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Post-Newtonian Gravitational Waves with cosmological constant Λ derived from Einstein-Hilbert theory

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Preambles of the Post-Newtonian approximation

- This is a method to consists to find an approximate solution of the field equations from some action whose dynamics variable is the metric.
- It consists in expand the components of the metric in small parameters that depends of factors of v/c.
- Metric expansion

$$\begin{aligned} g_{00} &= -1 + \overset{(2)}{g}_{00} + \overset{(4)}{g}_{00} + \cdots \\ g_{0i} &= 0 + \overset{(3)}{g}_{0i} + \overset{(5)}{g}_{0i} + \cdots \\ g_{ij} &= \delta_{ij} + \overset{(2)}{g}_{ij} + \overset{(4)}{g}_{ij} + \cdots \end{aligned} \tag{1}$$

- The parameters of the components of the metric will be fixed through the field equations.

$$R_{\mu\nu} - \frac{1}{2} R g_{\mu\nu} + \Lambda g_{\mu\nu} = \frac{8\pi G}{c^4} T_{\mu\nu} \tag{2}$$

Relaxed Einstein field equations

- The gothic metric is introduced (densitized) as $\mathfrak{g}^{\mu\nu} := \sqrt{-g} g^{\mu\nu}$.
- The metric is expanded as follows: $\mathfrak{g}^{\mu\nu} = \eta^{\mu\nu} + h^{\mu\nu}$.
- Making use of the Harmonic gauge $\partial_\mu \mathfrak{g}^{\mu\nu} = 0$, we obtain the relaxed Einstein field equations

$$\Box h^{\mu\nu} = \frac{16\pi G}{c^4} \mu^{\mu\nu}, \tag{3}$$

$$\mu^{\mu\nu} = (-g) T^{\mu\nu} + \frac{c^4}{16\pi G} \Lambda^{\mu\nu}_{\rm GR} \tag{4}$$

$$\begin{aligned} \Lambda^{\alpha\beta}_{\rm GR} &= \frac{16\pi G}{c^4} (-g) t^{\alpha\beta}_{LL} - 2\Lambda \mathfrak{g}^{-1/2} \mathfrak{g}^{\alpha\beta} \\ &+ \partial_\mu h^{\alpha\mu} \partial_\nu h^{\beta\nu} - h^{\mu\nu} \partial_\mu \partial_\nu h^{\alpha\beta}. \end{aligned} \tag{5}$$

Wave form at 1PN order

- The Waveform corresponding to the near zone reads

$$h_N^{ij}(x) = \frac{2G}{Rc^4}\frac{d^2}{dt^2}\sum_{l=0}^{\infty}\hat{N}_{k_1}\cdots\hat{N}_{k_l}I_{EW}^{ijk_1\cdots k_l}, \tag{6}$$

where the Epstein-Wagoner moments are given explicitly as

$$I_{\rm EW}^{ij} := \frac{1}{c^2}\int_M \mu^{00}x^i x^j\, d^3x, \tag{7}$$

$$I_{\rm EW}^{ijk} := \frac{1}{c^3}\int_M \left(2\mu^{0(i}x^{j)}x^k - \mu^{0k}x^i x^j\right) d^3x, \tag{8}$$

$$I_{\rm EW}^{ijk_1\cdots k_l} := \frac{2}{l!c^2}\frac{d^{l-2}}{d(ct)^{l-2}}\int_M \mu^{ij}x^{k_1}x^{k_2}\cdots x^{k_l}\, d^3x. \tag{9}$$

- At order 1PN the waveform acquires the following form

$$h_N^{ij}(x) = \frac{2G}{Rc^4}\frac{d^2}{dt^2}\left\{I^{ij} + \hat{n}_k I^{ijk} + \hat{n}_k\hat{n}_l I^{ijkl}\right\}_{\rm TT}. \tag{10}$$

- In order to compute the Epstein-Wagoner moments, it is necessary to compute the source $\mu^{\alpha\beta}$ at the necessary order.
- The relaxed Einstein field equations collapse to

$$
\begin{aligned}
\nabla^2 h^{00} &= \frac{16\pi G}{c^2}\sum_a m_a \delta^3(\vec{x}-\vec{x}_a(t)) + 2\Lambda \\
&+ O(\Lambda h, \frac{1}{c^4}), && (11)\\
\nabla^2 h^{0i} &= O(\Lambda h, \frac{1}{c^3}), && (12)\\
\nabla^2 h^{ij} &= -2\Lambda \eta^{ij} + O(\Lambda h, \frac{1}{c^4}). && (13)
\end{aligned}
$$

- From (11), we can say that Λ plays the role of a perturbation parameter (PN factor).
- Bearing in mind the Harmonic gauge $\partial_\mu h^{\mu\nu} = 0$, the solution yields

$$
\begin{aligned}
h^{00} &= -\frac{4G}{c^2}\sum_a \frac{m_a}{r_a} + \frac{\Lambda}{3}|\vec{x}|^2 + O(\Lambda h, c^{-4}), && (14)\\
h^{0i} &= O(\Lambda h, c^{-3}), && (15)\\
h^{ij} &= \delta^{ij}\left[-\frac{1}{2}\Lambda\left(|\vec{x}|^2 - x_i^2\right)\right] + O(\Lambda h, c^{-4}), && (16)
\end{aligned}
$$

where there is no sum over the index i of the term x_i.

$$
h^{\mu\nu} = \begin{pmatrix} -\frac{4G}{c^2}\sum_a \frac{m_a}{r_a} + \frac{\Lambda}{3}|\vec{x}|^2 & 0 & 0 & 0 \\ 0 & -\frac{1}{2}\Lambda(y^2+z^2) & 0 & 0 \\ 0 & 0 & -\frac{1}{2}\Lambda(x^2+z^2) & 0 \\ 0 & 0 & 0 & -\frac{1}{2}\Lambda(x^2+y^2) \end{pmatrix},
$$

- Considering the center of mass of a two particles frame $X^i_{\rm CM} := \frac{1}{m}\int_M \mu^{00} x^i d^3x$, with $X^i_{\rm CM} = 0$, yields

$$\vec{r}_1 = \frac{\mu}{m_1}\vec{r} + \frac{\mu\Delta m}{2m^2c^2}\left(v^2 - \frac{Gm}{r} - \frac{\Lambda c^2 r^2}{2}\right)\vec{r} + O(c^{-2}\Lambda, \Lambda^2, c^{-3}), \tag{17}$$

$$\vec{r}_2 = -\frac{\mu}{m_2}\vec{r} + \frac{\mu\Delta m}{2m^2c^2}\left(v^2 - \frac{Gm}{r} - \frac{\Lambda c^2 r^2}{2}\right)\vec{r} + O(\Lambda c^{-2}, \Lambda^2, c^{-3}), \tag{18}$$

$$\begin{aligned} \vec{v}_1 &= \frac{\mu}{m_1}\vec{v} + \frac{\mu\Delta m}{2m^2c^2}\left[\left(v^2 - \frac{Gm}{r} - \frac{\Lambda c^2 r^2}{2}\right)\vec{v} - \left(\frac{Gm}{r^2} + \frac{\Lambda c^2 r}{2}\right)\dot{r}\vec{r}\right] \\ &+ O(\Lambda c^{-2}, \Lambda^2, c^{-4}), \end{aligned} \tag{19}$$

$$\begin{aligned} \vec{v}_2 &= -\frac{\mu}{m_2}\vec{v} + \frac{\mu\Delta m}{2m^2c^2}\left[\left(v^2 - \frac{Gm}{r} - \frac{\Lambda c^2 r^2}{2}\right)\vec{v} - \left(\frac{Gm}{r^2} + \frac{\Lambda c^2 r}{2}\right)\dot{r}\vec{r}\right] \\ &+ O(\Lambda c^{-2}, \Lambda^2, c^{-4}). \end{aligned} \tag{20}$$

Wave form

- Substituting the Epstein-Wagoner moments, the positions and the velocities in the center of mass of two particles frame we obtain the following result

$$
\begin{aligned}
h^{ij}_{\mathrm{N,TT}} &= \frac{2G\mu}{Rc^4}\frac{d^2}{dt^2}\left\{\left[1+\frac{1}{2c^2}(1-3\nu)(v^2-\Lambda c^2 r^2)-\frac{Gm}{3rc^2}(2-9\nu)\right]r^i r^j\right. \\
&- \frac{\Delta m}{mc^2}\left(2v^{(i}r^{j)}(\hat{N}\cdot\vec{r})-(\hat{N}\cdot\vec{v})r^i r^j\right) \\
&+ \left.\frac{1}{c^2}(1-3\nu)(\hat{N}\cdot\vec{r})^2\left(v^i v^j-\frac{Gm}{3r^3}r^i r^j\right)\right\}_{\mathrm{TT}},
\end{aligned}
\tag{21}
$$

with $\nu := \frac{\mu}{m} = \frac{m_1 m_2}{m}$ as the mass ratio of the system.

Two body Lagrangian system

- The Lagrangian is obtained à la Droste-Fichtenholz.
- We insert the equation of motion of a point particle of mass m_1 from the geodesic equation

$$\begin{aligned} S &:= \int dt L_{m_1} = -m_1 c \int dt \Big(-g_{\mu\nu} \frac{dx^\mu}{dt} \frac{dx^\nu}{dt} \Big)^{1/2} \\ &= -m_1 c^2 \int dt \Big(-g_{00} - 2 g_{0i} \frac{v_1^i}{c} - g_{ij} \frac{v_1^i v_1^j}{c^2} \Big)^{1/2} \end{aligned} \qquad (22)$$

- Considering the center of mass frame, the Lagrangian acquires the following form

$$\begin{aligned} L = & -mc^2 + \frac{1}{2}\mu v^2 + \frac{G\mu m}{r} + \frac{1}{8c^2}\mu v^4 (1-3\nu) + \frac{G\mu m}{2c^2 r}\Big[(3+\nu)v^2 + \nu(\hat{n}\cdot\vec{v})^2 - \frac{Gm}{r}\Big] \\ & + \frac{\Lambda}{6} c^2 \mu r^2 \\ & - \frac{1}{6} G\Lambda\mu r(5+2\nu) + \frac{1}{6}\Lambda\mu(1-3\nu) r^2 v^2 + \frac{\Lambda}{3}\mu(\hat{n}\cdot\vec{v})^2 (1-3\nu) r^2 \\ & - \frac{11}{12}\Lambda\mu(1-3\nu)(x^2 v_x^2 + y^2 v_y^2 + z^2 v_z^2) + O(c^{-4}, \Lambda c^{-2}, \Lambda^2)\,, \end{aligned} \qquad (23)$$

with $\vec{r} = \vec{x}_1 - \vec{x}_2$, $\hat{n} = \frac{\vec{x}_1 - \vec{x}_2}{|\vec{x}_1 - \vec{x}_2|}$, and $\vec{v} := \vec{v}_1 - \vec{v}_2$.

Equation of motion of a binary compact system

- Making use of the Euler-Lagrange equations

$$\frac{d}{dt}\left(\frac{\partial L}{\partial v^i}\right) - \frac{\partial L}{\partial r^i} = 0\,, \tag{24}$$

we get the equations of motion corresponding to the interaction of two body compact system

$$\begin{aligned} a^i \quad = \quad & -\frac{Gm}{r^2}\hat{n}^i + \frac{Gm}{c^2 r^2}\left\{\left[\frac{Gm}{r}(4+2\nu) - v^2(1+3\nu) + \frac{3}{2}\nu(\hat{n}\cdot\vec{v})^2\right]\hat{n}^i + (4-2\nu)(\hat{n}\cdot\vec{v})v^i\right\} \\ & +\frac{\Lambda}{3}c^2 r\hat{n}^i \\ & +\Lambda(1-3\nu)\left[-\frac{5}{3}r(\hat{n}\cdot\vec{v}) + \frac{11}{3}(r_i\dot{r}_i)\right]v^i - Gm\Lambda\left[2(\frac{3}{4}+\nu) + \frac{11}{6}(1-3\nu)\frac{(r_i)^2}{r^2}\right]\hat{n}^i \\ & -\Lambda r(1-3\nu)\left[\frac{1}{2}v^2 + \frac{11}{6}(v_i)^2\right]\hat{n}^i + O(c^{-4}, \Lambda c^{-2}, \Lambda^2)\,. \end{aligned} \tag{25}$$

Waveform in a circular motion $\dot{r} = 0$

$$
\begin{aligned}
h^{ij}_{\mathrm{N}}(t,\vec{x}) \;=\;& \frac{2G\mu}{c^4 R}\Bigg\{2\left(v^i v^j - \frac{Gm}{r^3}r^i r^j\right) + \frac{\Lambda}{3}c^2 r^i r^j \\
+\;& \frac{\Delta m}{c}\Bigg[3\frac{Gm}{r^3}(\hat{n}\cdot\vec{r})\left(2v^{(i}r^{j)} - \frac{\dot{r}}{r}r^i r^j\right) + (\vec{v}\cdot\hat{n})\left(-2v^i v^j + \frac{Gm}{r^3}r^i r^j\right) \\
-\;& 2\Lambda c^2(\vec{n}\cdot\vec{r})v^{(i}r^{j)} - \frac{\Lambda}{3}c^2(\vec{n}\cdot\vec{v})r^i r^j\Bigg] \\
+\;& \frac{1}{c^2}\Bigg\{\frac{1}{3}\left[3(1-3\nu)v^2 - 2(2-3\nu)\frac{Gm}{r}\right]v^i v^j + \frac{4}{3}(5+3\nu)\frac{Gm}{r^2}\dot{r}v^{(i}v^{j)} \\
+\;& \frac{1}{3}\frac{Gm}{r^3}\left[-(10+3\nu)v^2 + 3(1-3\nu)\dot{r}^2 + 29\frac{Gm}{r}\right]r^i r^j \\
+\;& \frac{2}{3}(1-3\nu)(\vec{v}\cdot\hat{n})^2\left(3v^i v^j - \frac{Gm}{r^3}r^i r^j\right) \\
+\;& \frac{4}{3}(1-3\nu)(\vec{v}\cdot\hat{n})(\vec{r}\cdot\hat{n})\frac{Gm}{r^3}\left[-8v^{(i}r^{j)} + 3\frac{\dot{r}}{r}r^i r^j\right] \\
+\;& \frac{1}{3}(1-3\nu)(\vec{r}\cdot\hat{n})^2\frac{Gm}{r^3}\left[-14v^i v^j + 30\frac{\dot{r}}{r}v^{(i}r^{j)} + \left(3\frac{v^2}{r^2} - 15\frac{\dot{r}^2}{r^2} + 7\frac{Gm}{r^3}\right)r^i r^j\right]\Bigg\} \\
-\;& \frac{17\Lambda}{9}(1+3\nu)\frac{Gm}{r}r^i r^j - \Lambda\left[2\left(\frac{2}{3}-\nu\right)v^2 + (1-3\nu)\left(\frac{Gm}{r^3}(r_i)^2 + (v_i)^2\right)\right]r^i r^j \\
+\;& \Lambda\left[2(1-3\nu)r_i\dot{r}_i - (6-14\nu)r\dot{r}\right]v^{(i}r^{j)} - \Lambda(1-3\nu)r^2 r^i r^j \\
+\;& \frac{4}{3}\Lambda(1-3\nu)(\hat{n}\cdot\vec{r})^2 v^i v^j - \frac{13}{9}\Lambda(1-3\nu)\frac{Gm}{r^3}(\hat{n}\cdot\vec{r})^2 r^i r^j \\
+\;& \frac{8}{3}\Lambda(1-3\nu)(\hat{n}\cdot\vec{v})(\hat{n}\cdot\vec{r})r^{(i}v^{j)} + O(c^{-4}, c^{-2}\Lambda, \Lambda^2)\Bigg\}.
\end{aligned}
\tag{26}
$$

Polarizations h_+ and $h_\times$ of Gravitational Waves

- Defining the Post-Newtonian parameter $x := \left(\frac{\omega G m}{c^3}\right)^{1/3}$,

$$h_+ = \frac{2G\mu}{c^2 R}\left(\frac{Gm\omega}{c^3}\right)^{2/3}\left\{H_+^0 + xH_+^{1/2} + x^2 H_+^1 + O(x^3, \Lambda c^{-1}, \Lambda^2)\right\}, \qquad (27)$$

with

$$\begin{aligned}
H_+^0 &= -(1+\cos^2\iota)\cos 2\phi + \frac{\Lambda c^2}{\omega^2}\left(-\frac{1}{12}\sin^2\iota + \frac{5}{36}(1+\cos^2\iota)\cos 2\phi\right), \\
H_+^{1/2} &= -\frac{\Delta m}{m}\frac{1}{8}\sin\iota\left[(5+\cos^2\iota)\cos\phi - 9(1+\cos^2\iota)\cos 3\phi\right]\left(1-\frac{\Lambda c^2}{3\omega^2}\right), \\
H_+^1 &= \frac{1}{6}\{[19+9\cos^2\iota - 2\cos^4\iota] - \nu[19 - 11\cos^2\iota - 6\cos^4\iota]\}\cos 2\phi \\
&\quad -\frac{4}{3}\sin^2\iota(1+\cos^2\iota)(1-3\nu)\cos 4\phi \\
&\quad +\frac{\Lambda c^2}{\omega^2}\left\{\frac{13}{24} - \frac{9}{16}\cos^2\iota + \frac{1}{48}\cos^4\iota + \frac{275}{72}\nu\sin^2\iota + \cos 2\phi\left[-\frac{371}{432} - \frac{35}{144}\cos^2\iota - \frac{35}{108}\cos^4\iota\right.\right. \\
&\quad \left. +\nu\left(\frac{331}{144} + \frac{65}{144}\cos^2\iota + \frac{35}{36}\cos^4\iota\right)\right] \\
&\quad \left. +\cos 4\phi\left[\frac{5}{18} + \frac{11}{54}\cos^2\iota - \frac{13}{27}\cos^4\iota + \nu\left(-\frac{5}{6} - \frac{69}{72}\cos^2\iota + \frac{13}{9}\cos^4\iota\right)\right]\right\},
\end{aligned}$$

- Considering $\omega_0 = c\sqrt{5\Lambda}/6$, the amplitude $h+$ is canceled out at Newtonian order.

$$h_\times = \frac{2G\mu}{c^2 R}\left(\frac{Gm\omega}{c^3}\right)^{2/3}\left\{H_\times^0 + xH_\times^{1/2} + x^2 H_\times^1 + O(x^3, \Lambda c^{-1}, \Lambda^2)\right\}, \qquad (28)$$

with

$$H_\times^0 = -2\cos\iota\sin2\phi + \frac{\Lambda c^2}{9\omega^2}\cos\iota\sin2\phi\,,$$

$$H_\times^{1/2} = -\frac{\Delta m}{m}\frac{3}{8}\sin2\iota\left[\left(1+\frac{2}{9}\frac{\Lambda c^2}{\omega^2}\right)\sin\phi - \left(3-\frac{20}{9}\frac{\Lambda c^2}{\omega^2}\right)\sin3\phi\right],$$

$$\begin{aligned} H_\times^1 = {} & \cos\iota\left[\left\{\left(\frac{17}{3}-\frac{4}{3}\cos^2\iota\right)+\nu\left(-\frac{13}{3}+4\cos^2\iota\right)\right\}\sin2\phi - \frac{8}{3}(1-3\nu)\sin^2\iota\sin4\phi \right.\\ & \left. +\frac{\Lambda c^2}{\omega^2}\left\{\left(-\frac{92}{27}+\frac{1}{3}\cos^2\iota\right)+\nu\left(\frac{79}{18}-\frac{13}{6}\cos^2\iota\right)\right\}\sin2\phi + \frac{\Lambda c^2}{\omega^2}\left(\frac{359}{216}-\frac{359}{72}\nu\right)\sin^2\iota\sin4\phi\right].\end{aligned}$$

- Considering $\omega_0 = c\sqrt{2\Lambda}/6$, the amplitude of $h_\times$ is canceled out at Newtonian order.

Particular cases

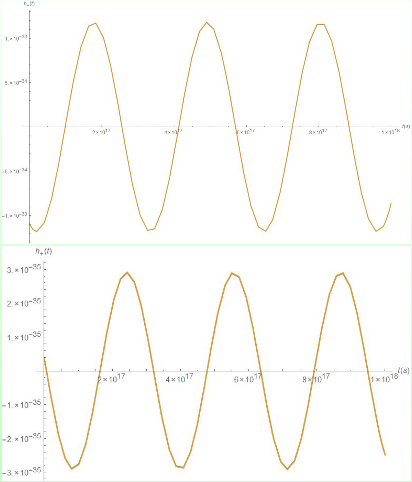

Figure: h_+ (top figure) and $h_\times$ (bottom figure) for a binary compact system of identical masses at 1PN order with parameter values $m = 10^{31}$kg, $R = 200 \times 10^{22}$m, $\omega = 10^{-17}\mathrm{s}^{-1}$, $\Lambda = 10^{-52}\mathrm{m}^{-2}$ and the inclination angle $\iota = \pi/2$ (top figure), $\iota = 0$ (bottom figure). The effect of Λ is negligible.

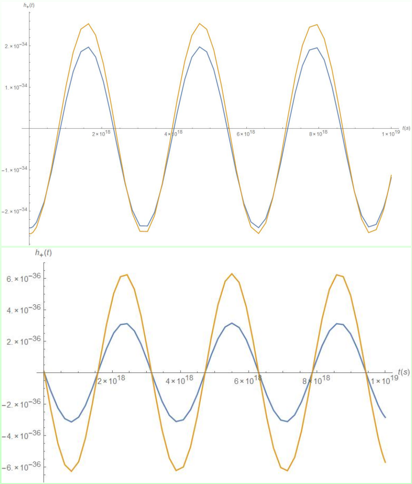

Figure: h_+ (top figure) and $h_\times$ (bottom figure) for a binary compact system of identical masses at 1PN order. The parameters are given by $m = 10^{31}$Kg, $R = 200 \times 10^{22}$m, $\omega = 10^{-18}\mathrm{s}^{-1}$ and the inclination angle $\iota = \pi/2$ (top figure), $\iota = 0$ (bottom figure). The blue line includes Λ, while the orange one does not ($\Lambda = 0$). Note that with this particular frequency, the effect of Λ starts to be observable.

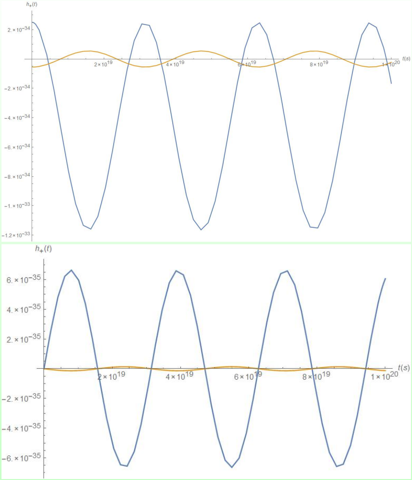

Figure: h_+ (top figure) and $h_\times$ (bottom figure) for a binary compact system of identical masses at 1PN order. The parameter values are given by $m = 10^{31}$kg, $R = 200 \times 10^{22}$m, $\omega = 10^{-19}\mathrm{s}^{-1}$ with inclination angle $\iota = \pi/2$ (top figure), $\iota = 0$ (bottom figure). The blue line includes Λ, while the orange one does not ($\Lambda = 0$). Note that with this particular frequency, the effect of Λ becomes very evident.

Conclusions and results

- The cosmological constant Λ can be interpreted as a Post-Newtonian factor.
- Using the Post-Newtonian approach at 1PN order, we compute the Lagrangian that describes a compact binary system for very small and positive values for Λ.
- We obtain the polarizations of the waveforms h_+ and $h_\times$ making use of the equations of motion of a binary compact system.
- For particular frequencies $\omega_0 = c\sqrt{5\Lambda}/6$ and $\omega_0 = c\sqrt{2\Lambda}/6$, the amplitudes of h_+ and $h_\times$ are canceled out at Newtonian order.