mathcal{L}^0_\varphi(g)$ $= \begin{cases} 0 & \text{if } \theta \in [0, 2\varphi] \\ \dfrac{i}{8\kappa \cos\varphi^2 \sin\theta} & \text{if } \theta \in [2\varphi, \pi/2] \end{cases}$	$i\mathrm{Im}\left(\mathcal{L}_\varphi(g)\right) = \mathcal{L}^0_\varphi(g)$ $= \begin{cases} 0 & \text{if } \theta \in [0, 2\varphi] \\ \dfrac{i}{4\kappa \cos\varphi^2 \sin\theta} & \text{if } \theta \in [2\varphi, \pi] \end{cases}$
$\varphi \in [\frac{\pi}{4}, \frac{\pi}{2}]$	Failure of cutting rule	$i\mathrm{Im}\left(\mathcal{L}_\varphi(g)\right) = \mathcal{L}^0_\varphi(g)$ $= \begin{cases} 0 & \text{if } \theta \in [0, 2\varphi] \\ \dfrac{i}{4\kappa \cos\varphi^2 \sin\theta} & \text{if } \theta \in [2\varphi, \pi] \end{cases}$;

Summary and Outlook

- Unitarity is barely investigated in Noncommutative theory and LQG. ([Gomis, Mehen, 2000], [Sasai,Sasakura,2009])
- Failure of cutting rule in SO(3) and SO(2,1) (agree with [Sasai,Sasakura,2009])
- Unitarity is verify in SU(2) and SU(1,1) at one loop level.
- Anstaz for SU(2) Feynman propagator $\Longrightarrow$ Can we build other Feynman propagators which verify cutting rule ?
- Unitarity at one loop level check $\Longrightarrow$ What for other orders ?

Thank you for your attention