Deformed Spheres in General Relativity

Khalid Saifullah

Work done with:

Rehana Rahim
Usman Gillani

Department of Mathematics
Quaid-i-Azam University, Islamabad

- Einstein's field equations and solutions
- Tests of general relativity
- Johannsen-Psaltis (non-Kerr) spacetime
- The Killing horizons
- Lorentz violations, closed time-like curves
- The charged generalization of Johannsen-Psaltis spacetime
- The non-Kerr spacetime with acceleration
- Conclusions

The Schwarzschild spacetime

The Schwarzschild spacetime is the simplest static vacuum solution of EFEs. Mathematically it is represented as

$$ds^2 = -\left(1 - \frac{2M}{r}\right) dt^2 + \left(1 - \frac{2M}{r}\right)^{-1} dr^2 + r^2 d\theta^2 + r^2 \sin^2\theta d\phi^2, \qquad (1)$$

where M is the mass of the black hole. This solution was given by Karl Schwarzschild in 1916 just one year after the development of Einstein field equations.

The Reissner-Nordström spacetime

It is the charged generalization of the Schwarzschild spacetime and is given as

$$ds^2 = -\left(1 - \frac{2M}{r} + \frac{q^2}{r^2}\right) dt^2 + \left(1 - \frac{2M}{r} + \frac{q^2}{r^2}\right)^{-1} dr^2 + r^2 d\theta^2 + r^2 \sin^2\theta d\phi^2. \qquad (2)$$

Here black hole's charge is represented by q.

The Kerr spacetime

The rotating generalization of the Schwarzschild spacetime is the Kerr metric. It is stationary, axisymmetric and vacuum solution of EFEs. Its mathematical expression is given by

$$ds^2 = -\left[1 - \frac{2Mr}{\Sigma}\right]dt^2 - \frac{4aMr\sin^2\theta}{\Sigma}dtd\phi + \frac{\Sigma}{\Delta}dr^2 + \Sigma d\theta^2$$
$$+ \sin^2\theta\left[r^2 + a^2 + \frac{2a^2Mr\sin^2\theta}{\Sigma}\right]d\phi^2, \quad (3)$$

where

$$\Sigma = r^2 + a^2\cos^2\theta, \Delta = r^2 + a^2 - 2Mr.$$

Here a denotes spin of the black hole. Setting $a = 0$, one gets the Schwarzschild spacetime.

The Kerr-Newman spacetime

The Kerr-Newman spacetime is the charged generalization of the Kerr metric as well as rotating generalization of the Reissner-Nordström spacetime. It is also stationary, axisymmetric and vacuum solution of EFEs. The mathematical expression is given by

$$ds^2 = -\left(1 - \frac{2Mr}{\Sigma} + \frac{q^2}{\Sigma}\right)dt^2 - \frac{2a(2Mr - q^2)\sin^2\theta}{\Sigma}dtd\phi + \frac{\Sigma}{\Delta}dr^2 + \Sigma d\theta^2$$
$$+ \sin^2\theta\left(r^2 + a^2 + \frac{a^2(2Mr - q^2)\sin^2\theta}{\Sigma}\right)d\phi^2,$$

where q and a denote charge and spin of the black hole respectively. Here Σ is same as in Kerr metric whereas Δ modifies to $r^2 + a^2 - 2Mr + q^2$.

Both Schwarzschild and Reissner-Nordström spacetimes are spherically symmetric. Here the surfaces of constant t and constant r are spheres. In fact, for such surfaces, Schwarzschild and Reissner-Nordström metrics reduce to

$$ds^2 = r^2 d\theta^2 + r^2 \sin^2\theta d\phi^2, \tag{4}$$

which is the line element for a sphere of radius r. Now, consider the Kerr black hole given in Eq. (3). Taking its limit as $M \to 0$, we get

$$g_{\mu\nu} = -dt^2 + \frac{r^2 + a^2 \cos^2\theta}{a^2 + r^2} dr^2 + \left(r^2 + a^2 \cos^2\theta\right) d\theta^2 + \left(a^2 + r^2\right) \sin^2\theta \, d\phi^2. \tag{5}$$

The above equation represents Minkowski spacetime in spheroidal coordinates.

It can also be written as

$$ds^2 = -dt^2 + dx^2 + dy^2 + dz^2, \tag{6}$$

where

$$x = \sqrt{r^2 + a^2}\sin\theta\cos\phi, \quad y = \sqrt{r^2 + a^2}\sin\theta\sin\phi, \quad z = r\cos\theta. \tag{7}$$

Here a can be considered as deviation from the spherical geometry. The spatial part of Eq. (6) can be written as

$$dx^2 + dy^2 + dz^2 = \frac{r^2 + a^2\cos^2\theta}{r^2 + a^2}dr^2 + \left(r^2 + a^2\cos^2\theta\right)d\theta^2 + \left(r^2 + a^2\right)\sin^2\theta\, d\phi^2. \tag{8}$$

The above line element describes a prolate spheroid for $a^2 > 0$, oblate spheroid for $a^2 < 0$ and sphere for $a^2 = 0$.

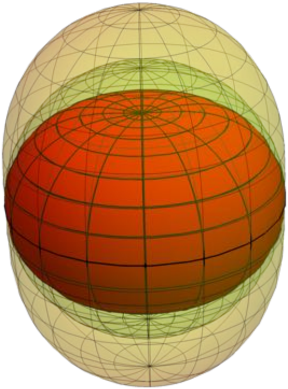


Figure: Spheroids: prolate spheroid with $a^2 > 0$ (in yellow) compared to oblate spheroid with $a^2 < 0$ (in red) and to the reference sphere $a^2 = 0$ (in green).

Historical tests (the weak gravitational field regime)

- The first experimental confirmation of the General Relativity was obtained in 1919 from measuring the bending of light in the vicinity of the surface of the Sun.
- Systematic tests of GR started much later.
- Experiments in the Solar system began in 1960s.
- Tests employing the observations of the binary pulsars started in the 1970s.
- Over the past many years, a large number of experiments have confirmed the predictions of GR in the weak gravitational field.

Testing in the strong gravitational field regime

It would be interesting to test the predictions of general relativity in the strong gravitational field regime. The ideal laboratory in this case is spacetime around the astrophysical black holes.

The no-hair theorem

Kerr black hole is the unique solution of Einstein's field equations having the following properties

- stationary
- asymptotically flat
- axisymmetric
- vacuum
- regular outside the event horizon
- no closed time-like curves

Except for the very short transient periods such as mergers, all the astrophysical black holes are expected to be described by the Kerr metric and its two parameters.

- Methods such as continuum-fitting and iron line have been proposed to measure the black hole spin under the assumption that the black hole metric is described by the Kerr solution of GR. The electromagnetic spectrum is measured in the Kerr spacetime and the results are compared with the observational data to see the best fit and measure the estimate in the spin values and other free parameters appearing in the model.
- Following the same approach, one can use some non-Kerr spacetime, determine the expected spectrum in this background, and do the comparison of the new predictions with the observational data. In this way one can check if the non-kerr metric gives the best fit than Kerr and constrain deviations from the Kerr case.
- One can expect the deviations from the Kerr solution, for example, from classical extensions of GR, macroscopic quantum gravity effects or the presence of exotic matter.

The top-bottom approach

Modification and parameterization of the action and constraining the deviations through observations.

The bottom-top approach

Here, phenomenological parameterization of the metric is employed. Possible deviations are in terms of the deviation parameters. The parameters are determined from the astrophysical data.

There are many metrics in the alternate gravity theories each with their advantages and disadvantages. One such example is the metric proposed by Johannsen and Psaltis:
[T. Johannsen and D. Psaltis, Phys. Rev. D **83** (2011) 124015]
Its further extensions are

* The charged Johannsen-Psaltis spacetime,
  R. Rahim and K. Saifullah, Annals Phys. **405** (2019) 220.
* The charged CPR black hole,
  R. Rahim and K. Saifullah, IJMPD. (2021) 2150123.
* The non-Kerr black hole with acceleration,
  U. A. Gilliani, R. Rahim and K. Saifullah, Astroparticle Phys. 138 (2022) 102684.
* The CPR black hole with acceleration.
  U. A. Gilliani and K. Saifullah, Eur. Phys. J. C **81** (2021) 841.

The Schwarzschild metric is given by

$$ds^2 = -f(r)dt^2 + f(r)^{-1}dr^2 + r^2 d\theta^2 + r^2 \sin^2\theta d\phi^2, \tag{9}$$

where $f = 1 - 2M/r$ and M is the mass of the central object. The $(t - r)$ sector is modified by multiplying the corresponding component by the expression of the form $1 + h(r)$ where $h(r)$ is given by

$$h(r) = \sum_{k=0}^{\infty} \epsilon_k \left(\frac{M}{r}\right)^k. \tag{10}$$

The deformed Schwarzchild metric thus takes the form

Change of coordinates

Change from (t, r, θ, ϕ) coordinates to (u', r', θ', ϕ') where

$$du' = dt - \frac{dr}{f}, \tag{11}$$

$$r = r', \theta = \theta', \phi = \phi'. \tag{12}$$

The result is (after removing the primes)

$$g_{\mu\nu} = -f\Big(1 + h(r)\Big)du^2 - 2\Big(1 + h(r)\Big)dudr + r^2 d\theta^2 + r^2 \sin^2\theta d\phi^2. \tag{13}$$

The Newman-Penrose formalism

The contravariant metric in Newman-Penrose formalism is

$$g^{\mu\nu} = -l^\mu n^\nu - l^\nu n^\mu + m^\mu \overline{m}^\nu + m^\nu \overline{m}^\mu. \tag{14}$$

The null vectors in our case are

$$
\begin{aligned}
l^\mu &= (0,1,0,0), \\
n^\mu &= (1+h(r))^{-1}(1,-f/2,0,0), \\
m^\mu &= \frac{1}{\sqrt{2}r}\left(0,0,1,\frac{i}{\sin\theta}\right), \\
\overline{m}^\mu &= \frac{1}{\sqrt{2}r}\left(0,0,1,-\frac{i}{\sin\theta}\right).
\end{aligned}
\tag{15}
$$

Complexification of the variables

Consider r to be complex. Changes in the expressions for r are

$$
\begin{aligned}
\frac{1}{r} &\longrightarrow \frac{1}{2}\left(\frac{1}{r}+\frac{1}{\overline{r}}\right), \\
\frac{1}{r^2} &\longrightarrow \left(\frac{1}{r\overline{r}}\right).
\end{aligned}
\tag{16}
$$

where the bar denotes the complex conjugate.

The null vectors are

$$l^\mu = \Big(0, 1, 0, 0\Big), \tag{17}$$

$$n^\mu = (1 + h(r, \bar{r}))^{-1}(1, -f(r, \bar{r})/2, 0, 0), \tag{18}$$

$$m^\mu = \frac{1}{\sqrt{2}r}\Big(0, 0, 1, \frac{i}{\sin\theta}\Big), \tag{19}$$

$$\bar{m}^\mu = \frac{1}{\sqrt{2}\bar{r}}\Big(0, 0, 1, -\frac{i}{\sin\theta}\Big), \tag{20}$$

where

$$f(r, \bar{r}) = 1 - M\left(\frac{1}{r} + \frac{1}{\bar{r}}\right), \tag{21}$$

$$h(r, \bar{r}) = \sum_{k=0}^{\infty}\left(\epsilon_{2k} + \epsilon_{2k+1}\frac{M}{2}\left(\frac{1}{r} + \frac{1}{\bar{r}}\right)\right)\left(\frac{M^2}{r\bar{r}}\right)^k. \tag{22}$$

Using the transformation

$$u' = u - ia\cos\theta, \quad r' = r + ia\cos\theta, \tag{23}$$
$$\theta = \theta', \quad \phi = \phi', \tag{24}$$

the null vectors in Eqs. (17)-(20) take the form (again the primes have been removed)

$$l^\mu = \Big(0, 1, 0, 0\Big), \tag{25}$$
$$n^\mu = \frac{1}{\Big(1 + h(r,\theta)\Big)}\Big(1, \frac{-f(r,\theta)}{2}, 0, 0\Big), \tag{26}$$
$$m^\mu = \frac{1}{\sqrt{2}r}\Big(ia\sin\theta, -ia\sin\theta, 1, \frac{i}{\sin\theta}\Big), \tag{27}$$
$$\overline{m}^\mu = \frac{1}{\sqrt{2}\bar{r}}\Big(-ia\sin\theta, ia\sin\theta, 1, -\frac{i}{\sin\theta}\Big). \tag{28}$$

Here the expressions for $f(r,\theta), h(r,\theta)$ are

$$f(r,\theta) = 1 - \frac{2Mr}{\Sigma}, \tag{29}$$

$$h(r,\theta) = \sum_{k=0}^{\infty}\left(\epsilon_{2k} + \epsilon_{2k+1}\frac{Mr}{\Sigma}\right)\left(\frac{M^2}{\Sigma}\right)^k, \tag{30}$$

where Σ is given by

$$\Sigma = r^2 + a^2\cos^2\theta. \tag{31}$$

Using Eqs. (25)-(31), the metric tensor $g_{\mu\nu}$ is written in terms of the coordinates (u, r, θ, ϕ) as

$$g_{00} = -f\,(1+h)\,, g_{01} = -\,(1+h)\,, \tag{32}$$

$$g_{03} = a\,(1+h)\,(f-1)\sin^2\theta, g_{13} = a\,(1+h)\sin^2\theta, \tag{33}$$

$$g_{22} = \Sigma, g_{33} = \sin^2\theta\left[\Sigma - a^2\,(1+h)\,(f-2)\sin^2\theta\right]. \tag{34}$$

Conversion into Boyer-Lindquist coordinates

Change the coordinates (u, r, θ, ϕ) to (t', r', θ', ϕ') by using the transformations

$$du = dt' + \frac{r'^2 + a'^2}{\Delta} dr', \quad d\phi = d\phi' - \frac{adr'}{\Delta},$$
$$r = r', \quad \theta = \theta',$$

where $\Delta = r^2 + a^2 - 2Mr$. The new metric is (after removing the primes)

$$g_{00} = -f(1+h), \quad g_{03} = a(1+h)(f-1)\sin^2\theta \tag{35}$$

$$g_{11} = \frac{\Sigma}{\Delta} + \frac{h(\Sigma\Delta - \Delta a^2 \sin^2\theta + a^4 \sin^4\theta - 2Mra^2 \sin^2\theta)}{\Delta^2}, \tag{36}$$

$$g_{13} = a(1+h)\sin^2\theta, \tag{37}$$

$$g_{22} = \Sigma, g_{33} = \sin^2\theta \left[\Sigma - a^2(1+h)(f-2)\sin^2\theta\right]. \tag{38}$$

In order to remove the off-diagonal term g_{13}, we use another transformation

$$dt = dt' + F(r', \theta')dr, \quad d\phi = d\phi' + G(r', \theta')dr, \tag{39}$$

$$r = r', \quad \theta = \theta', \tag{40}$$

where

$$F(r', \theta') = -\frac{g_{13}}{g_{00}}\left(\frac{g_{03}^2 - g_{33}g_{00}}{g_{00}g_{03}}\right)^{-1}, G(r', \theta') = \frac{g_{13}}{g_{03}}\left(\frac{g_{03}^2 - g_{33}g_{00}}{g_{00}g_{03}}\right)^{-1}. \tag{41}$$

This leads to the metric tensor given by (after removing the primes)

$$
\begin{aligned}
ds^2 = &-(1+h(r,\theta))\left(1-\frac{2Mr}{\Sigma}\right)dt^2 + \Sigma d\theta^2 \\
&- \frac{4aMr\sin^2\theta}{\Sigma}(1+h(r,\theta))dtd\phi + \frac{\Sigma(1+h(r,\theta))}{\Delta + a^2\sin^2\theta h(r,\theta)}dr^2 \\
&+ \left[\sin^2\theta\left(r^2+a^2+\frac{2a^2Mr\sin^2\theta}{\Sigma}\right)\right. \\
&\left. + h(r,\theta)\frac{a^2\sin^4\theta(\Sigma+2Mr)}{\Sigma}\right]d\phi^2, \qquad (42)
\end{aligned}
$$

where $\Sigma = r^2 + a^2\cos^2\theta$, $\Delta = r^2 + a^2 - 2Mr$ and $h(r,\theta)$ has the general expression given in Eq. (30). With $h = 0$, the Kerr spacetime is recovered.

The function $h(r, \theta)$ contains infinite number of parameters. The first two parameters ϵ_0 and ϵ_1 are set to zero by requiring that the metric must be asymptotically flat and the next parameter ϵ_2 is constrained at 10^{-4} by weak field tests of general relativity in the parameterized post-Newtonian approach. Thus ϵ_2 can also be set to zero. For the simplest case, we can set $\epsilon_k = 0$ for $k > 3$, which leads to $h(r, \theta)$ as

$$h(r, \theta) = \frac{\epsilon_3 M^3 r}{\Sigma^2}. \tag{43}$$

Since ϵ_3 is the only retained deviation parameter, it is represented as ϵ in the further analysis.

A Killing horizon is a null hypersurface on which there is a null Killing vector field. In a stationary and axisymmetric spacetime, the Killing horizon is given by

$$g_{tt}g_{\phi\phi} - g_{t\phi}^2 = 0, \tag{44}$$

which for metric (42) takes the value $(1+h)(\Delta + a^2 h \sin^2\theta)\sin^2\theta = 0$.

Kretschmann scalar and the Killing horizon

$1 + h(r,\theta)$ corresponds to spacetime singularity as the Kretschmann scalar

$$R_{\alpha\beta\gamma\delta}R^{\alpha\beta\gamma\delta} \propto (1 + h(r,\theta))^{-6}. \tag{45}$$

It diverges at the Killing horizon for the polar angles $0 < \theta < \pi$.

The graphs are drawn for $a = 0.9$ and $M = 1$ with varying values of ϵ.

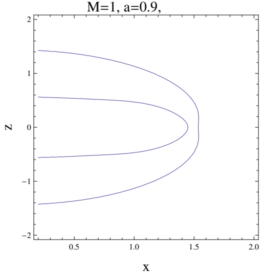


Figure: 1(a). The inner and outer Killing horizons have spherical topology.

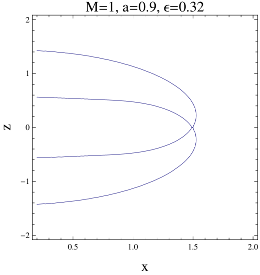


Figure: 1(b). The inner and outer Killing horizons merge at the equatorial plane.

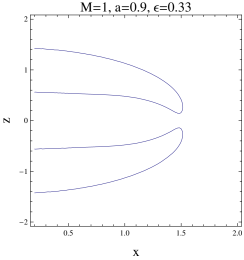


Figure: 1(c). Disjoint Killing horizons appear above and below the equatorial plane.

Killing horizons for negative deformation parameter

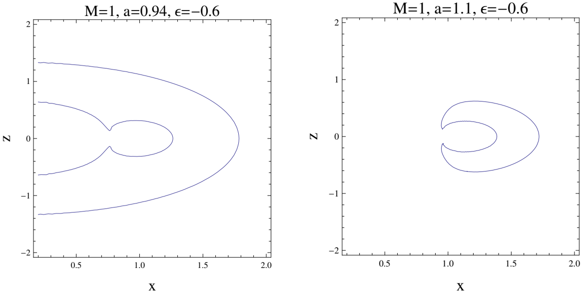


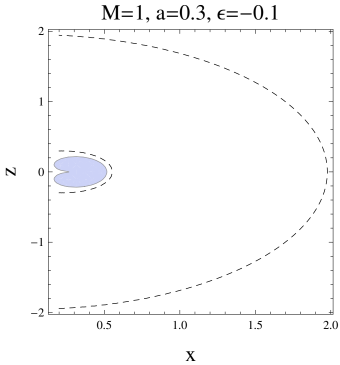
Figure: 2. The Killing horizon for negative ϵ. On the left $a = 0.94, \epsilon = -0.6$ and $M = 1$. Here the inner and outer Killing horizons are in the form of spherical surface. On the right $a = 1.1, \epsilon = -0.6$ and $M = 1$. The Killing horizon is in the shape of toroidal surface.

$$det(g_{\mu\nu}) = -\frac{\sin^2\theta}{64\Sigma^2}\Big[3a^4 + 8a^2r^2 + 8r^4 + 8\epsilon M^3 r + 4a^2(2r^2 + a^2)\cos 2\theta + a^4\cos 4\theta\Big]^2. \quad (46)$$

It is negative, semidefinite and becomes zero at two values of the radii for $\epsilon < -4r_+$ which coincide with the location of the Killing horizon as

$$det(g_{\mu\nu}) \propto g_{tt}g_{\phi\phi} - g_{t\phi}^2. \quad (47)$$

So, the metric does not contain any Lorentz violating regions.



The Killing horizons and closed time-like curves for $\epsilon = -0.1$ and $a = 0.3$. Mass M has been set to 1. The dashed curves denote the inner and outer Killing horizons and the solid region shows the closed time-like curve.

The closed time-like curves in the above graph lie within the inner Killing horizon. In general, for Johannsen-Psaltis metric they lie inside the outer Killing horizon. Thus it does not contain any closed time-like curves outside the central object.

The starting point is the Reissner-Nordström metric

$$ds^2 = -\left(1 - \frac{2M}{r} + \frac{q^2}{r^2}\right) dt^2 + \left(1 - \frac{2M}{r} + \frac{q^2}{r^2}\right)^{-1} dr^2 + r^2 d\theta^2 + r^2 \sin^2\theta d\phi^2, \tag{48}$$

where M is the mass of the central object having charge q. The 4-potential for the above metric is

$$A_\mu = \left(-\frac{q}{r}, 0, 0, 0\right). \tag{49}$$

The $(t - r)$ sector is modified by multiplying the corresponding component by the expression of the form $1 + h(r)$ where $h(r)$ is

$$h(r) = \sum_{k=0}^{\infty} \epsilon_k \left(\frac{M}{r}\right)^k. \tag{50}$$

Going through the Newman-Janis algorithm, the charged generalization of the Johannsen-Psaltis metric is

$$ds^2 = -(1+h(r,\theta))\left(1-\frac{2Mr}{\Sigma}+\frac{q^2}{\Sigma}\right)dt^2 + \Sigma d\theta^2 - \frac{2a(2Mr-q^2)\sin^2\theta}{\Sigma}(1+h(r,\theta))dtd\phi + \frac{\Sigma(1+h(r,\theta))}{\Delta + a^2\sin^2\theta h(r,\theta)}dr^2 + \left[\sin^2\theta\left(r^2+a^2+\frac{a^2(2Mr-q^2)\sin^2\theta}{\Sigma}\right) + h(r,\theta)\frac{a^2\sin^4\theta(\Sigma+2Mr-q^2)}{\Sigma}\right]d\phi^2, \quad (51)$$

where $\Sigma = r^2 + a^2\cos^2\theta$, $\Delta = r^2 + a^2 - 2Mr + q^2$ and $h(r,\theta)$ has the general expression given in Eq. (30). Setting $q = 0$ gives the Johannsen-Psaltis metric. With $h = 0$, the Kerr-Newman spacetime is recovered.
The function $h(r,\theta)$ is again determined by the astrophysical observations and has the same non-zero parameters as in Johannsen-Psaltis.

Applying the Newman-Janis algorithm on the potential A_μ in Eq. (49) leads to

$$A_\mu = \left(-\frac{qr}{\Sigma}, \frac{qr}{\Delta + a^2 h \sin^2\theta}, 0, \frac{aqr\sin^2\theta}{\Sigma} \right).$$

Here the A_r component depends on θ.

The Killing horizon, in this case has the equation

$$(1+h)(r^2+a^2+q^2-2Mr+a^2 h \sin^2\theta)\sin^2\theta = 0. \qquad (52)$$

The graphs show the similar behavior as in Johannsen-Psaltis metric when plotted for various values of the parameters ϵ, a, M and q.

The accelerating and rotating black hole solution is an important member of the Plebański and Demiański family of spacetimes. Here, acceleration of the black hole is measured by the parameter α. The metric represents the gravitational field of a pair of uniformly accelerating Kerr-type black holes. The charged non-Kerr accelerating spacetime is proposed as

$$ds^2=\frac{1}{\Omega^2}\Big\{-\left(\frac{Q}{\Sigma}-\frac{a^2P\sin^2\theta}{\Sigma}\right)(1+h)\,dt^2+\frac{\Sigma\,(1+h)}{Q+a^2h\sin^2\theta}dr^2+\frac{\Sigma}{P}d\theta^2$$
$$+\sin^2\theta\left(\frac{P(r^2+a^2)^2}{\Sigma}-\frac{Qa^2\sin^2\theta(1+h)}{\Sigma}\right)\Big\}d\phi^2$$
$$-\frac{2a\sin^2\theta(P(r^2+a^2)-Q)(1+h)}{\Sigma\Omega^2}dtd\phi, \tag{53}$$

where

$$\Omega=1-\alpha r\cos\theta, \tag{54}$$
$$\Sigma=r^2+a^2\cos^2\theta, \tag{55}$$
$$P=1-2\alpha M\cos\theta+\alpha^2\left(a^2+q^2\right)\cos^2\theta, \tag{56}$$
$$Q=(a^2+q^2-2Mr+r^2)(1-\alpha^2r^2). \tag{57}$$

Putting $\alpha = 0$ gives charged non-Kerr black hole. Setting $\alpha = 0 = h$ gives the Kerr-Newman black hole. This metric does not obey the usual Einstein-Maxwell equations due to presence of $h(r, \theta)$. So, we make the assumption that above spacetime might be an electro vacuum solution to the unknown field equations which are different from the Einstein-Maxwell equations for non vanishing $h(r, \theta)$.

In the CPR metric (named after the authors V. Cardoso, P. Pani and J. Rico) the approach of the non-Kerr metric was extended to construct a seed metric having two different deformation functions in the g_{tt} and g_{rr} components of the Schwarzschild black hole. The charged CPR metric is

$$\begin{aligned} ds^2 = & -(1+h^t)\left(1-\frac{2Mr}{\Sigma}+\frac{q^2}{\Sigma}\right)dt^2 - 2a\sin^2\theta\left[H-(1+h^t)\left(1-\frac{2Mr}{\Sigma}+\frac{q^2}{\Sigma}\right)\right]dtd\phi \\ & + \frac{\Sigma(1+h^r)}{\Delta+a^2\sin^2\theta h^r}dr^2 + \sin^2\theta\left\{\Sigma+a^2\sin^2\theta\left[2H-(1+h^t)\left(1-\frac{2Mr}{\Sigma}+\frac{q^2}{\Sigma}\right)\right]\right\}d\phi^2 \\ & + \Sigma d\theta^2, \end{aligned} \tag{58}$$

where $H = \sqrt{(1+h^t)(1+h^r)}, \Sigma = r^2 + a^2\cos^2\theta, \Delta = r^2 + a^2 - 2Mr + q^2$ and $h^i(r,\theta) = \epsilon^i M^3 r/\Sigma^2$ for $i = t, r$. Setting $\epsilon^t = \epsilon^r$ gives the charged Johannsen-Psaltis metric.

The charged CPR black hole metric has been extended to acceleration parameter α. The metric is

$$ds^2=\frac{1}{\Omega^2}\Bigg\{-\left(\frac{Q}{\rho^2}-\frac{a^2 P\sin^2\theta}{\rho^2}\right)\left(1+h^t\right)dt^2+\frac{\rho^2\left(1+h^r\right)}{Q+a^2h^r\sin^2\theta}dr^2+\frac{\rho^2}{P}d\theta^2$$
$$+\left[\left(\frac{P(r^2+a^2)^2}{\rho^2}-\frac{Qa^2\sin^2\theta\left(1+h^t\right)}{\rho^2}\right)\sin^2\theta+a^2\sin^4\theta\left(2(H-1)+\frac{a^2h^t\sin^2\theta}{\rho^2}\right)\right]d\phi^2\Bigg\}$$
$$-\frac{2a\sin^2\theta\left[H-\left(1+h^t\right)\left\{1-\frac{P\left(r^2+a^2\right)-Q}{\rho^2}\right\}\right]}{\Omega^2}d\phi dt, \quad (59)$$

where

$$\Omega=1-\alpha r\cos\theta, \quad (60)$$
$$\rho^2=r^2+a^2\cos^2\theta, \quad (61)$$
$$P=1-2\alpha M\cos\theta+\alpha^2\left(a^2+q^2\right)\cos^2\theta, \quad (62)$$
$$Q=(a^2+q^2-2Mr+r^2)(1-\alpha^2r^2). \quad (63)$$

The presentation focuses at the spacetimes suitable for testing the Kerr black hole hypothesis in the strong gravitational field. One such example is the metric proposed by Johannsen and Psaltis. Its salient features are summarized as

- It is an example of the bottom-top approach.
- It is stationary, axisymmetric and asymptotically flat.
- It does not correspond to any known gravity theory, but is a parametrization of deviation from the Kerr spacetime obtained by applying the Newman-Janis algorithm to a deformed Schwarzschild metric.
- It has infinite number of deviation parameters.
- These parameters are determined by the requirement that the new deformed metric is asymptotically flat and is consistent with observational weak-field constraints on deviations from general relativity in the parameterized post-Newtonian approach.

- It does not contain Lorentz-violating regions.
- It does not contain any closed time-like curves outside the central object.
- For the case of a single deformation parameter, the non-Kerr metric is regular and does not contain unphysical properties outside the event horizon and can represent a black hole a black hole upto the maximum range of the spin parameter.
- Kretschmann scalar diverges at the Killing horizon for the polar angles $0 < \theta < \pi$.
- The Killing horizons show dependence on the deviation parameter ϵ.
- The Johannsen-Psaltis metric has been extended to include the electric charge. Starting from the deformed Reissner-Nordström metric and subsequent application of the Newman-Janis algorithm leads to the charged analogue. It has also been extended to include acceleration as well. Starting from a deformed Reissner-Nordström metric having two different deformation functions, the charged CPR black hole has been developed through Newman-Janis algorithm. The accelerating CPR metric has also been developed.

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