Canonical vs. Covariant Hamiltonian formalism for gauge theories

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Motivation

- Classical Hamiltonian analysis of gauge theories has evolved along two (main) different paths, known as Covariant and Canonical methods. Even when they are based on different structures, they are expected to be equivalent in some form.
- Inclusions of boundaries introduces new challenges for both methods, but assumption of equivalence persists.
- Here we are interested in exploring the possible equivalence of symplectic structures, and the corresponding phase space.
- Simplest gauge theory (Maxwell) is too simple and may be misleading ...

Outline:

- Hamiltonian analysis: Canonical vs. covariant
- Inclusions of boundaries
- Equivalence of symplectic structures?

Canonical vs Covariant: Comparison for gauge theories

Canonical	*Covariant*
CanPS naturally arise as symplectic manifold	CovPS is space of solutions of the field equations
3+1 decomposition	Constructed from entire histories
There are constraints $(T^*\mathcal{Q},\Omega)$	Degenerate symplectic structure $(\mathrm{Sol}(M),\omega)$
H is non-unique For every H the HVF is unique	H is well defined HVFs are non-unique
In DiffInv theories H has BulkTerm + BoundaryTerm	In DiffInv theories H has only a BoundaryTerm
Starting point for quantization	Quantization?

(A. Ashtekar, L. Bombelli and O. Reula, 1991)

Common starting point

- Set of fields $\phi_a^A(x)$ and a covariant first order action

$$S[\phi^A] = \int_M \mathbf{L}(\phi^A, \nabla\phi^A) + S_{\rm BT}$$

- $M \subseteq \mathcal{M} = \mathbb{R} \times \Sigma$, is a region, with a boundary, of a globally hyperbolic spacetime.
- The boundary of M: $\partial M = \Sigma_1 \cup \Sigma_2 \cup \tau_\infty \cup \Delta$, where τ_∞ is an asymptotic region, Δ is an internal boundary, and Σ is a Cauchy surface.
- In order to have a 'well defined variational principle' (for given boundary conditions) or/and a finite action, one might need to add boundary terms to the action, $S_{\rm BT}$.

Canonical approach

- The starting point is a canonical action that is obtained after the 3+1 decomposition of a covariant action.

$$S[\Phi, P] = \int \{P[\mathcal{L}_t \Phi] - H\} \, \mathrm{d}t$$

where $P[\mathcal{L}_t \Phi]$ is the kinetic term,

$$P[\mathcal{L}_t \Phi] = \int_\Sigma \mathrm{d}^3x \; \tilde{P}_A \, \mathcal{L}_t \Phi^A + \int_{\partial\Sigma} \mathrm{d}^2y \; \tilde{\pi}_j \, \mathcal{L}_t \alpha^j$$

- $(\Phi^A(x), \tilde{P}_A(x))$ are the bulk canonical variables
- $(\alpha_j(y), \tilde{\pi}^j(y))$ are interpreted as boundary DOF (in some cases, not always, they are a pullback of the bulk variables to $\partial\Sigma$).

(A. C. and T. Vukasinac., 2020)

Symplectic structure and the Hamiltonian

- The kinetic term determines the pre-symplectic structure of the theory as

$$\Omega = \int_{\Sigma} \mathrm{d}^3x \, \mathbb{d}\tilde{P}_A \wedge\!\!\!\wedge \, \mathbb{d}\Phi^A + \int_{\partial\Sigma} \mathrm{d}^2x \, \mathbb{d}\tilde{\pi}^j \wedge\!\!\!\wedge \, \mathbb{d}\alpha_j$$

- Hamilton's equations of motion are generated by a preferred function, the Hamiltonian H

$$\mathbb{d}H(Y) = \Omega(Y, X_H)$$

- The form of the Hamiltonian:

$$H(\Phi^A, \tilde{P}_A) = \int_{\Sigma} (\mathcal{H}_c + u^j C_j) + H_{\mathrm{BT}}$$

where $C_j(\Phi^A, \tilde{P}_A) = 0$ are the bulk FC constraints.

- In DiffInv theories $\mathcal{H}_c = 0$.

Boundaries

There are two possibilities depending on the form of the canonical action:

- Ω has vanishing contribution from the boundary. This corresponds to the standard Regge-Teitelboim scenario. Examples:
  - Maxwell, Maxwell + Pontryagin
  - General relativity with asymptotically flat boundary conditions
- Ω has non vanishing contribution from the boundary. Then, there is a boundary contribution to HVF, gradients and Poisson brackets. Examples:
  - Maxwell + Chern-Simons
  - First order gravity with an isolated horizon as an internal boundary

Covariant approach: Construction of ω

$$\mathbb{d}S = \int_M E_A \wedge\!\!\!\wedge \mathbb{d}\phi^A + \int_{\partial M} \theta$$

- If the BT vanishes, due to boundary conditions, then $E_A = 0$ are (bulk) Euler-Lagrange EOM.
- Symplectic current, $J = \mathbb{d}\theta$, is conserved since

$$\mathbb{d}^2 S = 0 \quad \Rightarrow \quad \int_{\partial M} J = 0$$

- When the boundary conditions, on fields and their variations, guarantee that $\int_{\tau_\infty} J = 0$ and , $\int_\Delta J = 0$, then $\int_{\Sigma_1} J = \int_{\Sigma_2} J$, and we can define a (pre)-symplectic structure as

$$\omega = \int_\Sigma J$$

Subtleties

- In general, θ is of the form

$$\theta = B_A \wedge\!\!\!\wedge \mathrm{d\!\!d}\Phi^A + \mathrm{d}\tilde{\theta}$$

Then, $B_A = 0$ are boundary EOM or boundary conditions. In some cases we need to add a BT to the original covariant action.

- $\tilde{\theta}$ is arbitrary and does not contribute to ω, as well as any topological term that the action might have.
- In first order gravity, when Δ is an isolated horizon, one obtains $J|_\Delta = \mathrm{d}j$. In that case, ω acquires a boundary term

$$\omega = \int_\Sigma J + \int_{S_\Delta} j$$

where $S_\Delta = \Sigma \cap \Delta$. Also, in that case there is no contribution to ω from the asymptotic region, for asympt. flat spacetimes.

(AC, I. Rubalcava-García and T. Vukasinac, 2016; AC, J. D. Reyes and T. Vukasinac, 2017)

Hamiltonian

- Consider the vector field $X_\xi := \mathcal{L}_\xi \Phi^A$. It is a degenerate direction of ω if
$$\omega(X_\xi, X) = 0\,, \quad \forall X$$
For DiffInv theories this expression only has contributions from the boundaries. If $\xi = 0$ on the boundary, the corresponding HVF generates gauge transformations.
- In some cases, nonvanishing ξ at the boundary also defines a gauge direction.
- When $\xi \neq 0$ on ∂M we can obtain a Hamiltonian H_ξ as
$$\mathbb{d} H_\xi(Y) = \omega(Y, X_\xi)$$
- H_ξ is the Hamiltonian conserved charge associated to the symmetry generated by ξ.
- $H_{\text{can}} \approx H_{\text{cov}}$

Equivalence of the symplectic structures?

- In general, many aspects of the two approaches are equivalent, but the equivalence of Ω and ω has been proven only in very simple concrete examples (without boundaries). For instance, for Maxwell theory, both presimplectic Ω and ω look the same, even when they are defined on different spaces (space of solution vs initial data satisfying Gauss' law).

- Recently, a strong claim regarding the equivalence of both symplectic structures was put forward, for rather general first order theories in regions with boundaries.(J. Margalef-Bentabol and E. J. S. Villaseñor, 2022)

- Problem: What equivalence means is not well defined! Recent proposal for clarifying equivalence put forward (AC, J. D. Reyes and T. Vukasinac, 2023)

- The idea is to define a canonical mapping $\bar{\Pi} : \Gamma_{\rm cov} \to \bar{\Gamma}_{\rm can}$

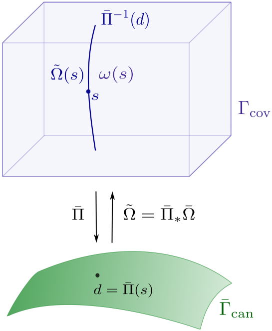

Figure: $\tilde{\Omega}$, the pullback under the canonical projection map of the pre-symplectic structure $\bar{\Omega}$ on the constraint surface, defines a pre-symplectic structure on the covariant phase space. Directions $\delta\mathbf{A}$ along the fiber $\bar{\Pi}^{-1}(d)$ are degenerate directions of $\tilde{\Omega}(s)$. Since $\tilde{\Omega}$ is closed, by Cartan's formula then $\mathcal{L}_{\delta\mathbf{A}}\tilde{\Omega} = 0$. So $\tilde{\Omega}$ also has a well defined projection.

- So to check for equivalence one needs to compare ω with $\tilde{\Omega}$.
- There are examples (of first order theories), where the Non-equivalence is clear: Maxwell + Pontryagin.
  In covariant theory there is only bulk term in ω, corresponding to Maxwell SS.
  In the canonical description we have an extra boundary term (Chern-Simons-like)! (AC, J. D. Reyes and T. Vukasinac, 2023)
- Even in a very simple topological theory, Pontryagin, both ω and Ω look very different. In covariant $\omega = 0$ on all possible configurations. The canonical Ω is non-zero in the kinematical PS. It is the pullback to the constraint surface that is fully degenerate (and therefore, zero).
- Even more, we have shown that, in case of gravity in first order formalism, the symplectic structures on WIH are not equivalent. (AC, J. D. Reyes and T. Vukasinac, 2024)
- Open question: Is it possible to understand when and why we have $\Omega \approx \omega$?

THANK YOU!

https://arxiv.org/abs/2312.10229