Uniform Asymptotic Approximation Method with Pöschl-Teller Potential

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Motivation

One of the major challenges in the study of cosmological perturbations in LQC is how to solve for the mode functions μ_k from the modified Mukhanov–Sasaki equation. So far, this has mainly been done numerically [Bojowald, M. 2008&2015, Ashtekar, A. ;Singh, P. 2011&2015……..]. However, this is often required to be conducted with high-performance computational resources [Agullo, I.; Morris, N.A. 2015], which are not accessible to the general audience.

Motivation

In particular, in the dressed metric approach, the mode function satisfies the following equation(Zhu, T 2017):

$$\mu_k''(\eta) + [k^2 - \mathscr{V}(\eta)]\mu_k(\eta) = 0, \tag{1}$$

in which $\mathscr{V}(\eta)$ serves as an effective potential.

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in which $\mathscr{V}(\eta)$ serves as an effective potential. During the bouncing phase, it is given by

$$\mathscr{V}_{\text{dressed}}(\eta) \equiv a_B^2 \frac{\gamma_{\text{B}} m_{\text{Pl}}^2 (3 - \gamma_{\text{B}} t^2/t_{\text{Pl}}^2)}{9(1 + \gamma_{\text{B}} t^2/t_{\text{Pl}}^2)^{5/3}} = \frac{a''}{a},$$

where $\gamma_{\text{B}} \equiv 24\pi\rho_c/m_{\text{Pl}}^4 \approx 30.9$ is a constant, and m_{Pl} and t_{Pl} are the Planck mass and time respectively.

Motivation

This potential can be well approximated by a PT potential

$$\mathscr{V}_{\mathsf{PT}}(\eta) = \frac{\mathscr{V}_0}{\cosh^2 \alpha(\eta - \eta_{\mathsf{B}})}, \tag{2}$$

with(Zhu, T 2017)

$$\mathscr{V}_0 = \frac{a_B^2 \gamma_{\mathsf{B}} m_{\mathsf{Pl}}^2}{3} = \frac{\alpha^2}{6}. \tag{3}$$

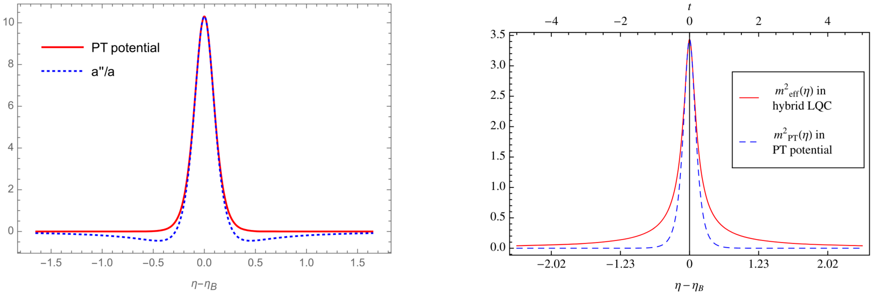
Here η is the conformal time related to the cosmic time t by $d\eta = dt/a(t)$.

On the other hand, in the hybrid approach, the effective potential during the bouncing phase is given by

$$\mathscr{V}_{\mathsf{Hybrid}}(\eta) = -\frac{a_B^2 \gamma_{\rm B} m_{\rm Pl}^2 (1 - \gamma_{\rm B} t^2/t_{\rm Pl}^2)}{9(1 + \gamma_{\rm B} t^2/t_{\rm Pl}^2)^{5/3}}, \tag{4}$$

which can be also modeled by the PT potential but now with (Wu, Q. 2018)

$$V_0 = \frac{a_B^2 m_{\rm Pl}^2 \gamma_B}{9}, \quad \alpha^2 = \frac{2}{3} a_B^2 m_{\rm Pl}^2 \gamma_B. \tag{5}$$



(a) Dressed metric (b) Hybrid

Figure 1: Comparison between a''/a and the PT potential given in (2)

The Liouville Transformation

In general, by properly choosing the variable y and $\mu_k(y)$, the second order differential equation can be written as

$$\frac{d^2 u_k(y)}{dy^2} = f(y)\mu_k(y),$$

and $f(y)$ can take the form

$$f(y) = \lambda^2 g(y) + q(y).$$

The function $g(y)$ has singularities and/or zeros in the interval of our interest. We call the zeros and singularities of $g(y)$ as turning points and poles, respectively.

The Liouville Transformation

For the equation

$$\frac{d^2\mu_k(y)}{dy^2} = [\lambda^2 g(y) + q(y)]\mu_k(y), \tag{6}$$

we can have the Liouville transformation[1] with two variables $U(\zeta)$ and $\zeta(y)$ (with its inverse $y = y(\zeta)$) satisfying

$$U(\zeta) = \dot{y}^{-\frac{1}{2}}\mu_k(y), \quad \dot{y} = \frac{dy}{d\zeta}.$$

[1] Olver F. W. J. 1974 THE LIOUVILLE-GREEN APPROXIMATION, *Introduction to Asymptotics and Special Functions*

The Liouville Transformation

In terms of $U(\zeta)$ and ζ, the equation (6) can be written as

$$\frac{d^2U(\zeta)}{d\zeta^2} = [\lambda^2\dot{y}^2 g + \psi(\zeta)]U(\zeta), \tag{7}$$

where,

$$\begin{aligned}\psi(\zeta) &\equiv \dot{y}^2 q - \dot{y}^{\frac{1}{2}}\frac{d^2}{d\zeta^2}\left(\dot{y}^{-\frac{1}{2}}\right) \\ &= \dot{y}^2 q - \dot{y}^{\frac{3}{2}}\frac{d^2}{dy^2}\left(\dot{y}^{\frac{1}{2}}\right) \equiv \psi(y).\end{aligned}$$

The Liouville Transformation

The advantage of form of Eq.(7) is that, by properly choosing q(y), the term $|\psi(\zeta)|$ can be much smaller than $|\lambda^2\dot{y}^2 g|$, that is

$$|\frac{\psi(\zeta)}{\lambda^2\dot{y}^2 g}| \ll 1, \tag{8}$$

so that the exact solution of Eq (6) can be very approximated by the first order solution of Eq(7) with $\psi(\zeta) = 0$.

Minimization of the Errors

First introduce the *error control function*

$$\mathscr{T}(\zeta) \equiv - \int \frac{\psi(\zeta)}{|\dot{y}^2 g|^{1/2}}. \qquad (9)$$

By analyzing the asymptotic behavier of $\mathscr{T}(\zeta)$ and properly choosing the $q(y)$, we can minimize the error control function, which characterize the difference between the approximate and the exact solutions.

Choise of Function $y(\zeta)$

The errors also depend on the choice of $y(\zeta)$, which in turn sensitively depends on the properties of the functions $g(y)$ and $q(y)$ near their poles and zeros. In addition, it must be chosen so that the resulting equation of the first-order approximation can be solved explicitly (in terms of known functions).

Choise of Function $y(\zeta)$

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Considering all the above, it has been found that $y(\zeta)$ can be chosen as

$$\dot{y}^2 g = \begin{cases} \text{sgn}(g), & \text{zero turning point,} \\ \zeta, & \text{one turning point,} \\ \zeta_0^2 - \zeta^2, & \text{two turning points,} \end{cases} \tag{10}$$

where sgn$(g) = 1$ for $g > 0$ and sgn$(g) = -1$ for $g < 0$.

UAA for two turning points

Here we consider only the cases with two turning points, we choose[2]

$$\dot{y}^2 g = \zeta_0^2 - \zeta^2, \tag{11}$$

where $\zeta(y)$ is an monotonically increasing function with the choices $\zeta(y_1) = -\zeta_0, \zeta(y_2) = \zeta_0$.

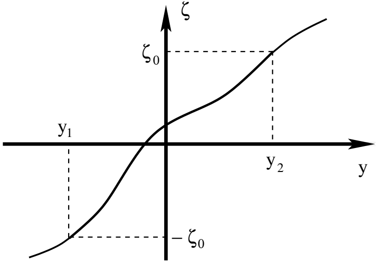

Figure 2: $\zeta(y)$

[2] Olver F. W. J. 1975 Second-order linear differential equations with two turning points

And it can be shown that

$$\zeta_0^2 = \pm \frac{2}{\pi} \left| \int_{y_1}^{y_2} \sqrt{g(y)} dy \right|,$$

where "+" and "−" corresponds to the cases with two real turning points and two complex conjugate turning points respectively.

UAA for two turning points

There are three conditions that are assumed to be satisfied when applying the UAA methods:

- When far away from any turning points:
  $|\frac{q(y)}{g(y)}| \ll 1,$
- When the two turning points are far way from each other ($|y_1 - y_2| \gg 1$), near any of these two points:
  $|\frac{q(y)(y-y_i)}{g(y)}| \ll 1$,
- When the two turning points are close to each other ($|y_1 - y_2| \approx 0$), near these points:
  $|\frac{q(y)(y-y_1)(y-y_2)}{g(y)}| \ll 1$.

UAA for two turning points

With the choice of (11), with $\psi(\zeta) \approx 0$ we find that (7) reduces to

$$\frac{d^2U}{d\zeta^2} = [\lambda^2(\zeta_0^2 - \zeta^2)]U.$$

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$$\frac{d^2U}{d\zeta^2} = [\lambda^2(\zeta_0^2 - \zeta^2)]U.$$

Then we have solutions in the form of the Weber parabolic cylinder functions $W(s,t)$:

$$U(\zeta) = \alpha_1 \left[W(\frac{1}{2}\lambda\zeta_0^2, \sqrt{2\lambda}\zeta) + \epsilon_1 \right] + \beta_1 \left[W(\frac{1}{2}\lambda\zeta_0^2, -\sqrt{2\lambda}\zeta) + \epsilon_2 \right], \quad (12)$$

UAA for two turning points

from which, we have the approximate solutions to the original equation (6):

$$\mu_k(y) = \alpha_1 \left(\frac{\zeta^2 - \zeta_0^2}{-g(y)}\right)^{\frac{1}{4}} \left[W(\frac{1}{2}\lambda\zeta_0^2, \sqrt{2\lambda}\zeta) + \epsilon_1\right] + \beta_1 \left(\frac{\zeta^2 - \zeta_0^2}{-g(y)}\right)^{\frac{1}{4}} \left[W(\frac{1}{2}\lambda\zeta_0^2, -\sqrt{2\lambda}\zeta) + \epsilon_2\right]. \tag{13}$$

UAA for two turning points

The errors ϵ have some bounded conditions[3]

$$\frac{|\epsilon_1|}{M(\frac{1}{2}\lambda\zeta_0^2,\sqrt{2\lambda}\zeta)},\frac{|\frac{\partial\epsilon_1}{\partial\zeta}|}{\sqrt{2}N(\frac{1}{2}\lambda\zeta_0^2,\sqrt{2\lambda}\zeta))}\leq\frac{1}{\lambda E(\frac{1}{2}\lambda\zeta_0^2,\sqrt{2\lambda}\zeta))}\left(e^{\lambda V_\zeta}-1\right),$$

$$\frac{|\epsilon_2|}{M(\frac{1}{2}\lambda\zeta_0^2,\sqrt{2\lambda}\zeta)},\frac{|\frac{\partial\epsilon_2}{\partial\zeta}|}{\sqrt{2}N(\frac{1}{2}\lambda\zeta_0^2,\sqrt{2\lambda}\zeta))}\leq\frac{E(\frac{1}{2}\lambda\zeta_0^2,\sqrt{2\lambda}\zeta))}{\lambda}\left(e^{\lambda V_\zeta}-1\right).$$

where V_ζ is the *associated error control function*,

$$V_\zeta=\int^\zeta\frac{|\psi(\zeta)|}{\sqrt{|\dot{y}^2g|}}d\zeta.$$

[3] Olver F. W. J. 1975 Second-order linear differential equations with two turning points

PT potential

To test the analytical approximate solutions, we consider equation (6) with a Pöschl-Teller (PT) potential

$$(\lambda^2 g + q) = -(k^2 - \frac{\beta_0^2}{\cosh^2 y}),$$

where k is the comoving wavenumber and β_0 is a real and positive constant.

PT potential

In this case, the equation (6) becomes

$$\frac{d^2\mu_k(y)}{dy^2} + (k^2 - \frac{\beta_0^2}{\cosh^2 y})\mu_k(y) = 0.$$

And with the variable transformations:

$$x = \frac{1}{1+e^{-2y}}, \quad \mathcal{Y} = [x(1-x)]^{\frac{ik}{2}}\mu_k,$$

PT potential

we find that (6) with PT potential reads

$$x(1-x)\frac{d^2\mathcal{Y}(x)}{dx^2}+(a_3-(a_1+a_2+1)x)\frac{d\mathcal{Y}(x)}{dx}-a_1a_2\mathcal{Y}(x)=0, \quad (14)$$

where

$$a_1=\frac{1}{2}(1+\sqrt{1-4\beta_0^2})-ik,$$
$$a_2=\frac{1}{2}(1-\sqrt{1-4\beta_0^2})-ik,$$
$$a_3=1-ik.$$

PT potential

The equation (14) is the standard hypergeometric equation, and has the general solution[4]

$$\mu_k^{(E)}(y) = a_1\left(\frac{x}{1-x}\right)^{\frac{ik}{2}} \times {}_2F_1(a_1 - a_3 + 1, a_2 - a_3 + 1, 2 - a_3, x)$$
$$+ b_1[x(1-x)]^{-\frac{ik}{2}} \times {}_2F_1(a_1, a_2, a_3, x).$$

[4] T. Zhu, A. Wang, G. Cleaver, K. Kirsten, and Q. Sheng, 2017, Pre-inflationary universe in loop quantum cosmology, Phys. Rev. D 96 083520

UAA solutions

First, we choose the $g(y)$ and $q(y)$ as

$$g(y) = \frac{\beta^2}{\cosh^2 y} - k^2 \quad q(y) = \frac{q_0^2}{\cosh^2 y},$$

then it turns out that

$$|\frac{q(y)}{g(y)}| \sim q_0^2 e^{-2|y|},$$
$$|\frac{q(y)(y-y_i)}{g(y)}| \sim \frac{q_0^2}{y+y_j},$$
$$|\frac{q(y)(y-y_1)(y-y_2)}{g(y)}| \sim q_0^2.$$

So as long as q_0 chosen to be small, we can use UAA method to get the approximate solution.

UAA solutions

Deponds on the turning points, $g(y)$ has three different cases:

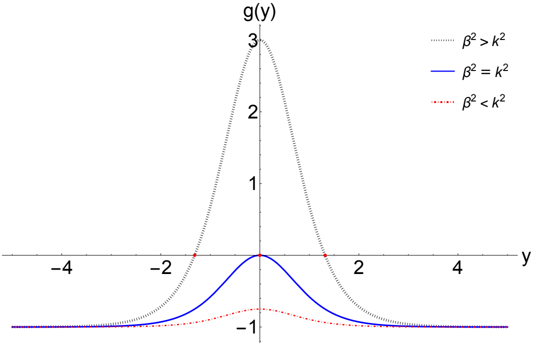


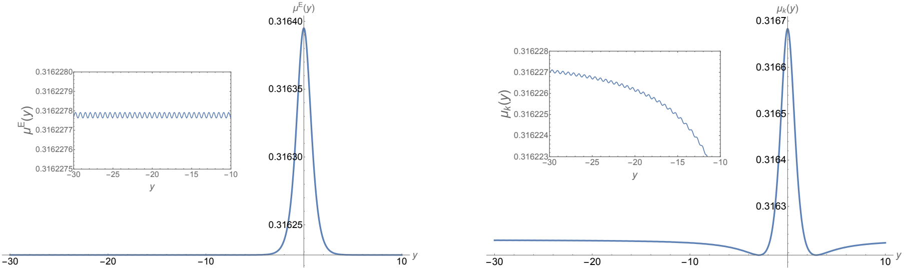
Figure 3: $\zeta(y)$

Applying the UAA method, we can have the approximate solutions:

$$\mu_k(y) = \alpha_k \left(\frac{\zeta^2 - \zeta_0^2}{-g(y)} \right)^{\frac{1}{4}} W(\frac{\zeta_0^2}{2}, \sqrt{2}\zeta) + \beta_k \left(\frac{\zeta^2 - \zeta_0^2}{-g(y)} \right)^{\frac{1}{4}} W(\frac{\zeta_0^2}{2}, -\sqrt{2}\zeta).$$

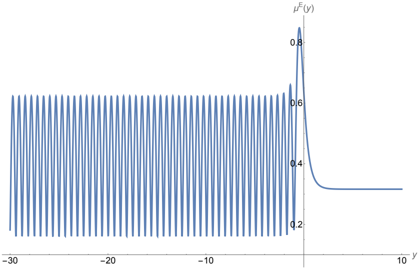
Compare Exact solution and UAA solution

Now we can compare the exact and approximate solutions. First we plot both solutions in the range of interest.

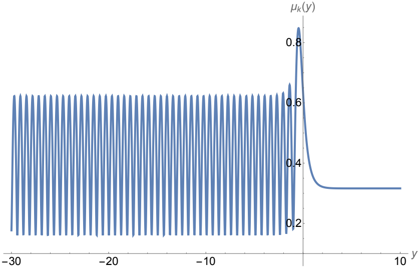


(a) Exact (b) UAA

Figure 4: $k = 5.0, \beta = 0.1$ and $q_0^2 = 1/24$



(a) Exact



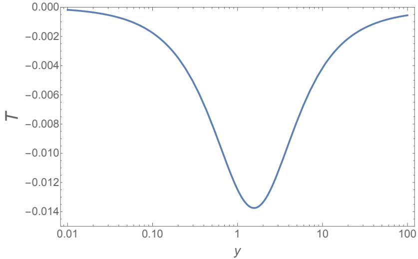
(b) UAA

Figure 5: $k = 5.0, \beta = 4.9$ and $q_0^2 = 1/4$

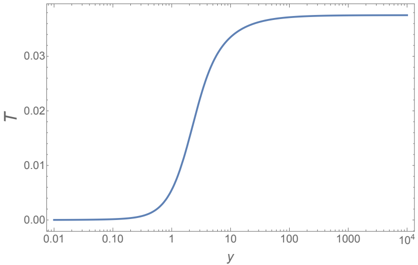
The error control function V_ζ is given by

$$V = -\int^y \left(\frac{q}{g} - \frac{5}{16}\frac{g'^2}{g^3} + \frac{1}{4}\frac{g''}{g^2} \right) \sqrt{-g}dy' + \frac{\zeta(\zeta^2 - 6\zeta_0^2)}{12\zeta_0^2(\zeta^2 - \zeta_0^2)^{3/2}},$$

and the plots are:



(a) $k \gg \beta$



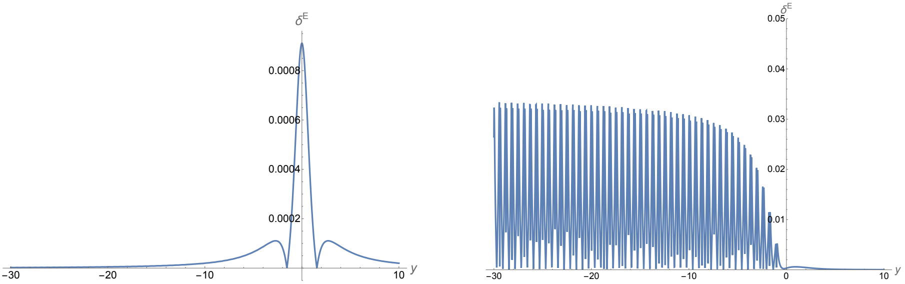
(b) $k > \beta$

Figure 6: Error control function V_ζ

We define the relative difference δ^E between approximate solution and exact solution as

$$\delta^E = \left| \frac{|\mu_k(y)| - |\mu_k^{(E)}(y)|}{|\mu_k^{(E)}(y)|} \right|,$$

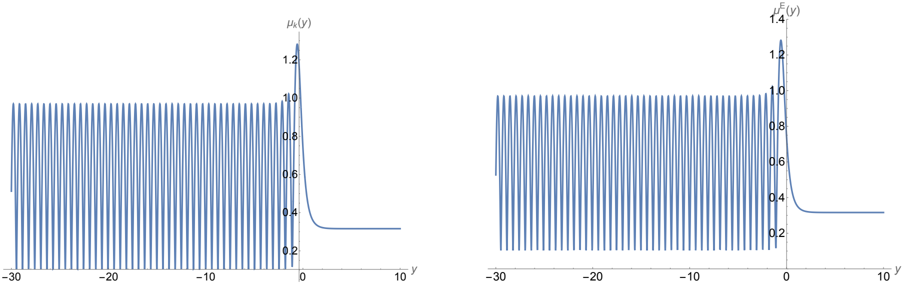
then we have,



(a) $k \gg \beta, \quad \delta < 0.09\%$

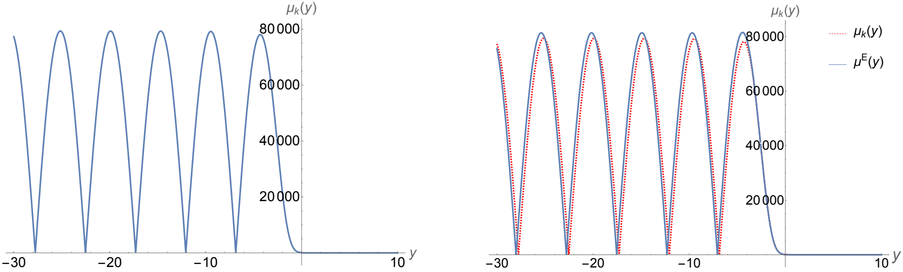
(b) $k > \beta, \quad \delta < 4\%$

Figure 7: $\delta^E(y)$



(a) UAA (b) Exact

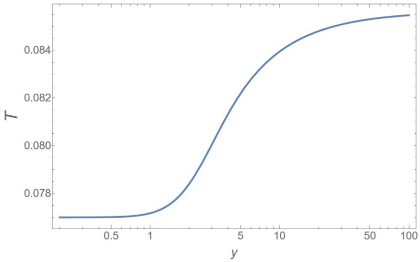
Figure 8: $k = 5.0, \beta = 5.1$ and $q_0^2 = 1/4$



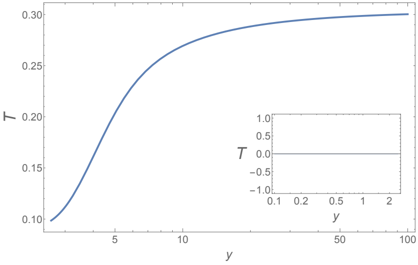
(a) UAA

(b) Exact

Figure 9: $k = 0.6, \beta = 4$ and $q_0^2 = 1/4$

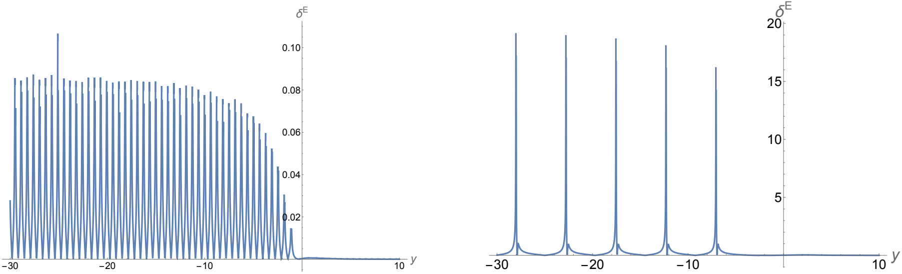


(a) $k < \beta$



(b) $k \ll \beta$

Figure 10: Error control function V_ζ



(a) $k < \beta$ (b) $k \ll \beta$

Figure 11: $\delta^E(y)$

Summary

- The UAA method provides a useful way to solve second-order differential equations by approximate solutions, associated with some particular bounded controlled errors, especially when the equation is hard to solve analytically;
- Applying to PT potential case indicates that UAA approximation has small errors, which may be accurate enough for studying the cosmology observation data;
- The differential equations for the quasi-normal modes of black holes usually also take the form of Equation (1), one can apply the UAA to the studies of black holes as well.

Thank You!