A new 2+1 coherent spin-foam vertex for quantum gravity

Based on

Biquaternions, Manjorana spinors and time-like spin-foams, ArXiv:2401.10324
A new 2+1 coherent spin-foam vertex for quantum gravity, ArXiv:2402.05993

[redacted]

FRIEDRICH-SCHILLER-UNIVERSITÄT JENA

José Diogo Simão
Friedrich-Schiller University of Jena

Emmy Noether-Programm
DFG Deutsche Forschungsgemeinschaft

The route for this talk

1

Spin-foams and time-like regions

On spin-foam models of quantum space-time,
Lorentzian signature and causal character

2

Constructing a 2+1 model

Spinors, coherent states and a vertex amplitude

1

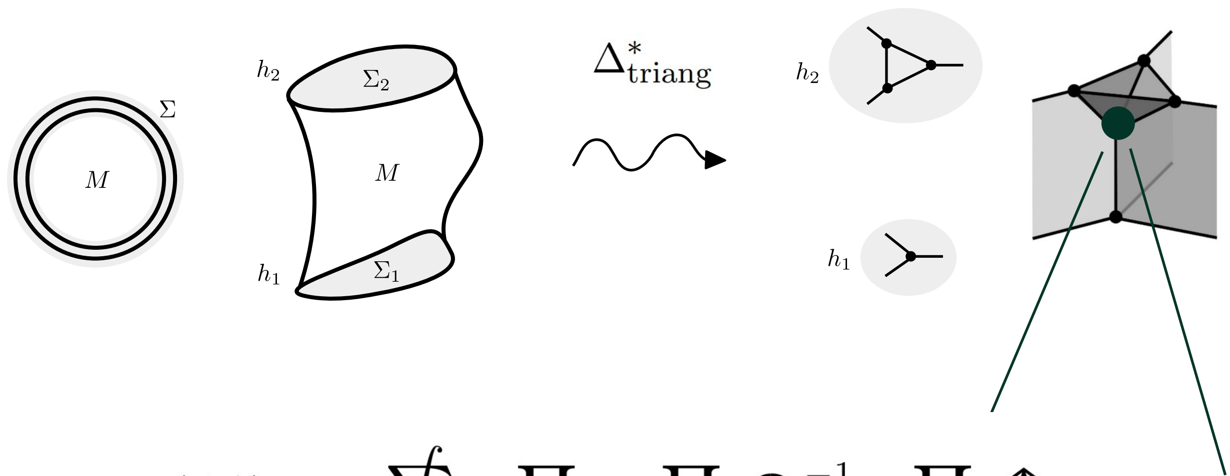
Spin-foams and time-like regions

On spin-foam models of quantum space-time,
Lorentzian signature and causal character

The general idea for spin-foams

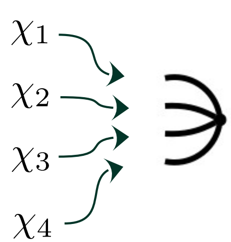
- Quantization of gravity via “sum over histories”, reminiscent of transition amplitudes between 3d boundaries.

Reisenberger, Baez, Freidel, Krasnov, Barret, Crane, Livine, Engle, Rovelli, …

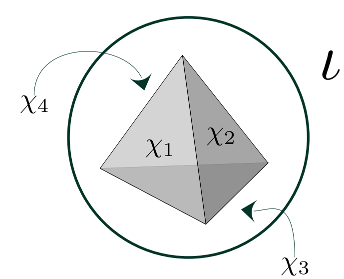


$$Z(\Delta^*) = \sum\!\!\!\!\!\!\int_{\chi\to f,\,\iota\to e} \prod_f d_f \prod_e \ominus^{-1}_{\iota,\{\chi\}} \prod_v ⛤_{v,\{\iota\},\{\chi\}}$$

- States are *graphs* decorated with objects from the representation theory of the Lorentz group: *spins* and *intertwiners*.



$$\longleftarrow \iota \in \mathrm{Inv} \bigotimes_{i=1}^{4} \mathcal{H}^{\chi_i}$$

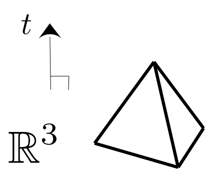


Simplicity constraints, causal character

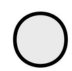
$$Z_{BF}(\Delta^*) = \sum\!\!\!\!\!\!\int_{\chi \to f,\, \iota \to e} \prod_f d_f \prod_e \ominus^{-1}_{\iota,\{\chi\}} \prod_v \star_{v,\{\iota\},\{\chi\}}$$

- Gravity = topological *BF* theory + *simplicity constraints.*
- Generically, *simplicity* requires picking a *causal character* for each tetrahedron.

Engle, Pereira, Livine, Rovelli '08



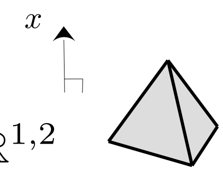
$\mathbb{R}^3$ space-like



SU(2) isometries

$\chi \longleftarrow$ spins $j \in \mathbb{N}/2$

Conrady, Hnybida ´10



$\mathbb{R}^{1,2}$ time-like

SU(1, 1) isometries

$\chi \longleftarrow$ spins $k \in -\mathbb{N}/2,\ s \in \mathbb{R}^+$

- *Simplicity* simply **restricts** the sum over $\mathrm{SL}(2,\mathbb{C})$ spins, yielding a model for quantum gravity:

$$Z_{G?}(\Delta^*) = \sum\!\!\!\!\!\!\int_{\chi \to f,\, \iota \to e} \Bigg|_{\text{simplicity}} \prod_f d_f \prod_e \ominus^{-1}_{\iota,\{\chi\}} \prod_v \star_{v,\{\iota\},\{\chi\}}$$

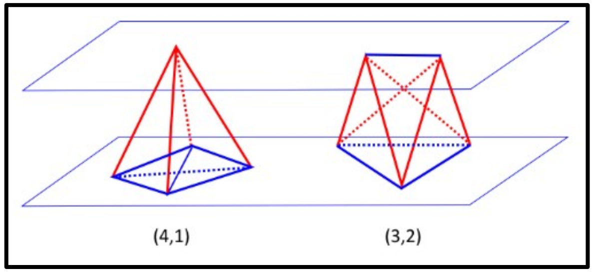
The role of time-like regions

- The EPRL model is most popular in the space-like case, where it is comparatively simpler. But!

We must better understand time-like amplitudes!

→ Sum-over-histories picture suggests considering every causal case;

→ Approaches like CDT requires time-like tetrahedra and triangles;



Loll ´19, *(4,1) and (3,2) moves*

→ Cosmological applications make use of time-like regions;

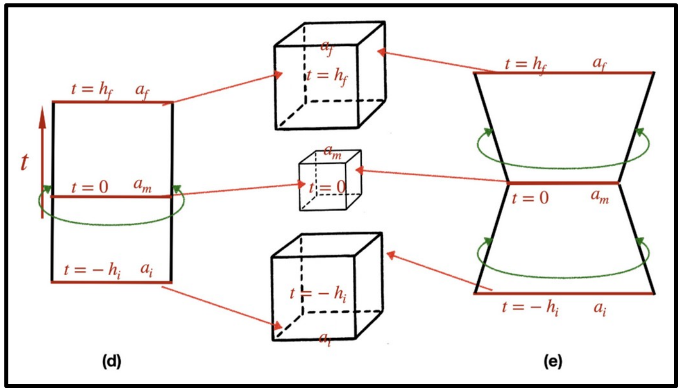
Han, Liu, Qu, Vidotto, Zhang '24

→ Avoiding causality violations requires both types;

Asante, Dittrich, Padua-Argüelles '21
Jercher, Steinhaus '24

Observation:

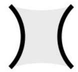
*EPRL time-like amplitude asymptotic limit is **not under control***



Han et al, '24, *hypercube discretization of bouncing Friedmann universe*

Asymptotics of EPRL time-like amplitudes

$$A_v = \int_{\mathrm{SL}(2,\mathbb{C})} \prod_{a=1}^{n} \mathrm{d}g_a \delta(g_n) \prod_{a<b} \int_{\mathbb{C}P} \omega(z_{ab}) f_{ab}(z_{ab}, g_a, g_b) e^{\Lambda S_{ab}(z_{ab}, g_a, g_b)}$$

- Define $\vec{n}^{\beta}_{ab} = \pi_{\mathrm{SU}(1,1)}(n_{ab}) \left[-\hat{e}_2 + \Im\beta_{ab}(\hat{e}_0 - \hat{e}_1)\right] \in H^{\mathrm{sl}}$ a space-like 3-vector  associated to the boundary data. One finds for the asymptotic critical points:

Liu, Han '18
JDS, Steinhaus '21

- **under-constrained:** *boundary does not select geometry.*

$$\pi(g_a)^{\wedge 2}(\vec{n}^{\beta}_{ab}, 0) \wedge (\vec{0}, 1) = \pi(g_b)^{\wedge 2}(\vec{n}^{\beta}_{ba}, 0) \wedge (\vec{0}, 1)$$

$$\textstyle\sum_{b:\,\mathrm{s.l.}\,ab} j_{ab}\vec{n}_{ab} + \sum_{b:\,\mathrm{t.l.}\,ab} s_{ab}\vec{n}^{\beta}_{ab} = 0$$

$$\Re\beta_{ab} = 0$$

→ critical line $\Im\beta_{ab} \in \mathbb{R}$

- **critical points:** *coalesce with branch cut of the integrand.*

$$f_{ab}(z_{ab}, g_a, g_b) = \frac{1}{2}\overline{\tilde{A}}^{-\frac{1}{2}+i\Lambda s_{ab}}_{-\gamma\Lambda s_{ab}} A^{-\frac{1}{2}+i\Lambda s_{ab}}_{-\gamma\Lambda s_{ab}} \Theta(\Re\beta_{ab})\Theta(\Re\beta_{ba})$$

$$\cdot\ |\alpha_{ab}\alpha_{ba}|^{-2} (\Re\beta_{ab})^{-\frac{1}{2}} (\Re\beta_{ba})^{-\frac{1}{2}} \left[(2\Lambda s_{ab} - i)\,\Im\beta_{ab} - 2\gamma\Lambda s_{ab}\,\Re\beta_{ab}\right]$$

Asymptotics of EPRL time-like amplitudes

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Hörmander's theorem on asymoptotic approximation of integrals is ***not applicable!***

Hörmander '03 *The Analysis of Linear Partial Differential Operators I*

↓

Understand time-like amplitudes by <u>studying a lower-dimensional model</u>

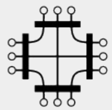
2

Constructing a 2+1 model

Spinors, coherent states and a vertex amplitude

The coherent Ponzano-Regge model as template

- The Ponzano-Regge model of 3d Riemannian gravity is very well understood:



$$= \int_{\mathrm{SU}(2)} \prod_{a=1}^{4} \mathrm{d}g_a \prod_{a\neq b} \langle j_{ab}, n_{ab} | D^{j_{ab}}(g_a)^\dagger D^{j_{ab}}(g_b) | j_{ab}, n_{ba} \rangle$$

Ponzano, Regge '68

- Each diagram edge yields a pairing of boundary states through the inner product on the SU(2) unitary irrep. of spin j.

$$\mathcal{H}^j \subset \bigotimes_{i=1}^{2j} \mathcal{H}^{\frac{1}{2}} \quad \Rightarrow \quad \langle j_{ab}, n_{ab} | D^{j_{ab}}(g_a)^\dagger D^{j_{ab}}(g_b) | j_{ab}, n_{ba} \rangle = \left\langle \tbinom{1}{0} \middle| n_{ab}^\dagger g_a^\dagger g_b n_{ba} \middle| \tbinom{1}{0} \right\rangle^{2j_{ab}}$$

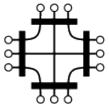
with $g_a \in \mathrm{SU}(2)$ integrated over
and $n_{ab} \in \mathrm{SU}(2)$ boundary data

- Defining $|z_{ab}\rangle := n_{ab} \tbinom{1}{0} \in \mathbb{C}^2$, each term $\langle z_{ab} | g_a^\dagger g_b | z_{ba} \rangle^{2j_{ab}}$ amounts to:

1) An SU(2) pairing of Weyl spinors, each corresponding to an element of the sphere;

$$|z\rangle \;\; \text{s.t.} \;\; \langle z|z\rangle^2 = 1 \;\mapsto\; v_z^i = \langle z|\sigma^i|z\rangle \in S^2$$

2) Weighted by the degree of classicality: the spins.



$$\underset{j\to\infty}{\sim} \cos S_{\mathrm{Regge}}$$

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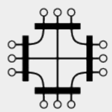

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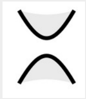
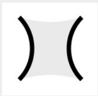
with $g_a \in \mathrm{SU}(2)$ integrated over
and $n_{ab} \in \mathrm{SU}(2)$ boundary data

The strategy

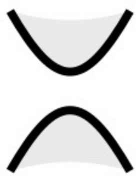
Find which spinors correspond to $\mathbb{R}^{1,2}$ vectors

$H^\pm \sim$  $H^{\mathrm{sl}} \sim$ 

Construct SU(1,1) boundary states inducing such spinors

Weyl and Majorana for Minkowski 3-space

- Underlying structure of mapping from spinors to sphere is *complexified quaternions*. Use this to find:



$$\mathbb{C}^2/\mathrm{U}(1) \;\to\; \mathbb{C}^{2\times 2}_{\substack{\det q=0 \\ \sigma_3 q \sigma_3 = -q^\dagger}} \;\to\; \left(\mathbb{R}^{1,2}_{|\cdot|^2\geq 0,\, v^1\geq 1}, \eta_{(1,2)}\right)$$

$$|z\rangle \;\mapsto\; q_z = -i|z\rangle\langle z|\sigma_3 \;\mapsto\; v^i_z = \frac{i}{2}\mathrm{Tr}\left[\varsigma^i q_z\right]$$

$$\varsigma := (\sigma_3, i\sigma_2, -i\sigma_1)$$

$$|z\rangle := \begin{pmatrix} z_1 \\ z_2 \end{pmatrix}$$

$$(\mathbb{C}_{\mathrm{Majorana}} \oplus \mathbb{C}_{\mathrm{Majorana}})/\mathrm{U}(1) \;\to\; \mathbb{C}^{2\times 2}_{\substack{\det q=0 \\ RqR^{-1}=q}} \;\to\; \left(\mathbb{R}^{1,2}_{|\cdot|^2\leq 0}, \eta_{(1,2)}\right)$$

$$|z^1\rangle, |z^2\rangle \;\mapsto\; q_z = -i|z^1\rangle\langle z^2|\sigma_3 \;\mapsto\; v^i_z = \frac{i}{2}\mathrm{Tr}\left[\varsigma^i q_z\right]$$

$$R: \;\mathbb{C}^2 \to \mathbb{C}^2$$

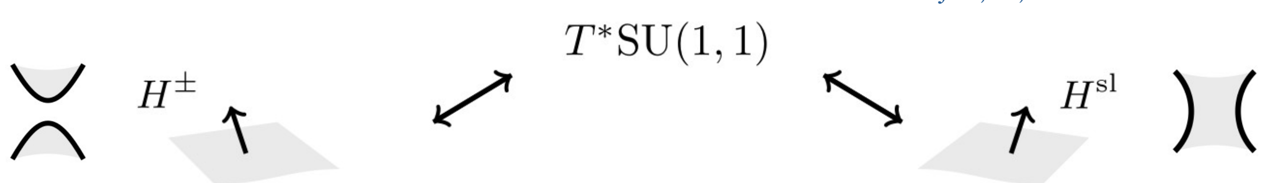
$$|z\rangle \mapsto \sigma_1|\overline{z}\rangle$$

$$|z_i\rangle = \begin{pmatrix} z_i \\ \overline{z}_i \end{pmatrix}$$

$$R|z_i\rangle = |z_i\rangle$$

charge conjugation

JDS, '24, ArXiv:2401.10324



In the spirit of *twisted geometries*, Speziale, Wieland, Speziale, Freidel, Livine, ...

$$\omega = i(\mathrm{d}z_1 \wedge \mathrm{d}\overline{z}_1 - \mathrm{d}z_2 \wedge \mathrm{d}\overline{z}_2) \qquad\qquad \omega = i(\mathrm{d}z_1 \wedge \mathrm{d}\overline{z}_2 - \mathrm{d}\overline{z}_1 \wedge \mathrm{d}z_2)$$

K^2 generalized eigenbasis

- SU(1,1) continuous series in K^2 generalized eigenbasis:

$$K^2|j,\lambda,\sigma\rangle = \lambda|j,m\rangle\,, \quad \lambda\in\mathbb{C}\,,$$
$$P|j,\lambda,\sigma\rangle = (-1)^\sigma|j,\lambda,\sigma\rangle\,, \quad \sigma\in\{0,1\}$$

- Consider coherent states at $\lambda = ij$, minimizing $\langle\Delta|(L^3,K^1,K^2)|\rangle = -s^2 - \frac{1}{4} + |\lambda|^2\,,$

$$|j,g\rangle := D^j(g)|j,ij,\sigma\rangle\,, \quad g\in\mathrm{SU}(1,1)$$

spans continuous series of spin j

$\mathcal{H}^j = \mathrm{span}\,\{|j,g\rangle\}_{g\in\mathrm{SU}(1,1)}$

- Matrix elements are *very complicated*, and diverge at $\lambda=\lambda'$:

$$\langle j,\overline{\lambda},\sigma|D^j(e^{-i\alpha L^3})|j,\lambda',\sigma'\rangle_{\mathrm{reg}} =$$
$$\lim_{\epsilon\to0}\frac{1}{2\pi}\left\{\frac{\Gamma(\frac{-\overline{j}+\sigma-i\lambda}{2})\Gamma(\frac{-j+\sigma'+i\lambda'}{2})}{\Gamma(\frac{-j+\sigma+i\lambda}{2})\Gamma(\frac{-\overline{j}+\sigma'-i\lambda'}{2})}\Gamma(i\Delta\lambda+\epsilon)\psi_-(\alpha)\right.$$
$$\left.+(-1)^{2\delta}\frac{\Gamma(\frac{-\overline{j}+2\delta+(-1)^{2\delta}\sigma+i\lambda}{2})\Gamma(\frac{-j+2\delta+(-1)^{2\delta}\sigma'-i\lambda'}{2})}{\Gamma(\frac{-j+2\delta+(-1)^{2\delta}\sigma-i\lambda}{2})\Gamma(\frac{-\overline{j}+2\delta+(-1)^{2\delta}\sigma'+i\lambda'}{2})}\Gamma(-i\Delta\lambda-\epsilon)\psi_+(\alpha)\right\}$$
$$\cdot\cos\frac{\pi}{2}(i\Delta\lambda+\sigma-\sigma')\,,$$

$$\psi_\pm(\alpha) = \cos\left(\frac{\alpha}{2}\right)^{-2j-2}\left|2\tan\frac{\alpha}{2}\right|^{\pm i\Delta\lambda}$$
$$\cdot\,{}_2F_1\left(j+1\pm i\lambda, j+1\mp i\lambda', 1\pm i\Delta\lambda; -\tan^2\frac{\alpha}{2}\right)\mathrm{sgn}^{\sigma-\sigma'}\alpha$$

but we can regularize it!

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spans continuous series of spin j

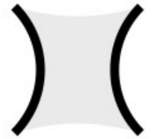
$$\mathcal{H}^j = \mathrm{span}\,\{|j,g\rangle\}_{g\in\mathrm{SU}(1,1)}$$

- Matrix elements are *very complicated*, and diverge at $\lambda=\lambda'$:

SU(1,1)-invariant pairing

note
complex conjugation

$$\langle j,\overline{ij},\sigma|D^{j\dagger}(g')D^j(g)|j,ij,\sigma\rangle_{\mathrm{reg}} = -\frac{\gamma}{\pi}[g'\cdot l^+|g\cdot l^-]^{2j}$$
$$= -\frac{\gamma}{\pi}\left(-i[z^+_{g'}|z^-_g]\right)^{2j}$$



H^{sl}

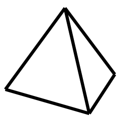
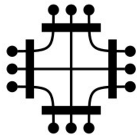
$$\wr$$
$$\{|z^+]\,,|z^-]\}$$

Majorana spinors,
weighted by the spin

The new model

- Tetrahedron with fully **time-like** edges:

Perelomov coherent states

 $\sim$ 

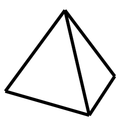
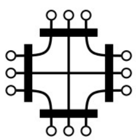
$$= \int_{\mathrm{SU}(1,1)} \prod_{a=1}^{3} g_a \prod_{a<b} d_{k_{ab}} \cdot \langle q k_{ab}, n_{ab} | D^{k_{ab}(q)_{ab}}(g_a)^{\dagger} D^{k_{ab}(q)_{ab}}(g_b) | q k_{ab}, n_{ba} \rangle$$

Discrete series inner product

$$= \int_{\mathrm{SU}(1,1)} \prod_{a=1}^{3} g_a \prod_{a<b} d_{k_{ab}} [g_a \cdot (-q)_{ab} | g_b \cdot (-q)_{ba}]^{2k_{ab}}$$

- Tetrahedron with fully **space-like** edges:

New coherent states

 $\sim$ 

$$= \int_{\mathrm{SU}(1,1)} \prod_{a=1}^{3} g_a \prod_{a<b} d_{s_{ab}} \mathcal{C}_{g_a n_{ab}, g_b n_{ba}} \cdot [j_{ab}, n_{ab} | D^{j_{ab}(0)}(g_a)^{\dagger} D^{j_{ab}(0)}(g_b) | j_{ab}, n_{ba} \rangle$$

Continuous series inner product

$$= \int_{\mathrm{SU}(1,1)} \prod_{a=1}^{3} g_a \prod_{a<b} \left(-\frac{\gamma}{\pi} d_{s_{ab}} e^{s_{ab} [g_a \cdot l_{ab}^{+} | g_b \cdot l_{ba}^{+}]^2} \right) [g_a \cdot l_{ab}^{+} | g_b \cdot l_{ba}^{-}]^{-1+2is_{ab}}$$

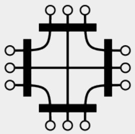
Can include any combination of space- and time-like edges

Can be generalized to higher polyhedra

JDS, '24, ArXiv:2402.05993

The new model

- Tetrahedron with fully space-like edges: *(analog of EPRL 4d time-like amplitudes)*



$$= \int \prod_{a=1}^{3} g_a \prod_{a<b} d_{s_{ab}} \mathcal{C}_{g_a n_{ab}, g_b n_{ba}} \cdot [j_{ab}, n_{ab}| D^{j_{ab}(0)}(g_a)^\dagger D^{j_{ab}(0)}(g_b) | j_{ab}, n_{ba}\rangle$$

$$= \int \prod_{a=1}^{3} g_a \prod_{a<b} \left(-\frac{\gamma}{\pi} d_{s_{ab}} e^{s_{ab}[g_a \cdot l_{ab}^+ | g_b \cdot l_{ba}^+]^2} \right) [g_a \cdot l_{ab}^+ | g_b \cdot l_{ba}^-]^{-1+2is_{ab}}$$

Gaussian peaked at glued triangles

observations

1) space-like pairing needs additional gluing constraint;

2) large-spin limit yields the exp Regge action. *Generically* no cosine in asymptotics!

Jercher, JDS, Steinhaus '24

3) continuous series coherent states need both *regularization* and additional *constraint*. Same behavior in 4d.

4) *Preliminary:* topological invariance requires summing over **both** space- and time-like bulk edges.

Overview and outlook

- Both space- and time-like regions should be considered in the spin-foam sum;
- The time-like EPRL amplitude may be ill-defined;
- An analog 3d model can be constructed which elucidates some of the obstacles;

Moving forward

→ 2+1 cosmology with space- and time-like regions;

→ Absence of 'cosine problem' allows identification with effective spin-foams;

→ Generalization to cosmological constant;

→ Inclusion of fermions *a la* Fairbairn from first principles;