Axial perturbations in Kantowski-Sachs spacetimes

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Introduction

- The application of LQC to black holes has attracted a lot of attention.
- The interest has been revitalized by Ashtekar, Olmedo, and Singh, who introduced a model for nonrotating black holes.
- The interior can be described as a Kantowski-Sachs (KS) cosmology.
- To go beyond this simple scenario, we consider perturbation theory.
- We will truncate our perturbations at second order in the action.
- Physical perturbations correspond to perturbative gauge invariants.
- We will construct a Hamiltonian formulation for this perturbed system and proceed to its hybrid quantization within LQC.

The background

- The metric in the interior region is of KS form

$$ds^2 = -\frac{V^2 \underline{N}^2(\tau)}{4\pi^2} d\tau^2 + \frac{p_b^2(\tau)}{|p_c(\tau)|} dx^2 + |p_c(\tau)| \left(d\theta^2 + \sin^2\theta \, d\phi^2\right).$$

- The spatial sections have a volume $V = 8\pi^2 p_b \sqrt{|p_c|}$.

- The geometry has **two canonical pairs** of degrees of freedom, with connection variables such that $\{b, p_b\} = \gamma$, $\{c, p_c\} = 2\gamma$.

- We include a (homogeneous) scalar field, $\{\Phi, \Pi_\Phi\} = 1$.

- This KS background is subject ONLY to the **Hamiltonian** constraint

$$\underline{N} H_{KS} = -\underline{N}\left(\Omega_b^2 + 2\Omega_b \Omega_c + p_b^2 - 4\Pi_\Phi^2\right), \quad \Omega_j = \frac{j\, p_j}{\gamma}, \quad j = b, c.$$

Perturbations

- We consider compact sections $S^1 \times S^2$. Zero-modes are treated exactly.
- We expand in REAL spherical harmonics and Fourier modes.
- Spherical harmonics, of eigenvalue $-l(l+1)$ for the S^2-Laplacian, split in polar and axial under parity [with eigenvalue $(-1)^l$ or $(-1)^{l+1}$].
- Using capital Latin letters for S^2-indices, we can decompose any symmetric spatial tensor as $T_{ab}\, dx^a\, dx^b = T_{xx}\, dx^2 + 2\mathrm{T}_{xA}\, dx\, dx^A + T_{AB}\, dx^A\, dx^B$.
- We use a real Regge-Wheeler-Zerilli basis of harmonics and restrict the study to axial modes with $l \geq 2$.
- We call $\left\{ X^m_{l\;A}, X^m_{l\;AB} \right\}$ our basis of vector and tensor axial harmonics. Scalar harmonics are polar.

Axial perturbations

- We call $\{Q_{n,\lambda}\}$ our Fourier basis of sines $(\lambda=-1)$ and cosines $(\lambda=+1)$, with frequency $\omega_n=2\pi n$ and n any natural number.

- Let $\{\nu\}=\{n,\lambda,l,m\}$. We can expand the pertubations as

$$\Delta h_{ab}\, dx^a\, dx^b=-2\sum h_1^\nu(t)\, X^m_{l\ A}(\theta,\phi)\, Q_{n,\lambda}(x)\, dx\, dx^A+\sum h_2^\nu(t)\, X^m_{l\ AB}(\theta,\phi)\, Q_{n,\lambda}(x)\, dx^A\, dx^B,$$

$$V\,\Delta\left[\frac{p_{ab}}{\sqrt{h}}\, dx^a\, dx^b\right]=-\sum\frac{4\pi}{l(l+1)}\left\{p_b^2\, p_1^\nu(t)\, X^m_{l\ A}(\theta,\phi)\, dx-\frac{2\,p_c^2}{(l+2)}\, p_2^\nu(t)\, X^m_{l\ AB}(\theta,\phi)\, dx^B\right\} dx^A\, Q_{n,\lambda}(x),$$

$$N_a\, dx^a=-16\pi\sum h_0^\nu(t)\, X^m_{l\ A}(\theta,\phi)\, Q_{n,\lambda}(x)\, dx^A.$$

- At second order, the contribution of these perturbations to the action is

$$\frac{1}{16\pi}\int dt\ \sum\left(\dot{h}_1^\nu\, p_1^\nu+\dot{h}_2^\nu\, p_2^\nu-h_0^\nu\, C^{ax}_\nu-\underline{N}\, H^{ax}_\nu\right).$$

Perturbative diff. constraints

Hamiltonian constraint

Gauge invariants

- With fixed background, we can perform a linear canonical transformation so that perturbations are described by gauge invariant pairs, and perturbative constraints and variables conjugated to them,

$$\{h_1^\nu, p_1^\nu, h_2^\nu, p_2^\nu\} \rightarrow \{Q_1^\nu, P_1^\nu, Q_2^\nu, P_2^\nu \propto C_\nu^{ax}\}.$$

- The perturbative Hamiltonian changes by a time variation, given by the bracket of the generating function with the background Hamiltonian,

$$H_\nu^{ax} = (P_1^\nu)^2 + \left[(l+2)(l-1)\, p_b^2 + \omega_n^2\, p_c^2\right](Q_1^\nu)^2$$

$$+ \frac{(l+2)(l-1)\, p_b^2}{(l+2)(l-1)\, p_b^2 + \omega_n^2\, p_c^2}\left[p_b^2 - \frac{(l+2)(l-1)\, p_b^2 - 2\,\omega_n^2\, p_c^2}{(l+2)(l-1)\, p_b^2 + \omega_n^2\, p_c^2}(\Omega_b - \Omega_c)^2\right](Q_1^\nu)^2.$$

- Resemblance with a scalar field in KS suggests a new transformation $\{Q_1^\nu, P_1^\nu\} \rightarrow \{\boldsymbol{Q}_1^\nu, \boldsymbol{P}_1^\nu\}$ with manageable high-frequency limit, of wavenumber

$$k^2 = (l+2)(l-1) + \omega_n^2.$$

Total system

- We then arrive to a perturbative Hamiltonian corresponding to oscillators with background-dependent masses,

$$\boldsymbol{H}_{\nu}^{ax}=b_{\hat{l}}\left[(\boldsymbol{P}_{1}^{\nu})^{2}+(k^{2}+s_{\hat{l}})(\boldsymbol{Q}_{1}^{\nu})^{2}\right], \qquad b_{\hat{l}}^{2}=\hat{l}^{2}p_{b}^{2}+\frac{\omega_{n}^{2}}{k^{2}}p_{c}^{2}, \qquad \hat{l}=\frac{\sqrt{(l+2)(l-1)}}{k}.$$

$$s_{\hat{l}}=\frac{4}{b_{\hat{l}}^{2}}\left[p_{b}^{2}+\Omega_{b}^{2}\right]-\frac{\hat{l}^{2}}{b_{\hat{l}}^{4}}p_{b}^{2}\left[4(\Omega_{b}-\Omega_{c})^{2}+p_{b}^{2}+2(\Omega_{b}^{2}-\Omega_{c}^{2})\right]+2\frac{\hat{l}^{4}}{b_{\hat{l}}^{6}}p_{b}^{4}(\Omega_{b}-\Omega_{c})^{2}.$$

- Our canonical transformation can be extended to the background. The new geometric zero-modes contain quadratic perturbative terms. Denoting these new variables as before, the truncated action becomes

$$\int dt\left[\left(\dot{\Phi}\,\Pi_{\Phi}-\frac{\dot{p}_{c}}{2\gamma}c-\frac{\dot{p}_{b}}{\gamma}b+\frac{1}{16\pi}\sum\dot{\boldsymbol{Q}}_{1}^{\nu}\boldsymbol{P}_{1}^{\nu}+\sum\dot{Q}_{2}^{\nu}P_{2}^{\nu}\right)+\sum 2\tilde{h}_{0}^{\nu}P_{2}^{\nu}-\underline{\tilde{N}}\left(H_{KS}+\sum\boldsymbol{H}_{\nu}^{ax}\right)\right].$$

- This system is canonical, and subject only to a nontrivial global (Hamiltonian) constraint, on the background and the gauge invariants.

Hybrid quantization

- We adopt e.g. a loop representation of the background, built on an extended phase space with polymerization parameters δ_j (Andrés talk).

- Adopting a triad-representation for the scaled variables $\tilde{p}_j = p_j / \delta_j$, calling $\boldsymbol{N}_{2\delta_j} = e^{i\delta_j j}$, and using an MMO prescription, we define

$$\hat{\Omega}_j = \frac{1}{4i\gamma} \left|\hat{\tilde{p}}_j\right|^{1/2} \left[\left(\hat{\boldsymbol{N}}_{2\delta_j} - \hat{\boldsymbol{N}}_{-2\delta_j} \right) \widehat{sign}(\tilde{p}_j) + \widehat{sign}(\tilde{p}_j) \left(\hat{\boldsymbol{N}}_{2\delta_j} - \hat{\boldsymbol{N}}_{-2\delta_j} \right) \right] \left|\hat{\tilde{p}}_j\right|^{1/2},$$

$$\hat{H}_{KS} = -\left[\hat{\Omega}_b^2 + 2\hat{\Omega}_b \hat{\Omega}_c + \hat{\delta}_b^2 \hat{\tilde{p}}_b^2 + 4\partial_\Phi^2 \right].$$

- We also quantize the time factors $\hat{b}_{\hat{l}}$ and masses $\hat{s}_{\hat{l}}$ of the perturbative Hamiltonian, using a symmetric factor ordering and that factors of H_{KS} vanish at this perturbative level.

- Finally, we adopt the (essentially) unique Fock quantization of the perturbations that respects the symmetries of the axial dynamics and allows for a unitary Heisenberg evolution (Álvaro's talk).

Conclusions

- We considered perturbations around a KS spacetime, describing the interior of a nonrotating black hole.
- We truncated the action at second perturbative order and constructed a Hamiltonian formulation for axial pertubations.
- We obtained a canonical system made of background zero-modes, gauge invariants, and perturbative constraints with their momenta.
- There is only a global Hamiltonian constraint on the (sub-)system formed by the background and the perturbative gauge invariants.
- We combined a loop quantization of the background with a (unique) Fock quantization of the perturbations.
- In future research, we will derive the corrected perturbation equations and study the relation between interior and exterior modes.

Bibliography

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- See also:
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