Cross-section Continuity of Angular Momentum

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The Metric Near Null Infinity

Class of metrics near $\mathscr{I}^+$ can be written down in $1/r$ expansion $(x^A = (\theta, \phi))$:

$$ds^2 = -du^2 - 2dudr + r^2 h_{AB}dx^A dx^B + \frac{2\mathbf{m}}{r}du^2 + r\mathbf{C_{AB}}dx^A dx^B + D^B C_{AB}dudx^A + \frac{4}{3r}\mathbf{N_A}dudx^A + ...$$

- Bondi mass aspect $m(u)$
- Shear tensor $C_{AB}(u, x^A)$
- Angular momentum aspect $N_A(u, x^A)$

NB: a 'cross-section' of $\mathscr{I}^+$ is 2-sphere labelled by u (Bondi frame).

The Supertranslation Ambiguity

$$BMS_4 = SO(3,1) \ltimes S$$

Supertranslations are coupled to rotations. Cannot distinguish between 'rotation' and 'rotation + supertranslation'. Cannot recover unique Lorentz subgroup, rather infinite number, each differing by supertranslation.

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Ambiguity Example

On $u = 0$ cross-section call $X = Y^A \frac{\partial}{\partial x^A}$ 'rotation' (Y^A is rotational Killing field on $u =$const sphere).

On $u' = 0$ cross-section, where $u' = u - f(x^A)$ and $x'^A = x^A$, symmetry is $X = Y^A \frac{\partial}{\partial x'^A} - Y^A D_A f \frac{\partial}{\partial u'} \to$ 'rotation + supertranslation' $-Y^A D_A f$.

Why? Different foliations!

Immediate consequence: cannot distinguish ‘angular momentum’ and ‘angular momentum + supertranslation charge’. How to proceed?

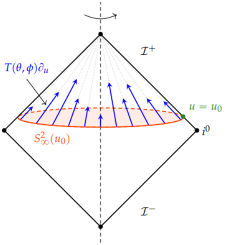

Figure: *Compere*, 1801.07064v4. Supertranslations T act by “wiggling” cross-sections of $\mathscr{I}$.

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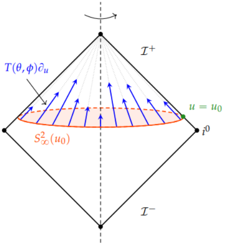

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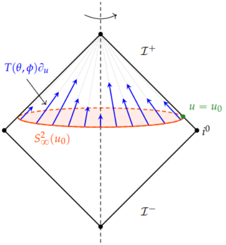
2. Use data (physical fields) on cross-section to determine what should be pure rotation.



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The Dray Streubel Angular Momentum

DS angular momentum

$$J^{DS}|_{u=0} = \frac{1}{8\pi}\int Y^A\Big(N_A - \frac{1}{4}C_{AB}D_D C^{DB}\Big)$$

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All combined into one-parameter family by Compere and Nichols:

$$J^{CN}|_{u=0} = \frac{1}{8\pi}\int Y^A\Big(N_A - \frac{1}{4}C_{AB}D_D C^{DB}\Big) - \frac{\alpha-1}{32\pi}\int Y^A C_{AB}D_C C^{BC}$$

The Chen-Wang-Yao Angular Momentum

Arose from attempts to eliminate supertranslation ambiguity:

$$J^{CWY}|_{u=0} = \frac{1}{8\pi}\int Y^A\Big(N_A - \frac{1}{4}C_{AB}D_D C^{DB}\Big) + \frac{1}{8\pi}\int mY^A D_A c$$

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Note: CWY angular momentum does not reduce to Komar formula for arbitary cross-section in presence of axial Killing field.

The Cross-section Continuity Criterion

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Cross-section Continuity Condition

Let $J : \mathscr{C} \to \mathbb{R}$ be map from space, $\mathscr{C}$, of smooth cross-sections of $\mathscr{I}$ into $\mathbb{R}$. Then J satisfies cross-section continuity condition at $u = 0$ if for any sequence $\{f_n\}$ of smooth functions on sphere such that $f_n \to 0$ uniformly as $n \to \infty$ we have $J|_{u=f_n} \to J|_{u=0}$.

Flux of Angular Momentum

Due to presence of gravitational radiation, these “conserved” quantities not actually conserved: difference between two cross-sections should be encoded in flux of angular momentum.

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DS angular momentum has well-defined flux and obeys balance law:

$$J^{DS}|_{\mathscr{C}_2} - J^{DS}|_{\mathscr{C}_1} = \int_R \mathscr{F} = -\frac{1}{32\pi} \int N^{AB} \pounds_Y C_{AB}$$

Balance law is necessary criteria and cross-section continuity can be used as tool to check if satisfied.

Results and Conclusion

Dray-Streubel definition is c.c. since it arises from local flux. So use bootstrapping argument for other definitions: investigate differences $J^{CN} - J^{DS}$ and $J^{CWY} - J^{DS}$.

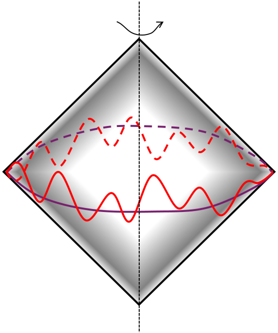

Figure: Cross section continuity means $J|_{u=0}$ approaches $J|_{u=f_n}$ as $n \to \infty$.

$$\{\mathbf{f_n}\} = \mathbf{F}(\theta)\frac{\mathbf{1}}{\mathbf{n}}\sin(\mathbf{n}\phi)$$

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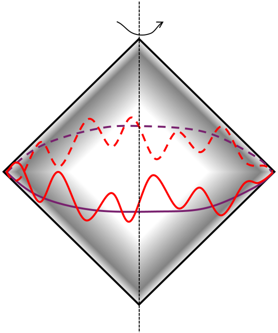

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$$\{\mathbf{f_n}\} = \mathbf{F}(\theta)\frac{\mathbf{1}}{\mathbf{n}}\sin(\mathbf{n}\phi)$$

CN one-parameter family only c.c. when $\alpha = 1$. Only DS definition is viable, generalizations are not. CWY definition is also c.c.

Final Remarks

- Cross-section continuity is natural condition to impose: $u =$const cross-sections on $\mathscr{I}$ represent 'instances of time'.
- Not trivial to see functions on $\mathscr{I}$ satisfy c.c. due to nonlinear transformation laws, unless local flux exists.
- Criterion introduced via angular momentum, but should be useful more broadly to generic functions of cross-sections in dynamical (radiative) eras (c.f. *S. Hollands, R. M. Wald, and V. G. Zhang*).