Relational Lorentzian Asymptotically Safe Quantum Gravity: Showcase model

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based on work in collaboration with Thomas Thiemann
2404.18224 [hep-th]

May 6th 2024, Fort Lauderdale, Florida, USA

Loops '24 International Conference on Quantum Gravity

FRIEDRICH-ALEXANDER
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ASQG & CQG

CQG

- Lorentzian signature
- Manifestly background-independent
- No truncations performed

[Rovelli's book (2004),
Thiemann's book (2019)]

ASQG

- QFT-approach: non-perturbative RG flow with UV fixed point
- Mostly uses Euclidean signature
- Background dependence?
- Truncations of the flow equations

[Percacci's book (2017)
Reuter and Saueressig book (2019)]

[Manrique, Rechenberger, Saueressig 1102.5012 (2011),
D'Angelo, Pinamonti, Rejzner 2202.0758 (2022)]

Lorentzian version of the flow equation possible

Background-independent via the background field method

Truncations required in practice in any RG as an approximation scheme

[Thiemann 2003.13622 (2022)]

Relational formulation

Idea

1. a) Reduced phase space formulation of **CQG**
   b) Construction of time-ordered correlation functions as a path integral
2. Path integral treated with methods of **ASQG** in the Lorentzian version

Framework

Machinery

First application

3. • Einstein-Hilbert action coupled to 4 massless scalar fields
   • Development of Lorentzian heat kernel cutoff functions

1.a) CQG derivation

The model: $$S = G_N^{-1} \int_M d^dx \, \sqrt{-\det(g)} \, [R[g] - 2\Lambda - \frac{G_N}{2} S_{IJ} \, g^{\mu\nu} \, \phi^I_{,\mu} \, \phi^J_{,\nu}]$$

[Giesel, Thiemann 1206.3807 (2012), [Giesel, Vetter 1610.07422 (2016)]]

- **Reduced phase space approach**: impose d gauge fixing conditions on the configuration variables and solve the constraint for the d conjugate momenta
- Convenient set of gauge

$$G^I := \phi^I - k^I = 0 \, , \qquad \det(\partial k / \partial x) \neq 0 \, , \qquad \partial k^I / \partial x^\mu \text{ independent of } x^0$$

- Gauge dofs. $(\phi^I), (\pi_I)$ and **true dofs.** (q_{ab}, p^{ab})
- Impose **gauge stability** condition: solved by $N^\mu := N^\mu_*$
- **Reduced Hamiltonian**: function of the true dofs which generate the eoms as the primary Hamiltonian when restricted to the reduced phase space

$$\{H, F\} = \{H_{\text{primary}}, F\}_*$$

- Equipped with the reduced phase space and the reduced Hamiltonian, we can **quantize** the system

1.b) CQG derivation

- The cyclic rep. correspond to states ω wrt. which we can compute **time-ordered correlation functions** [Brattelli, Robison (1997)]
- They are obtained from a **generating functional**

$$Z_s[f] = \int [dq\ dp]\ J_\omega[q]\ e^{-\int\ dx^0\ (i <p,\dot{q}>+(-i)^s\ H[p,q])}\ e^{i^s\ \int\ dx^0\ <f,q>}$$

- **Problem: H involves a square root**
- Solution: unfolding the reduced phase space path integral to the unreduced phase space [Henneaux, Teitelboim's book (1992)]
  - Extend the integration to ϕ^I, π^I through $\delta(C)\ , \delta(G)$

$$C_I = \pi_I + h_I = 0\ , \qquad G = \phi^I - k^I = 0 \qquad \longrightarrow \qquad H = \dot{k}^I h_I = -\dot{\phi}^I \pi_I$$

$\longrightarrow$ this forces us to work with **Lorentzian signature** $s = 1$.

$$Z_1[f] = \int [dg]\ J_\omega[g]\ e^{-iS_1[g]}\ \ e^{i \int_M\ d^dx\ f^{ab} q_{ab}}, \quad S_1[g] = \frac{1}{G_N}\int_M d^dx\sqrt{-det(g)}[R[g] - 2\Lambda - \frac{G_N}{2} g^{\mu\nu}\ S_{IJ}\ \ k^I_{,\mu}\ k^J_{,\nu}]$$

- Ghost matrix K_ω

$$J_\omega[g] = |\int [d\rho\ d\eta]\ e^{-i\int\ d^dx\ \eta^\mu\ [K_\omega]^I_\mu(g)\ \rho_I}|$$

2. ASQG treatment

[Wetterich (1992), Reuter (1996)]

- Effective action $C'[F] = i^{-1}\ \ln(Z'[F]),\ \Gamma'[\hat{g}] := [L \cdot C'][\hat{g}] = \text{extr}_F(< F, \hat{g} > -C'[F])$
- **Background field method** $g \to \bar{g} + h$
- **Effective average action** $\bar{Z}'_k[F, \bar{g}] = \int [dh]\ J_\omega[\bar{g} + h]\ e^{-iS[\bar{g}+h]}\ e^{i<F,h>} e^{-i\frac{1}{2}<h, R_k(\bar{g})\cdot h>},$

$$\bar{C}'_k[F, \bar{g}] = i^{-1}\ \ln(\bar{Z}'[F, \bar{g}]),$$

$$\bar{\Gamma}'_k[\hat{g}, \bar{g}] = \text{extr}_F\ (< F, \hat{g} > -C'_k(F, \bar{g})) - \frac{1}{2} < \hat{g}, R_k(\bar{g}) \cdot \hat{g} >$$

where $k \to R_k(\bar{g})$ is a 1-parameter family of integral kernels which only depend on the background d'Alembertian ⟶ **Lorentzian: oscillating with $R_k = 0$ for $k = 0$, s.t.** $\Gamma'[\hat{g}] := \bar{\Gamma}'[\hat{g}'; \bar{g}]_{\hat{g}'=0,\ \bar{g}:=\hat{g}}$

- **Lorentzian version of the Wetterich equation:**

$$k\ \partial_k\ \bar{\Gamma}'_k[\hat{g}, \bar{g}] = \frac{1}{2i}\ \text{Tr}([R_k(\bar{g}) + \bar{\Gamma}'^{(2)}_k(\hat{g}, \bar{g})]^{-1}\ [k\ \partial_k R_k(\bar{g})]),\ \bar{\Gamma}'^{(2)}[\hat{g}, \bar{g}] := \frac{\delta^2 \bar{\Gamma}'^{(2)}[\hat{g}, \bar{g}]}{\delta \hat{g} \otimes \delta \hat{g}}$$

Exact and non-perturbative identity and **can be used to construct a well defined Γ rather than using $Z_1[f]$**

To solve it : Taylor expand both LHS and RHS in powers of $g_{\mu\nu}$ and compare coefficients

2. Lorentzian heat kernel

$$\bar{\Gamma}'^{(2)}_k(\hat{g},\bar{g}) + R_k(\bar{g}) =: P_k + U_k + R_k$$

- Consider the geometric series

$$\mathrm{Tr}([P_k+R_k+U_k]^{-1}\;[k\partial_k R_k]\;[P_k+R_k+U_k]^{-1}\;V_k^1\;[P_k+R_k+U_k]^{-1}\;..\;[P+R_k+U_k]^{-1}\;V_k^M)$$

- Traces rewritten by the spectral theorem (involving both minimal and non-minimal operators)

$$O_k(\bar{\Box}) = \int_{-\infty}^{\infty} dt\; \hat{O}_k(t)\; H_t, \quad H_t := e^{it\bar{\Box}}$$

[Benedetti, Groh, Machado, Saueressig 1012.3081 (2010), Groh, Saueressig, Zanusso 1112.4856 (2011)]

- **Heat kernel expansion on a general manifold**

$$\mathrm{Tr}[H_t(\bar{\Box})] = \frac{e^{i\pi/4\mathrm{sgn}(t)(2-d)}}{(4\pi|t|^{d/2})}\left(1+\frac{i}{6}\bar{R}t+\cdots\right)$$

[Christensen (1976), Fulling's book (1989), Moretti (1999), Decanini, Folacci (2006), Parker, Toms' book (2009)]

Using the Schwinger proper time integral

$$P_k^{-1} = B^{-1}\;[\bar{\Box}+C_k]^{-1} = -\frac{i}{B}\;[\int_0^{\infty} dt\; e^{it\bar{\Box}-t\epsilon}]_{\epsilon\to -iC_k}$$

and choosing the cutoff function

$$R_k(z) = f_k\;k^2\;r(z/k^2),\; r(y) = \int_0^{\infty} dt\; e^{-t^2-t^{-2}}\; e^{ity}$$

we get convergent traces. **Price to pay: complex flow.**

- **Admissable trajectories**: flow to real valued dimensionful coupling constants when $k\to 0$

3. ASQG model

[Baldazzi, Falls, Ferrero 2112.02118 (2021)]

- Ansatz $$\bar{\Gamma}'_k(\hat{g},\bar{g}) := \{\frac{1}{G_{N,k}}\int d^dx\,[-\det(g)]^{1/2}[R[g]-2\Lambda_k-\frac{\kappa_k G_{N,k}}{2}g^{\mu\nu}\,\delta_{\mu\nu}]\}_{g=\bar{g}+\hat{g}}$$

where we specialized to the gauge $\phi^I = k^I$, s.t. $\kappa^I_\mu = k^I_{,\mu} = \text{const.}$ and $\kappa_{\mu\nu} := S_{IJ}\kappa^I_\mu \kappa^J_\nu = \kappa\delta_{\mu\nu}$

- Order of expansion of the RHS $\frac{1}{2i}\mathrm{Tr}\{[k\partial_k R_k]\,P_k^{-1}(1-[(R_k+U_k)\,P_k^{-1}]+[((R_k+U_k)\,P_k^{-1}]^2)$

- Use Lorentzian heat kernel with the cutoff function and solve the **proper time integrals**

$$I_{m,n} = \int_0^\infty dt_1\cdots dt_m \frac{e^{-t_1^2-t_1^{-2}}\cdots e^{-t_m^2-t_m^{-2}}}{(t_1+\cdots+t_m)^n}\,,$$
$$J_{m,n} = \int_0^\infty dt_1\cdots dt_m \frac{\frac{d}{dt_1}(e^{-t_1^2-t_1^{-2}})\cdots e^{-t_m^2-t_m^{-2}}}{(t_1+\cdots+t_m)^n}$$

3. ASQG model: results

- At the level of our truncation, the coupling constant κ_k is not flowing and the flow of the gravitational couplings completely disentangles

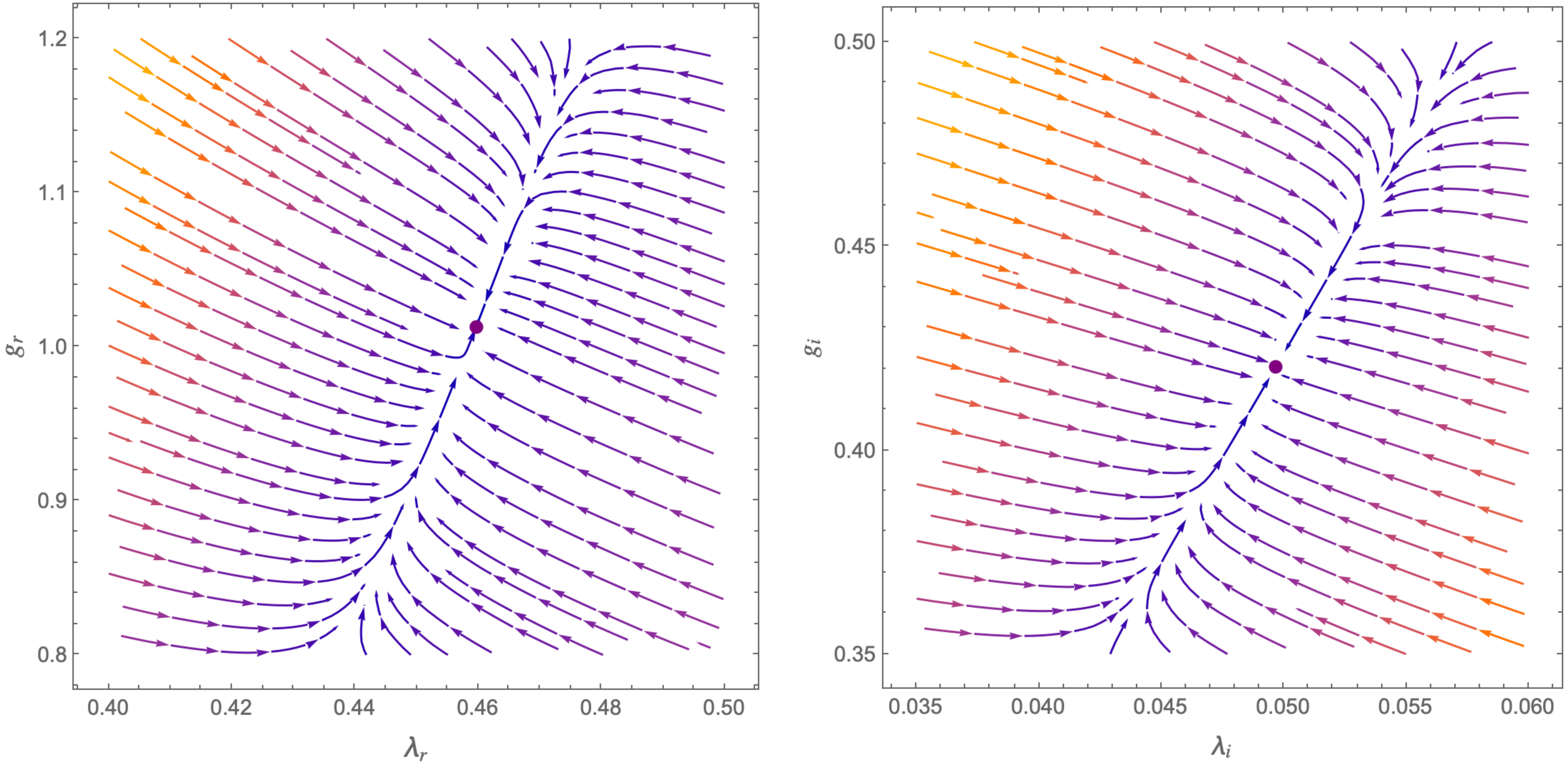
- **Beta functions:**

$$\begin{aligned} k\partial_k\lambda_k = \quad & -4\lambda_k + \eta_N\lambda_k - \frac{g}{4\pi}\frac{1}{2\lambda_k}\left((2-\eta_N)\left(I_{1,2} + \frac{1}{2\lambda_k}\left(I_{2,2} - 5I_{1,3} + \frac{4}{3}\lambda_k I_{1,2}\right)\right.\right. \\ & + \frac{1}{(2\lambda_k)^2}\left(I_{3,2} + 2\left(I_{2,2} - 5I_{1,3} + \frac{4}{3}\lambda_k I_{1,2}\right) - i\frac{16}{9}\lambda_k^2 I_{1,2} + \frac{2}{3}I_{1,3}\right)\Bigg) \\ & + 2\left(J_{1,2} + \frac{1}{2\lambda_k}\left(J_{2,2} - 5J_{1,3} + \frac{4}{3}\lambda_k J_{1,2}\right)\right. \\ & \left.\left. + \frac{1}{(2\lambda_k)^2}\left(J_{3,2} + 2\left(I_{2,2} - 5J_{1,3} + \frac{4}{3}\lambda_k J_{1,2}\right) - i\frac{16}{9}\lambda_k^2 J_{1,2} + \frac{2}{3}J_{1,3}\right)\right)\right) \end{aligned}$$

$$\begin{aligned} k\partial_k g_k = \quad & 2g_k - \frac{g_k^2}{2\pi}\frac{1}{2\lambda_k}\left((2-\eta_N)\left(\frac{i}{6}I_{1,1} + \frac{1}{2\lambda_k}\left(\frac{i}{6}I_{2,1} + 3I_{1,2} + i\frac{2}{9}\lambda_k I_{1,1} - i\frac{5}{12}I_{1,2}\right)\right.\right. \\ & + \frac{1}{(2\lambda_k)^2}\left(\frac{i}{6}I_{3,1} + 2\left(\frac{i}{6}I_{2,1} + 3I_{2,2} + i\frac{2}{9}\lambda_k I_{2,1} - i\frac{5}{12}I_{2,2}\right) - i\frac{8}{27}I_{1,1} + \frac{3}{2}I_{1,3} + i\frac{1}{18}\lambda_k I_{1,2}\right)\Bigg) \\ & + 2\left(\frac{i}{6}J_{1,1} + \frac{1}{2\lambda_k}\left(\frac{i}{6}J_{2,1} + 3J_{1,2} + i\frac{2}{9}\lambda_k J_{1,1} - i\frac{5}{12}J_{1,2}\right)\right. \\ & \left.\left. + \frac{1}{(2\lambda_k)^2}\left(\frac{i}{6}J_{3,1} + 2\left(\frac{i}{6}J_{2,1} + 3J_{2,2} + i\frac{2}{9}\lambda_k I_{2,1} - i\frac{5}{12}J_{2,2}\right) - i\frac{8}{27}J_{1,1} + \frac{3}{2}J_{1,3} + i\frac{1}{18}\lambda_k J_{1,2}\right)\right)\right) \end{aligned}$$

3. ASQG model: results

- **UV fixed point** $\lambda_* = 0.460 + 0.050\,i\,,\quad g_* = 1.013 + 0.420\,i$ very close to the Reuter's fixed point
- Critical exponents $\theta_1 = 12.24 - 0.07\,i\,,\quad \theta_2 = 0.95 + 0.017\,i$ **two relevant directions**
- Projection into real and imaginary part:



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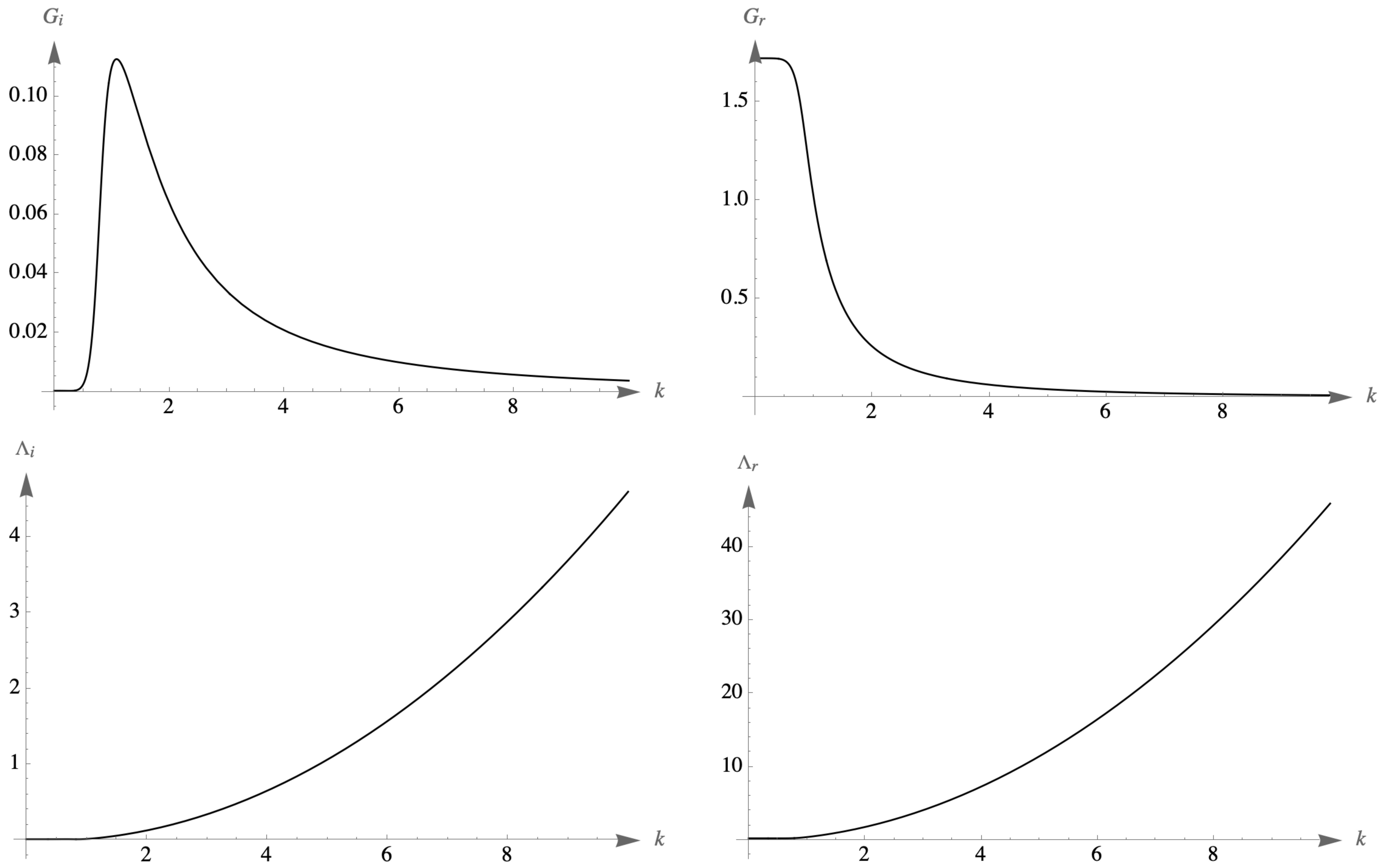

- **Admissible trajectories**: we found trajectories with $\mathrm{Im}\,[G_{k=0}] = 0\,, \mathrm{Im}\,[\Lambda_{k=0}] = 0$ which flow into the fixed point in the UV

- **The effective action can be argued to be a complete definition of the theory.**

Conclusions

- **CQG**: input how to define the effective average action
    1. State underlying the Hamiltonian theory
    2. Restiction on correlation functions of the true dofs
- **ASQG**: systematic procedure
    1. Well defined construction of the effective action
    2. Machinery to compute correlators of the Hamiltonian theory
- Einstein-Klein-Gordon theory as a showcase model to demonstrate that ASQG and CQG can be fruitfully combined
- Techn. development: regularized **Lorentzian** heat kernel proper time
- **Results**: UV fixed point

  Existence of admissible trajectories

Outlook

- Classification of Lorentzian cutoff functions
- Numerical analysis of the admissible trajectories
- Incorporation of ghost matrix term
- Higher order truncations
- More realistic matter coupling where the Euclidean version is possible