UNIVERSITY OF ALBERTA

Fermions in Loop Quantum Gravity

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Objective

- Study constraints in the presence of fermions coupled to gravity

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- Modified Ashtekar-Barbero variables

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- Analyze possible issues
  - Modifications to fermionic field and constraints

Vacuum

Holst action

$$S = \frac{1}{2\kappa} \int d^4x |e| e_I^a e_J^b P_{KL}^{IJ} F_{ab}^{KL}(\omega)$$

where

$$P^{IJ}{}_{KL} = \delta_K^{[I} \delta_L^{J]} - \frac{1}{2\gamma} \varepsilon^{IJ}{}_{KL} \,, \qquad \kappa = 8\pi G \,,$$

$$F^{IJ}(\omega) = d\omega^{IJ} + \omega^{IK} \wedge \omega_K{}^J \,, \qquad \omega_a^{IJ} = e^{bI} \nabla_a e_b^J$$

with $\varepsilon^{0ijk} = \varepsilon_{tabc} = 1$.

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with $\varepsilon^{0ijk} = \varepsilon_{tabc} = 1$.

- From variation of the action w.r.t. ω_a^{IJ}, compatibility condition

$$P^{KL}{}_{IJ} \mathcal{D}_b \left(|e| e_K^{[a} e_L^{b]} \right) = 0$$

- Using the parametrization

$$e^a_I = \mathcal{E}^a_I - n^a n_I$$

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- Introducing the following variables

$$E^a_i := \sqrt{\det h}\,\mathcal{E}^a_i$$ **densitized triad**

$$K^i_a := \omega^{0i}_a\,,$$ **extrinsic curvature**

$$\Gamma^i_a := \frac{1}{2}\varepsilon_j{}^i{}_k\omega^{jk}_a$$ **spin connection**

$$A^i_a := \gamma K^i_a + \Gamma^i_a$$ **Ashtekar-Barbero connection**

where γ is the Barbero-Immirzi parameter

Vacuum constraints

- $\mathcal{G}_i^{grav}[A,E] = \frac{1}{\kappa\gamma}\mathcal{D}_a^{(A)}E_i^a = \frac{1}{\kappa\gamma}\left(\partial_a E_i^a + \varepsilon_{ij}{}^k A_a^j E_k^a\right)$
- $\mathcal{S}_i^{grav}[A,E] = \frac{1}{\kappa\gamma}\varepsilon_{ij}{}^k K_a^j E_k^a$
- $$\mathcal{H}^{grav}[A,E] = \frac{1}{2\kappa}\frac{E_i^a E_j^b}{\sqrt{\det h}}\varepsilon^{ij}{}_k\Big\{\mathcal{F}_{ab}^k(A) - (1+\gamma^2)\varepsilon^k{}_{mn}K_a^m K_b^n - 2\frac{1+\gamma^2}{\gamma}\mathcal{D}_{[a}^{(\Gamma)}K_{b]}^k\Big\}$$
- $\mathcal{H}_a^{grav}[A,E] = \frac{1}{\kappa\gamma}E_j^b\mathcal{F}_{ab}^j(A) - \frac{(1+\gamma^2)}{\kappa\gamma}\varepsilon^j{}_{kl}E_j^b K_a^k K_b^l$

where $\mathcal{F}_{ab}^i(A) = 2\partial_{[a}\Gamma_{b]}^i + \varepsilon^i{}_{jk}\Gamma_a^j\Gamma_b^k + +2\gamma\mathcal{D}_{[a}K_{b]}^i + \gamma^2\varepsilon^i{}_{jk}K_a^j K_b^k$.

Vacuum constraints

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where $\mathcal{F}^i_{ab}(A) = 2\partial_{[a}\Gamma^i_{b]} + \varepsilon^i{}_{jk}\Gamma^j_a\Gamma^k_b + +2\gamma\mathcal{D}_{[a}K^i_{b]} + \gamma^2\varepsilon^i{}_{jk}K^j_a K^k_b$.

Hence

$$H = \int d^3x\left(-\Lambda^i\mathcal{G}_i^{grav} + N\mathcal{H}^{grav} + N^a\mathcal{H}_a^{grav} - (1+\gamma^2)\omega_t^{0i}\mathcal{S}_i^{grav}\right)$$

with $\Lambda^i = \gamma\omega_t^{0i} - \frac{1}{2}\varepsilon^i{}_{jk}\omega_t^{jk}$.

Spin connection

- Taking the variation of H w.r.t. Γ_a^i
- Using $\mathcal{G}_{\mathrm{i}}^{\mathrm{grav}} \approx 0$

$\Longrightarrow$ expression for the spin connection in terms of triads

$$\Gamma_a^i = \frac{1}{2}\varepsilon^{ijk} e_k^b \left(\partial_a e_{bj} - \partial_b e_{aj} + e_a^l e_j^c \partial_c e_{bl}\right)$$

Also S_i 2nd class constraint $\rightarrow$ constraints reduce to

- $\mathcal{G}_i^{grav}[A,E] = \frac{1}{\kappa\gamma}\mathcal{D}_a^{(A)}E_i^a \approx 0$
- $\mathcal{S}_i^{grav}[A,E] = \frac{1}{\kappa\gamma}\varepsilon_{ij}{}^{k}K_a^j E_k^a = 0$
- $\mathcal{H}^{grav}[A,E] = \frac{1}{2\kappa}\frac{E_i^a E_j^b}{\sqrt{det\ h}}\varepsilon^{ij}{}_{k}\left\{\mathcal{F}_{ab}^k(A) - (1+\gamma^2)\varepsilon^k{}_{mn}K_a^m K_b^n\right\} \approx 0$
- $\mathcal{H}_a^{grav}[A,E] = \frac{1}{\kappa\gamma}E_j^b\mathcal{F}^j{}_{ab}(A) \approx 0$

Weyl Fermions in curved space-time

Covariant derivative in curved space-time

$$\mathfrak{D}_a\Psi = \partial_a\Psi + \frac{1}{2}\omega_a^{IJ}\sigma_{IJ}\Psi$$

with $\sigma_{IJ} = \frac{1}{4}[\gamma_I, \gamma_J]$, Dirac spinors $\Psi = \begin{pmatrix}\psi_L \\ \psi_R\end{pmatrix}$ and gamma matrices defined as

$$\gamma^\mu = \begin{pmatrix} 0 & \tilde{\sigma}^\mu \\ \sigma^\mu & 0 \end{pmatrix}$$

i.e.

$$\gamma^0 = \begin{pmatrix} 0 & \mathbb{1} \\ \mathbb{1} & 0 \end{pmatrix} \qquad \gamma^i = \begin{pmatrix} 0 & -\sigma^i \\ \sigma^i & 0 \end{pmatrix}$$

which satisfy the Clifford algebra $\{\gamma^\mu, \gamma^\nu\} = -2\eta^{\mu\nu}$.

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$\Longrightarrow$ *fermionic action* in curved space-time

$$S_F = \int_M d^4x|e|\frac{-i}{2}\left(\overline{\Psi}\gamma^I e_I^a\mathfrak{D}_a\Psi - \overline{\mathfrak{D}_a\Psi}\gamma^I e_I^a\Psi\right)$$

Fermionic constraints

Decomposing the space-time as before and summing the result with the vacuum constraints,

- $\mathcal{G}_i^{tot}[A, E, \Psi, \Pi] = \mathcal{G}_i^{grav}[A, E] - \frac{1}{2}\sqrt{det\ h}\, J_i$
- $\mathcal{S}_i^{tot}[A, E, \Psi, \Pi] = -\frac{1+\gamma^2}{\gamma}\mathcal{S}_i^{grav}[A, E] + \frac{1}{2}\sqrt{\det h} J_i$
- $\mathcal{H}^{tot}[A, E, \Psi, \Pi] = \mathcal{H}^{grav}[A, E] + \frac{i}{2}\sqrt{det\ h}\, e_i^a \left(\overline{\Psi}\gamma^i\mathfrak{D}_a\Psi - \overline{\mathfrak{D}_a\Psi}\gamma^i\Psi\right)$
- $\mathcal{H}_a^{tot}[A, E, \Psi, \Pi] = \mathcal{H}_a^{grav}[A, E] - \frac{i}{2}\sqrt{det\ h}\left(\overline{\Psi}\gamma^0\mathfrak{D}_a\Psi - \overline{\mathfrak{D}_a\Psi}\gamma^0\Psi\right)$

with $J^I = \overline{\Psi}\gamma_5\gamma^I\Psi$ fermionic axial current.

From $\frac{\delta H_{TOT}}{\delta \omega_a^{IJ}} = 0$, ***new compatibility condition***:

$$\mathcal{D}_b(|e| e_K^{[a} e_L^{b]}) = -\frac{\kappa}{4} \frac{\gamma^2}{1+\gamma^2} |e| \left(\varepsilon^M{}_{KLN} e_M^a J^N - \frac{1}{\gamma} \left(e_K^a J_L - e_L^a J_K \right) \right)$$

$\Longrightarrow$ connection must be modified

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$\Longrightarrow$ connection must be modified

- Variation of the total action w.r.t. Γ_a^i
- 2nd class constraints given by $\frac{\delta H_{TOT}}{\delta \omega_t^{ij}} = 0$ and $\frac{\delta H_{TOT}}{\delta \omega_t^{0i}} = 0$

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- **New spin connection**

$$\tilde{\Gamma}_a^i = \underbrace{-\frac{1}{2} \frac{\varepsilon^{dbc}}{\sqrt{det\ h}} \left(e_a^i e_d^k \partial_b e_{ck} - 2 e_a^k e_d^i \partial_b e_{ck} \right)}_{\Gamma_a^i}$$

$$\underbrace{-\frac{\kappa}{4} \frac{\gamma^2}{1+\gamma^2} \left(e_a^i J^0 - \frac{1}{\gamma} \varepsilon^i{}_{jk} e_a^j J^k \right)}_{C_a^i}$$

New Ashtekar connection and constraints

- Modified Ashtekar connection

$$\tilde{A}^i_a = \underbrace{\gamma K^i_a + \Gamma^i_a}_{A^i_a} \underbrace{- \frac{\kappa}{4} \frac{\gamma^2}{1+\gamma^2} \left(e^i_a J^0 - \frac{1}{\gamma} \varepsilon^i{}_{jk} e^j_a J^k \right)}_{C^i_a}$$

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- New 2nd class constraints

$$\mathcal{S}_i = \frac{1}{2} \frac{\gamma}{1+\gamma^2} \sqrt{\det h} J_i \quad \text{and} \quad \varepsilon_{ij}{}^k K^j_a E^a_k = \gamma \varepsilon_{ij}{}^k C^j_a E^a_k$$

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- Fermionic covariant derivatives in terms of the new Ashtekar connection

$$\mathfrak{D}_a \Psi = \partial_a \Psi - i \tilde{A}^i_a \gamma_5 \sigma_{0i} \Psi + K^i_a \left(1 + i\gamma\gamma_5\right) \sigma_{0i} \Psi =$$
$$= \mathfrak{D}^{(\tilde{A})} \Psi + K^i_a \left(1 + i\gamma\gamma_5\right) \sigma_{0i} \Psi$$
$$\overline{\mathfrak{D}_a \Psi} = \partial_a \Psi^\dagger \gamma^0 + i \tilde{A}^i_a \Psi^\dagger \gamma_5 \sigma_{0i} \gamma^0 + K^i_a \Psi^\dagger \left(1 - i\gamma\gamma_5\right) \sigma_{0i} \gamma =$$
$$= \overline{\mathfrak{D}^{(\tilde{A})} \Psi} + K^i_a \Psi^\dagger \left(1 - i\gamma\gamma_5\right) \sigma_{0i} \gamma^0$$

Thus, constraints in terms of the new Ashtekar connection

- $\mathcal{G}_i^{tot}[\tilde{A}, E, \Psi, \Pi] = \mathcal{G}_i^{grav}[\tilde{A}, E] - \frac{1}{2}\sqrt{det\, h} J_i \approx 0$
- $\mathcal{S}_i^{tot}[\tilde{A}, E, \Psi, \Pi] = -\frac{1+\gamma^2}{\gamma}\mathcal{S}_i^{grav}[\tilde{A}, E] + \frac{1}{2}\sqrt{\det h}\, J_i = 0$
- $\mathcal{H}^{tot}[\tilde{A}, E, \Psi, \Pi] = \mathcal{H}^{grav}[\tilde{A}, E] + \frac{\gamma}{2}\frac{E_i^a}{\sqrt{\det h}}\mathcal{D}^{(\Gamma)}\left(\sqrt{\det h}\, J^i\right)$
  $+ \frac{i}{2}E_i^a\left(\overline{\Psi}\gamma^i\mathfrak{D}_a^{(\tilde{A})}\Psi - \overline{\mathfrak{D}_a^{(\tilde{A})}\Psi}\gamma^i\Psi\right) + \frac{3}{4}\varepsilon^i{}_{jk}E_i^a K_a^j J^k \approx 0$
- $\mathcal{H}_a^{tot}[\tilde{A}, E, \Psi, \Pi] = \mathcal{H}_a^{grav}[\tilde{A}, E] - \frac{i}{2}\sqrt{det\, h}\left(\overline{\Psi}\gamma^0\mathfrak{D}_a^{(\tilde{A})}\Psi - \overline{\mathfrak{D}_a^{(\tilde{A})}\Psi}\gamma^0\Psi\right)$
  $+ \frac{\gamma}{2}\sqrt{\det h}\, K_a^i J_i \approx 0$

Issues

- Until now the canonical pairs are $(\tilde{A}_a^i, E_j^b)$, (Ψ, Π) with $\Pi = i\sqrt{\det h}\Psi^\dagger$
- In this case, the fermionic symplectic term is

$$\Theta = \int d^4x \Pi \dot{\Psi} + \underbrace{\frac{i}{2}\kappa\gamma \int d^4x \Psi^\dagger \Psi e_a^i \dot{E}_i^a}_{} - \int d^4x \mathcal{L}_t(\Pi\Psi)$$

$\Longrightarrow \tilde{A}_a^i$ **acquires an imaginary correction**

Half density fermions

- Problem solved by half-density fermions

$$\Xi = \sqrt[4]{\det h}\,\Psi = \begin{pmatrix} \xi_L \\ \xi_R \end{pmatrix} \implies \Pi_{\Xi} = i\Xi^{\dagger}$$

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- New fermionic symplectic term

$$\Theta = \int d^4x \left(\Pi_{\xi_L}\dot{\xi_L} + \Pi_{\xi_R}\dot{\xi_R} \right)$$

and anti-Poisson brackets

$$\{\xi_{L_\alpha}(x), \Pi_{\xi_{L_\beta}}(y)\}_+ = \delta_{\alpha\beta}\delta(x,y)$$
$$\{\xi_{R_\alpha}(x), \Pi_{\xi_{R_\beta}}(y)\}_+ = \delta_{\alpha\beta}\delta(x,y)$$

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$$\{\xi_{L_\alpha}(x), \Pi_{\xi_{L_\beta}}(y)\}_+ = \delta_{\alpha\beta}\delta(x,y)$$
$$\{\xi_{R_\alpha}(x), \Pi_{\xi_{R_\beta}}(y)\}_+ = \delta_{\alpha\beta}\delta(x,y)$$

- Components of the densitized fermionic axial current

$$\overline{J}^i = \sqrt{\det h}\,J^i = 2\left(\Pi_{\xi_R}\tau^i\xi_R + \Pi_{\xi_L}\tau^i\xi_L\right)$$
$$\overline{J}^0 = \sqrt{\det h}\,J^0 = -\xi_R^\dagger\xi_R + \xi_L^\dagger\xi_L$$

Thus, new constraints

- $\mathcal{G}_i^{tot}[\tilde{A}, E, \Xi, \Pi_\Xi] = \mathcal{G}_i^{grav}[\tilde{A}, E] - \left(\Pi_{\xi_R}\tau^i\xi_R + \Pi_{\xi_L}\tau^i\xi_L\right)$
- $\mathcal{H}^{tot}[\tilde{A}, E, \Xi, \Pi_\Xi] = \mathcal{H}^{grav}[\tilde{A}, E] + \gamma\frac{E_i^a}{\sqrt{\det h}}\mathcal{D}^{(\Gamma)}\left(\Pi_{\xi_R}\tau^i\xi_R + \Pi_{\xi_L}\tau^i\xi_L\right)$
$$+ i\frac{E_i^a}{\sqrt{\det h}}\Big(-\Pi_{\xi_R}\tau^i\partial_a\xi_R + \Pi_{\xi_L}\tau^i\partial_a\xi_L + \partial_a(\Pi_{\xi_R})\tau^i\xi_R - \partial_a(\Pi_{\xi_L})\tau^i\xi_L\Big)$$
$$+ \frac{i}{2}\frac{E_i^a}{\sqrt{\det h}}\tilde{A}_a^i\left(-\Pi_{\xi_L}\xi_L + \Pi_{\xi_R}\xi_R\right) + \frac{3}{2}\varepsilon^i{}_{jk}\frac{E_i^a}{\sqrt{\det h}}K_a^j\left(\Pi_{\xi_L}\tau^k\xi_L + \Pi_{\xi_R}\tau^k\xi_R\right)$$
- $\mathcal{H}_a^{tot}[\tilde{A}, E, \Xi, \Pi_\Xi] = \mathcal{H}_a^{grav}[\tilde{A}, E] - \tilde{A}_a^i\left(\Pi_{\xi_L}\tau_i\xi_L + \Pi_{\xi_R}\tau_i\xi_R\right)$
$$- \frac{1}{2}\Big(\Pi_{\xi_R}\tau^i\partial_a\xi_R + \Pi_{\xi_L}\tau^i\partial_a\xi_L - \partial_a(\Pi_{\xi_R})\tau^i\xi_R - \partial_a(\Pi_{\xi_L})\tau^i\xi_L\Big)$$
$$+ \gamma K_a^i\left(\Pi_{\xi_L}\tau_i\xi_L + \Pi_{\xi_R}\tau_i\xi_R\right)$$

Work in progress

- Consistency check: Dirac EOM in curved space-time
- Adding photon field
- Polymerize fermion field keeping gravity classical and vice versa
- Transition to loop representation

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Thank you!

Constraints

Consider

$$S = \int_{t_1}^{t_2} L(q, \dot{q}) dt$$

If $det\left(\frac{\partial^2 L}{\partial \dot{q}^{n'} \partial \dot{q}^n}\right) = 0 \rightarrow$ Singular system $\implies p_n = \frac{\partial L}{\partial \dot{q}^n}$ not all independent

Hence, there are some relations

$$\phi_m(q, p) = 0 \text{ with } m = 1, ..., M \qquad \textbf{primary constraints}$$

that follow from the definition of the momenta.
From the consistency condition

$$\dot{\phi} = [\phi_m, H] + u^{m'}[\phi_m, \phi_{m'}] = 0 \rightarrow \varphi_k = 0 \text{ with } k = M+1, ..., M+K$$

secondary constraints

with u^m Lagrange multipliers.

Weak Equality

A function f is weakly equal to a function g

$$f \approx g$$

if f and g are equal on the subspace defined by the primary constraints $\phi_m = 0$.

First and Second Class Constraints

A constraint is called "*first class*" if its Poisson bracket with all the constraints Ω_A vanishes weakly,

$$\{\Omega^{(1)}_{A_1}, \Omega_B\} \approx 0 \ ; \ A_1 = 1, ..., N^{(1)} \ , \ B = 1, ..., N$$

First class constraints generate gauge transformations.
A constraint that is not first class is called "*second class*".